DOI: 10.24996/ijs.2024.65.8.29

ISSN: 0067-2904

The Migration Effect on an Eco-toxicant Model

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Received: 16/4/2023 Accepted: 9/7/2023 Published: 30/8/2024

Abstract

This paper proposes and studies an ecotoxicant system with Lotka-Volterra functional response for predation including prey protective region. The equilibrium points and the stability of this model have been investigated analytically both locally and globally. Finally, numerical simulations and graphical representations have been utilized to support our analytical findings.

Keywords: Toxin , Refuge region , Predatory region.

تأثير الهجرة على نظام بيئي سمي

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قسم الرياضيات , كلية العلوم, جامعة بغداد, بغداد, العراق

الخلاصة

في هذا البحث تم اقتراح ودراسة نظام بيئي – سمي مع دالة استجابة من نوع لوتيكا فولتيرا متضمن وجود مناطق محمية لمجتمع الفريسة . نقاط الاتزان و الاستقرارية (المحلية والشاملة) لهذا النظام درست باستخدام الطرق التحليلية , اخيرا المحاكاة العددية والرسوم التوضيحية استخدمت لتأكيد النتائج التحليلية.

1. Introduction:

According to the biological theory of animal migration, living creatures migrate to locations with better living conditions. Together with mammals, many birds, fish, and insects move frequently to avoid unfavorable changes, whether in climate or food sources, as people migrate, however, their migration is often political or social, and they may also migrate for biological reasons.

Biologists use the term immigration to describe many movements. Biologists, especially entomologist, consider leaving without return, as some animals leave from somewhere in search of better living conditions which means they do not necessarily return with their children to the places from which they migrated. Other biologists describe historical changes as migrations, put, most of them know that migrations are regular circular movements that animals perform between two regions, each of which provides a stage of life for those animals better living conditions than others. These migrations are made on the ground or in water or air. Some animals migrate for short distances only and others for long distances.

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Most migratory animals perform two types of migration daily. The zooplankton in the oceans performs this type of migration, where they swim hundreds of meters under the water during the day and return during the night to the surface, and seasonal migration, as these migrations occur twice a year . They are related to seasonal changes such as temperature or the level of rainfall. There are three types of seasonal migration:, namely migrations across latitudes, migrations anchored to highlands and local migrations and animals migrate for several reasons, including reproduction, hibernation, escape from the harsh weather and the search for food.

Many researchers discussed the effect of migration on the stability of biological systems, for more details see [1-8].

Toxins in biology are substances that cause disturbances to living organisms, that often occur either through chemical reactions or others at the molecular level when ingested in sufficient quantities.

In the field of medicine and animals, the word poison differs from toxin and creep. Toxins are toxins that are produced by living organisms in nature, while rattles mean toxins that are injected by a bite or sting. The difference between creep and other toxins is by the way of conducting it inside the body.

Toxicology, then, is the branch of science that investigates the potentially detrimental interactions between substances and biological systems. The categories of toxic chemicals include phytotoxins, zootoxins, bacteria toxins and human-made substances. Toxicology can be considered the oldest science, due to the ancient person who had to distinguish between substances that could be eaten and harmful substances.

Many researchers have proposed articles that have been touched upon to study the effect of toxins on biological systems, be they ecosystems or epidemiological systems, for example [9-14].Also, others proposed the environmental to study the effect of different factors [15-17].

The aim of this study is to determine the effect of the transition from the protected area to the schistosomal zone on the dynamics of an environmental model where it is assumed that predation is carried out according to a function of Lotka - Volterra. In this study, we also assume that the prey is in the traitors and predatory region, each of which secretes a toxic substance on the other. The prey excretes the toxic substance as a defense against predation and the predator sorts the toxic substance on the prey to facilitate its killing and predation. This model have been studied analytically and numerically, Furthermore, some graphics are shown to explain the effect of these factors on the behavior of the solution to the system.

2. Mathematical Model :

Consider the following ecotoxicant- model:-

$$\begin{aligned}
\frac{dZ_1}{dT} &= s_1 Z_1\left(1-\frac{Z_1}{L_1}\right) - \delta_1 Z_1 + \delta_2 Z_2 - m_1 Z_1, \\
\frac{dZ_2}{dT} &= s_2 Z_2\left(1-\frac{Z_2}{L_2}\right) + \delta_1 Z_1 - \delta_2 Z_2 - m_2 Z_2 - \gamma_1 Z_2 Z_3 - b Z_2^2 Z_3, \qquad (2.1)\\
\frac{dZ_3}{dT} &= \beta_1 Z_2 Z_3 - c Z_2 Z_3^2 - m_3 Z_3.
\end{aligned}$$

The following table provides an illustration of the variables and parameters of the aforementioned system:

Table 1: System (2.1) variables and parameters

Parameter	Representation
$Z_1(T)$	Prey density in the refuge area at time T
$Z_2(T)$	Prey density at time T in the area of the predator
$Z_3(T)$	Predator density at time T
$s_i > 0, i = 1,2$	The rate at which a prey grows naturally in the predatory and sanctuary regions respectively
$L_i > 0, i = 1,2$	Carrying capacity in the refuge region and the predatory region respectively
$\delta_i > 0, i = 1,2$	The migration and emigration rates of the prey population in the refuge region respectively
$\gamma_1 > 0$	Maximum attack rate for prey in the predatory region
$0 < \beta_1 < 1$	The uptake rate of food from the prey in the predatory region
$m_i > 0, i = 1,2,3$	Prey fatality rates in areas of sanctuary and predatory areas and predator mortality rate when not being fed
$b > 0$ and $c > 0$	Rates of toxicity for the prey and the predator, respectively

The following dimensionless variables and constants are attained in order to lower the number of parameters in system (2.1):

$t = s_1 T, z_1 = \frac{Z_1}{L_1}, z_2 = \frac{Z_2}{L_1}, z_3 = \frac{Z_3}{L_1}. k_1 = \frac{\delta_1}{s_1}, k_2 = \frac{\delta_2}{s_1}, k_3 = \frac{m_1}{s_1}, k_4 = \frac{s_2}{s_1},$
$k_5 = \frac{s_2 L_1}{s_1 L_2}, k_6 = \frac{m_2}{s_1}, k_7 = \frac{\gamma_1 L_1}{s_1}, k_8 = \frac{b L_1^2}{s_1}, k_9 = \frac{\beta_1 L_1}{s_1}, k_{10} = \frac{c L_1^2}{s_1}, k_{11} = \frac{m_3}{s_1}.$

Then the non dimension system is:

$$\begin{aligned}
\frac{dz_1}{dt} &= z_1(1 - z_1) - k_1 z_1 + k_2 z_2 - k_3 z_1, \\
\frac{dz_2}{dt} &= k_4 z_2 \left(1 - \frac{k_5}{k_4} z_2\right) + k_1 z_1 - k_2 z_2 - k_6 z_2 - k_7 z_2 z_3 - k_8 z_2^2 z_3, \\
\frac{dz_3}{dt} &= k_9 z_2 z_3 - k_{10} z_2 z_3^2 - k_{11} z_3.
\end{aligned} \tag{2.2}$$

with the next initial condition $z_1(0) \geq 0$, $z_2(0) \geq 0$ and $z_3(0) \geq 0$.

It is simple to verify that the solution to system (2.2) is both unique and existing.

Theorem (2.1): The system**'s** (2.2) solutions that begin in R_+^3 are uniformly bounded.

Proof.

Let $(z_1(t), z_2(t), z_3(t))$ be any system**'s** (2.2) solution with non-negative initial condition $(z_1(0), z_2(0), z_3(0))$. Now consider the following map,

$Z(t) = z_1(t) + z_2(t) + z_3(t)$.

Therefore,

$$\frac{dZ}{dt} < z_1(1 - z_1) + k_4 z_2 \left(1 - \frac{k_5}{k_4} z_2\right) - (k_7 - k_9) z_2 z_3 - k_3 z_1 - k_6 z_2 - k_{11} z_3.$$

Now, hence from the natural fact $k_7 > k_9$, thus

$$\frac{dZ}{dt} \leq \aleph - \rho Z, \quad \text{where } \aleph = \frac{1}{4} + \frac{k_4^2}{4k_5}, \text{and } \rho = \min\{k_3, k_6, k_{11}\}.$$

Now, by the theorem of comparison [10], we get:

$$Z(t) \leq \frac{\aleph}{\rho} + \left(Z(0) - \frac{\aleph}{\rho}\right) e^{-\rho t} \quad .$$

Thus $0 \leq Z(t) \leq \frac{\aleph}{\rho}$ as $t \to \infty$. ■

3 The equilibrium points (E.Ps.) :

System (2.2) has three **(E.Ps.)** that are listed below.

1) The **(E.P.)** $Q_0 = (0,0,0)$ always present.

2) The **(E.P.)** $Q_1 = (\bar{z}_1, \bar{z}_2, 0)$ present *by solving the next equations :*

$$z_1(1-z_1) - k_1 z_1 + k_2 z_2 - k_3 z_1 = 0 \tag{3.1a}$$

$$k_4 z_2\left(1 - \frac{k_5}{k_4} z_2\right) + k_1 z_1 - k_2 z_2 - k_6 z_2 = 0 \tag{3.1b}$$

From eq.(3.1a) we have:

$$z_2 = \frac{z_1^2 + (k_1 + k_3 - 1)z_1}{k_2}. \tag{3.1c}$$

By substitute equations (3.1c) in (3.1b), we get,

$$\bar{L}_1 z_1^3 + \bar{L}_2 z_1^2 + \bar{L}_3 z_1 + \bar{L}_4 = 0. \tag{3.1d}$$

Where:

$\bar{L}_1 = -k_5 < 0,$

$\bar{L}_2 = -2k_5(k_1 + k_3 - 1),$

$\bar{L}_3 = k_2 k_4 - k_5(k_1 + k_3 - 1) - k_2(k_2 + k_6),$

$\bar{L}_4 = k_2(k_1 + k_3 - 1)[k_4 - (k_2 + k_6)] + k_1.$, and

Now, eq. $(3.1d)$ has unique positive root say $\bar{z}_1$, according to the discard rule, if and only if

$$k_1 > 1 - k_3, \tag{3.1e}$$

$$k_4 > (k_2 + k_6). \tag{3.1f}$$

So, The **(E.P.)** $Q_1 = (\bar{z}_1, \bar{z}_2, 0)$ *where* $\bar{z}_2 = z_2(\bar{z}_1)$ present under condition's (3.1e and 3.1f).

3) The **(E.P.)** $Q_2 = (z_1^*, z_2^*, z_3^*)$ exists by resolving the set of equations below:

$$z_1(1-z_1) - (k_1 + k_3)z_1 + k_2 z_2 = 0, \tag{3.2a}$$

$$k_4 z_2 - k_5 z_2^2 + k_1 z_1 - k_2 z_2 - k_6 z_2 - k_7 z_2 z_3 - k_7 z_2^2 z_3 = 0, \tag{3.2b}$$

$$k_9 z_2 z_3 - k_{10} z_2 z_3^2 - k_{11} z_3 = 0. \tag{3.2c}$$

Equation (3.2c) gives us,

$$z_3 = \frac{k_9 z_2 - k_{11}}{k_{10} z_2}. \tag{3.2d}$$

Also, eq.(3.2a) gives ,

$$z_2 = \frac{z_1^2 + \sigma_1 z_1}{k_2} \tag{3.2e}$$

Replacing eq. (3.2e) in eq. (3.2d) results:

$$z_3 = \frac{k_9(z_1^2 + \sigma_1)z_1 - k_{11}}{k_{10}(z_1^2 + \sigma_1)z_1}. \tag{3.2f}$$

By placing eqs. (3.2e) and (3.2f) in eq. (3.2b) results:

$$L_1^* z_1^5 + L_2^* z_1^4 + L_3^* z_1^3 + L_4^* z_1^2 + L_5^* z_1 + L_6^* = 0 \tag{3.2g}$$

where:

$L_1^* = -[k_4 k_{10} + k_8 k_9] < 0,$

$L_2^* = -3\sigma_1[k_5 k_{10} + k_8 k_9] < 0,$

$L_3^* = k_2 k_{10}\sigma_2 - 3{\sigma_1}^2(k_5 k_{10} + k_8 k_9) - k_2 k_7 k_{10} + k_8 k_{11},$

$L_4^* = 2k_2\sigma_1[k_{10}\sigma_2 - k_7 k_9] - {\sigma_1}^3(k_5 k_{10} + k_8 k_9) + k_1 k_2^2 k_{10} + 2k_8 k_{11}\sigma_1,$

$L_5^* = k_2 k_{10}\sigma_1(k_4 + k_1 k_2) - {\sigma_1}^2[k_2(k_2 + k_6) - k_8 k_{11}] + k_2 k_7[k_{11} - k_9{\sigma_1}^2],$

$L_6^* = k_2 k_7 k_{11}\sigma_1 > 0.$

where:

$\sigma_1 = (k_1 + k_3 - 1), \sigma_2 = k_4 - (k_2 + k_6)$

So, eq. $(3.2g)$ has a unique positive root, namely z_1^* if in addition to conds.(3.1d) and (3.1e) the following conditions hold:

$$3{\sigma_1}^2(k_5k_{10}+k_8k_9)+k_2k_7k_{10} > k_2k_{10}\sigma_2 + k_8k_{11}, \quad (3.2h)$$
$$\sigma_2 > \frac{k_7k_9}{k_{10}}, \quad (3.2i)$$
$$2k_2\sigma_1[k_{10}\sigma_2 - k_7k_9] + k_1k_2^2k_{10} + 2k_8k_{11}\sigma_1 > {\sigma_1}^3(k_5k_{10}+k_8k_9), \quad (3.2j)$$
$$k_2(k_2+k_6) < k_8k_{11}, \quad (3.2k)$$
$$k_9 < \frac{k_{11}}{{\sigma_1}^2}. \quad (3.2l)$$

So, the **(E.P.)** $Q_2 = (z_1^*, z_2^*, z_3^*)$ where $z_2^* = z_2(z_1^*)$ *and* $z_3^* = z_3(z_1^*)$ present if in addition to the above conditions the following condition holds:

$$z_1^2 + \sigma_1 z_1 > \frac{k_{11}}{k_9}. \quad (3.2m)$$

4 Local Stability Analysis (L.S.A.).

The stability of system (2.2) has been discussed in this subsection: -

The Jacobian matrix (J.M.) $D(z_1, z_2, z_3)$ of the system(2.2) can be written:

$$D = [d_{ij}]_{1\le i,j\le 3}, \quad (4.1)$$

where

$d_{11} = 1 - 2z_1 - (\mathrm{k}_1 + k_3), d_{12} = \mathrm{k}_2, d_{13} = 0, d_{21} = k_1$,
$d_{22} = k_4 - 2k_5z_2 - (k_2+k_6) - k_7z_3 - 2k_8z_2z_3$, $d_{23} = -z_2(k_7 + k_8z_2)$
$d_{31} = 0$, $d_{32} = z_3(k_9 - k_{10}z_3)$, $d_{33} = k_9z_2 - 2k_{10}z_2z_3 - k_{11}$.

4. 1 Local stability of Q_0

At Q_0 the (J.M.) is:

$$D_0 = D(Q_0) = [d^{\circ}_{ij}]_{1\le i,j\le 3}, \quad (4.1a)$$

where:

$\mathrm{d}^{\circ}_{11} = 1 - (\mathrm{k}_1 + k_3), \mathrm{d}^{\circ}_{12} = \mathrm{k}_2, \mathrm{d}^{\circ}_{13} = 0, \mathrm{d}^{\circ}_{21} = k_1$, $\mathrm{d}^{\circ}_{22} = k_4 - (k_2 + k_6)$,
$\mathrm{d}^{\circ}_{23} = \mathrm{d}^{\circ}_{31} = \mathrm{d}^{\circ}_{32} = 0$, $\mathrm{d}^{\circ}_{33} = -k_{11}$.

Then the characteristic equation (Ch.E.) of $D(Q_0)$ is given by:

$[\lambda^2 - \mathrm{tr}(\mathrm{A}^{\circ})\lambda + \mathrm{Det}(\mathrm{A}^{\circ})]\,[-k_{11} - \lambda] = 0$,

where:

$tr(\mathrm{A}^{\circ}) = \lambda_{0z_1} + \lambda_{0z_2} = (1 - (\mathrm{k}_1 + k_3)) + (k_4 - (k_2 + k_5))$
$Det(\mathrm{A}^{\circ}) = \lambda_{0z_1}.\lambda_{0z_2} = (1 - (\mathrm{k}_1 + k_3)\big(k_4 - (k_2 + k_5)\big) - k_1\,\mathrm{k}_2$

So, either

$[\lambda^2 - \mathrm{tr}(\mathrm{A}^{\circ})\lambda + \mathrm{Det}(\mathrm{A}^{\circ})] = 0$,

which gives the two eigenvalues λ_{0z_1} *and* λ_{0z_2} are negative provided that:

$$\mathrm{k}_1 + k_3 > 1, \quad (4.1b)$$
$$k_2 + k_5 > k_4, \quad (4.1c)$$
$$(1 - (\mathrm{k}_1 + k_3)\big(k_4 - (k_2 + k_5)\big) > k_1\,\mathrm{k}_2. \quad (4.1d)$$

Or

$-k_{11} - \lambda = 0$, which gives $\lambda_{0z_3} = -k_{11} < 0$.

Therefore, Q_0 is stable under conditions (4.1b)-(4.1d) . It is unstable otherwise.

4.2 Local stability of Q_1
At Q_1 the (J.M.) become

$$D_1 = D(Q_1) = \left[\bar{d}_{ij}\right]_{1\le i,j\le 3}, \qquad (4.2a)$$

where:
$\bar{d}_{11} = 1 - 2\bar{z}_1 - (k_1 + k_3), \bar{d}_{12} = k_2, \bar{d}_{13} = 0, \bar{d}_{21} = k_1$,
$\bar{d}_{22} = k_4 - 2k_5\bar{z}_2 - (k_2 + k_6), d_{23} = -\bar{z}_2(k_7 + k_8\bar{z}_2)$
$\bar{d}_{31} = \bar{d}_{32} = 0, d_{33} = k_9\bar{z}_2 - k_{11}$.
Then the (Ch.E.) of $D(Q_1)$ is given by:
$[\lambda^2 - \text{tr}(\bar{A})\lambda + \text{Det}(\bar{A})][k_9\bar{z}_2 - k_{11} - \lambda] = 0,$
where:
$tr(\bar{A}) = \lambda_{1z_1} + \lambda_{1z_2} = (1 - 2\bar{z}_1 - (k_1 + k_3)) + (k_4 - 2k_5\bar{z}_2 - (k_2 + k_6))$
$Det(\bar{A}) = \lambda_{1z_1}.\lambda_{1z_2} = \left(1 - 2\bar{z}_1 - (k_1 + k_3)\right).(k_4 - 2k_5\bar{z}_2 - (k_2 + k_6)) - k_1 k_2$
So, either
$[\lambda^2 - \text{tr}(\bar{A})\lambda + \text{Det}(\bar{A})] = 0,$
which gives the two eigenvalues λ_{1z_1} *and* λ_{1z_2} are negative provided that:

$$2\bar{z}_1 + (k_1 + k_3) > 1, \qquad (4.2b)$$
$$2k_5\bar{z}_2 + (k_2 + k_6) > k_4, \qquad (4.2c)$$
$$\left(1 - 2\bar{z}_1 - (k_1 + k_3)\right).(k_4 - 2k_5\bar{z}_2 - (k_2 + k_6)) > k_1 k_2. \qquad (4.2d)$$

Or
$k_9\bar{z}_2 - k_{11} - \lambda = 0$, which gives
$\lambda_{1z_3} = k_9\bar{z}_2 - k_{11}$, Therefore,
Q_0 is stable If the following condition holds in addition to conditions (4.2a) - (4.2d):

$$\bar{z}_2 < \frac{k_{11}}{k_9}. \qquad (4.2e)$$

It is unstable otherwise.

4.3 Local stability of Q_2

$$D_2 = D(Q_2) = \left[d_{ij}^*\right]_{1\le i,j\le 3}, \qquad (4.3a)$$

where:
$d_{11}^* = 1 - 2z_1^* - (k_1 + k_3), d_{12}^* = k_2, d_{13}^* = 0, d_{21}^* = k_1$,
$d_{22}^* = k_4 - 2k_5z_2^* - (k_2 + k_6) - k_7z_3^* - 2k_8z_2^*z_3^*, d_{23}^* = -z_2^*(k_7 + k_8z_2^*)$
$d_{31}^* = 0, d_{32}^* = z_3^*(k_9 - k_{10}z_3^*), d_{33}^* = z_2^*(k_9 - 2k_{10}z_3^*) - k_{11}$.
Then the (Ch.E.) of $D(Q_2)$ is given by:

$$\lambda^3 + \omega_1\lambda^2 + \omega_2\lambda + \omega_3 = 0. \qquad (4.3b)$$

Where:
$\omega_1 = -(d_{11}^* + d_{22}^* + d_{33}^*),$
$\omega_2 = -[d_{12}^*d_{21}^* + d_{23}^*d_{32}^* - d_{22}^*d_{33}^* - d_{11}^*(d_{22}^* + d_{33}^*)],$
$\omega_3 = d_{12}^*d_{21}^*d_{33}^* - d_{11}^*(d_{22}^*d_{33}^* - d_{23}^*d_{32}^*).$
The roots of equation (4.3b), according to the Routh-Hurwitz criterion, contain negative real portions if and only if $\omega_i > 0, i = 1,3$ and $\Delta = (\omega_1\omega_2 - \omega_3)\omega_3 > 0$.
$Now, \omega_i > 0, i = 1,3$, provided that

$$2z_1^* + (k_1 + k_3) > 1, \qquad (4.3c)$$
$$z_2^*(2k_5 + 2k_8z_3^*) + (k_2 + k_6) + k_7z_3^* > k_4 \qquad (4.3d)$$
$$z_3^* > \frac{k_9}{k_{10}}. \qquad (4.3e)$$

Further, it is easy to check that:
$\Delta = d_{11}^*[d_{12}^*d_{21}^* - d_{11}^*(d_{22}^* + d_{33}^*)] + d_{33}^*[d_{23}^*d_{32}^*) - d_{22}^*d_{33}^* - d_{11}^*(d_{22}^* + d_{33}^*)]$

$d_{22}^*[d_{12}^*d_{21}^* + d_{23}^*d_{32}^* - d_{22}^*d_{33}^* - d_{11}^*(d_{22}^*.d_{33}^*)]$

Hence, $\Delta > 0$, if in addition to conditions $(4.3c)$ and $(4.3e)$, the following conditions

$$d_{11}^*(d_{22}^* + d_{33}^*) > max\{d_{12}^*d_{21}^*, d_{23}^*d_{32}^* - d_{22}^*d_{33}^*\}, \quad (4.3f)$$

$$d_{12}^*d_{21}^* + d_{23}^*d_{32}^* > d_{22}^*d_{33}^*(1 + d_{11}^*). \quad (4.3g)$$

So, Q_2 is (L.S.) under conditions (4.3c)-(4.3g) , it is unstable otherwise.

5 Global stability analysis (G.S.A.):

This section examines the (G.S.A.) of the system (2.2) for the nearby stable points.

Theorem (5.1):

The point $Q_0 = (0,0,0)$ is globally asymptotically stable (G.A.S.) if the following requirements are met by the $Int.R_+^3$ Basin of attraction:

$$z_1 > 1, \quad (5.1a)$$

$$z_2 > \frac{k_4}{k_5}. \quad (5.1b)$$

Proof: Consider the function

$G_0(z_1, z_2, z_3) = z_1 + z_2 + z_3$.

It is easy to see that $G_0(z_1, z_2, z_3) \epsilon\, C^1(R_+^3, R)$, and $G_0(Q_0) = 0$, and $G_0(z_1, z_2, z_3) > 0$; $\forall (z_1, z_2, z_3) \neq Q_0$, By using the equations of the system's equations and differentiating G_0 with respect to time t, the following results are obtained:

$$\frac{dG_0}{dt} = z_1(1 - z_1) - k_3z_1 + k_4z_2\left(1 - \frac{k_5}{k_4}z_2\right) + -k_6z_2 - (k_{7-}k_9)z_2\, z_3 - k_8z_2^2\, z_3$$
$$-k_{10}z_2z_3^2 - k_{11}z_3.$$

Now, rendering to the accepted fact , $(k_7 > k_9)$, we get:

$\frac{dG_0}{dt} < z_1(1 - z_1) + k_4z_2\left(1 - \frac{k_5}{k_4}z_2\right) - k_{11}z_3.$

So, by conditions $(5.1a)$ and $(5.1b)$, $\frac{dG_0}{dt} < 0$. Hence Q_0 is (G.A.S.).

Theorem (5.2) :

The point $Q_1 = (\bar{z}_1, \bar{z}_2, 0)$ of system (2.2) is (G.A.S.) if the following requirements are met by the $Int.R_+^3$ Basin of attraction:

$$\frac{k_2(z_1 + z_2)}{z_1z_2} \leq 2\sqrt{\left(1 + \frac{k_2\bar{z}_2}{z_1\bar{z}_1}\right)\left(k_5 + \frac{k_1\bar{z}_1}{z_2\bar{z}_2}\right)}, \quad (5.2a)$$

$$\left[\sqrt{1 + \frac{k_2\bar{z}_2}{z_1\bar{z}_1}}(z_1 - \bar{z}_1) - \sqrt{k_5 + \frac{k_1\bar{z}_1}{z_2\bar{z}_2}}(z_2 - \bar{z}_2)\right]^2 > k_8\, z_2\, \bar{z}_2\bar{z}_2. \quad (5.2b)$$

Proof: Consider the function

$$G_1(z_1, z_2, z_3) = \left(z_1 - \bar{z}_1 - \bar{z}_1 ln\frac{z_1}{\bar{z}_1}\right) + \left(z_2 - \bar{z}_2 - \bar{z}_2 ln\frac{z_2}{\bar{z}_2}\right) + z_3.$$

It is clear that . $G_1(z_1, z_2, z_3) \epsilon\, C^1(R_+^3, R)$, and $G_1(Q_1) = 0$, and $G_1(z_1, z_2, z_3) > 0$; $\forall (z_1, z_2, z_3) \neq Q_1$, By using the equations of the system and differentiating G_1 with respect to time t, the following results are obtained:

$$\frac{dG_1}{dt} < -\left(1 + \frac{k_2\bar{z}_2}{z_1\bar{z}_1}\right)(z_1 - \bar{z}_1)^2 + \left(\frac{k_2(z_1 + z_2)}{z_1z_2}\right)(z_1 - \bar{z}_1)(z_2 - \bar{z}_2)$$
$$-\left(k_5 + \frac{k_1\bar{z}_1}{z_2\bar{z}_2}\right)(z_2 - \bar{z}_2)^2 - (k_{7-}k_9)z_2\, z_3 - k_8z_2^2\, z_3 - k_{10}z_2z_3^2 - k_{11}z_3 + k_8\, z_2\, \bar{z}_2\bar{z}_2.$$

Now, rendering to the accepted fact , $(k_7 > k_9)$ and condition (5.2a), we get:

$$\frac{dG_1}{dt} < -\left[\sqrt{1+\frac{k_2\bar{z}_2}{z_1\bar{z}_1}}(z_1-\bar{z}_1) - \sqrt{k_5+\frac{k_1\bar{z}_1}{z_2\bar{z}_2}}(z_2-\bar{z}_2)\right]^2 + k_8\, z_2\, \bar{z}_2\bar{z}_2.$$

Then $\frac{dG_1}{dt} < 0$ under $cond.$ (5.2b), and then Q_1 is (G.A.S.).

Theorem (5.3):

The positive equilibrium point $Q_2 = (z_1^*, z_2^*, z_3^*)$ of the system (2.2) is (G.A.S.) if the following requirements are met by the $Int.R_+^3$ Basin of attraction:

$$\frac{k_1z_1+k_2z_2}{z_1z_2} \le 2\sqrt{\frac{1}{2}\left(1+\frac{k_2z_2^*}{z_1z_1^*}\right)\left(k_5+k_8z_3+\frac{k_1z_1^*}{z_2z_2^*}\right)}, \quad (5.3a)$$

$$k_9-(k_7+k_8z_2^*) \le 2\sqrt{\frac{1}{2}k_{10}\left(k_5+k_8z_3+\frac{k_1z_1^*}{z_2z_2^*}\right)(z_2+z_2^*)}, \quad (5.3b)$$

Proof: Consider the function

$$G_2(z_1,z_2,z_3) = \left(z_1-z_1^*-z_1^*\ln\frac{z_1}{z_1^*}\right)+\left(z_2-z_2^*-z_2^*\ln\frac{z_2}{z_2^*}\right)+\left(z_3-z_3^*-z_3^*\ln\frac{z_3}{z_3^*}\right).$$

It is clear that $G_2(z_1,z_2,z_3)\epsilon\, C^1(R_+^3,R)$, and $G_2(Q_2)=0$, and $G_2(z_1,z_2,z_3) > 0\,; \forall(z_1,z_2,z_3) \neq Q_2$, By using the equations of the system and differentiating G_2 with respect to time t, the following results are obtained:

$$\frac{dG_2}{dt} < -\left(1+\frac{k_2z_2^*}{z_1z_1^*}\right)(z_1-z_1^*)^2 + \frac{k_1z_1+k_2z_2}{z_1z_2}(z_1-z_1^*)(z_2-z_2^*)$$
$$-\frac{1}{2}\left(k_5+k_8z_3+\frac{k_1z_1^*}{z_2z_2^*}\right)(z_2-z_2^*)^2 - \frac{1}{2}\left(k_5+k_8z_3+\frac{k_1z_1^*}{z_2z_2^*}\right)(z_2-z_2^*)^2 + k_9$$
$$-(k_7+k_8z_2^*)(z_2-z_2^*)(z_3-z_3^*) - k_{10}(z_2+z_2^*)(z_3-z_3^*)^2.$$

Now, by conditions (5.3a) and (5.3b) we get

$$\frac{dG_2}{dt} < -\left[\sqrt{1+\frac{k_2z_2^*}{z_1z_1^*}}(z_1-z_1^*) - \sqrt{\frac{1}{2}\left(k_5+k_8z_3+\frac{k_1z_1^*}{z_2z_2^*}\right)}(z_2-z_2^*)\right]^2$$
$$-\left[\sqrt{\frac{1}{2}\left(k_5+k_8z_3+\frac{k_1z_1^*}{z_2z_2^*}\right)}(z_2-z_2^*) - \sqrt{k_{10}(z_2+z_2^*)}(z_3-z_3^*)\right]^2.$$

Then G_2 is negative definite. Hence, Q_2 is a (G.A.S.) ■

6 Numerical simulation:

To verify the aforementioned analytic results, the behavior of the system (2.2) is numerically examined in this section. For the following set of parameters, the phase portrait and graphic representation of the system's solutions (2.2) are drawn.

$$\left.\begin{array}{c} k_1=0.25\,, k_2=0.26\,, k_3=0.1\,, k_4=1\,, k_5=0.2\,, k_6=0.1\,, k_7=0.5 \\ k_8=1, k_9=0.4\,, k_{10}=1, k_{11}=0.1 \end{array}\right\} \quad (6)$$

and with the primary point (2,1,1).

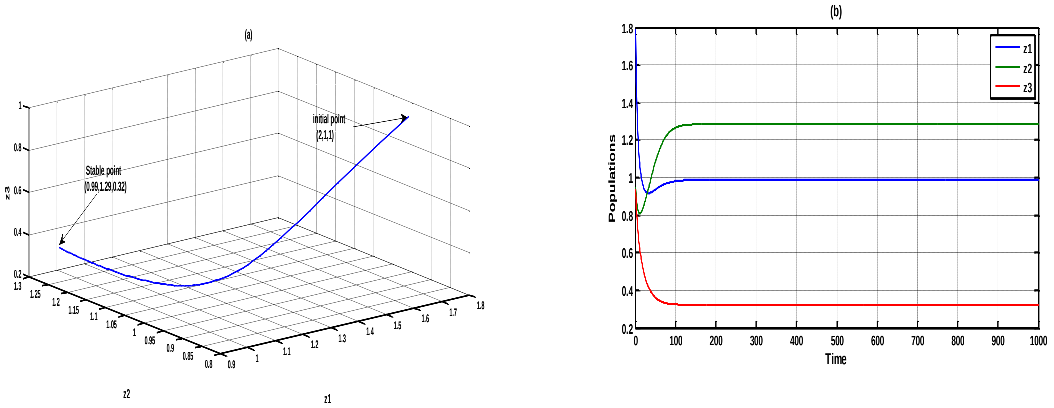

Figure (6.1): (a) The solution of system (2.2) with the parameter data (6) with respect to z_1, z_2, z_3
(b) Graphical representation of solution which close to the point $Q_2 = (0.99,1.29,0.32)$.

The suggested model is now numerically solved for the set given in Eq. (6) and various one factor each time, and the results are reported in the following table:

Table 2: Numerical solutions of the system (2.2)

The parameter's range	The stable point	The bifurcation point	The parameter's range	The stable point	The bifurcation point
$0.1 < k_1 \leq 3$	Q_2.		$0.5 \leq k_7 \leq 2$	Q_2.	
$0.1 < k_2 \leq 2$	Q_2.		$0.1 < k_8 \leq 2$	Q_2	
$0.1 < k_3 \leq 1$	Q_2.		$0.01 \leq k_9 \leq 0.039$ $0.039 < k_9 \leq 0.4$	Q_1 Q_2	$k_9 = 0.039$
$0.1 < k_4 \leq 2$	Q_2.		$0.1 \leq k_{10} \leq 2$	Q_2	
$0.1 < k_5 \leq 2$	Q_2.		$0.1 \leq k_{11} \leq 1$	Q_2	
$0.1 \leq k_6 \leq 1$	Q_2.				

Next, for the following parameters $k_1 = 1, k_3 = 0.5, k_4 = 0.5, k_{11} = 0.78$ with the rest of parameters in eq.(6) which satisfied the conditions of Theorem (5.2) in addition of conditions (3.1e),(3.1f),and (4.2b)-(4.2e) the solution close to the point Q_1, see Figure (6.2)(a,b).

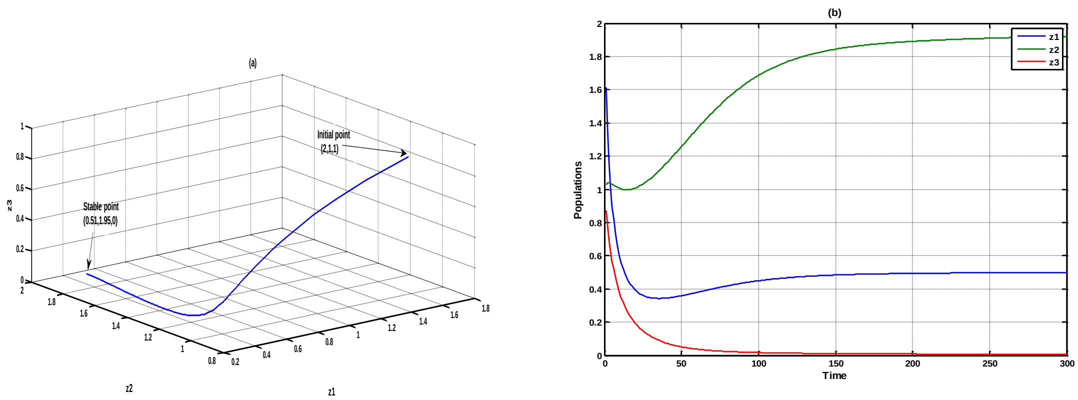

Figure (6.2):- (a)The system's (2.2) solution with the parameters in eq.(6) with respect to z_1, z_2, z_3 which close to the point Q_1=(0.51,1.95,0) (b)The Graphical representation of solution with respect to z_1, z_2, z_3 .

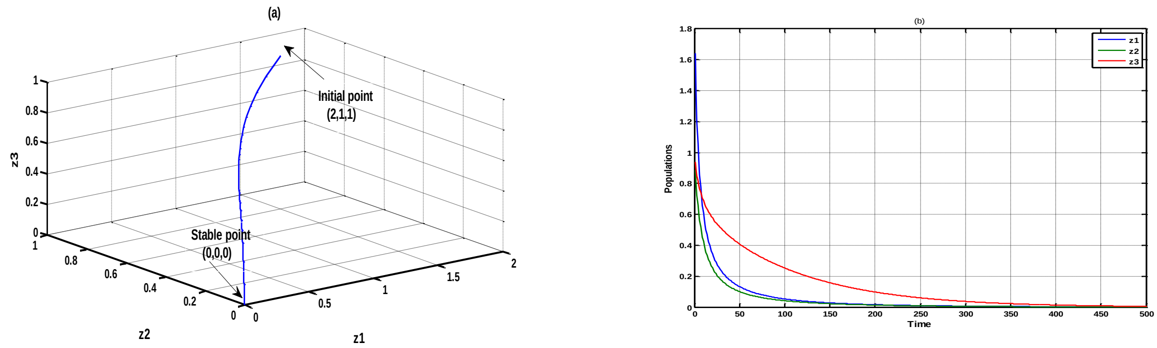

Figure (6.3):- (a)The solution of system(2.2) with the parameter data (6) with respect to z_1, z_2, z_3 which close to the point Q_0=(0,0,0) (b)The graphical representation of system(2.2) with the parameter data (6) with respect to z_1, z_2, z_3 .

Finally, for the following parameters $k_1 = 0.6, k_3 = 0.7 , k_4 = 0.1 , k_6 = 0.6$ with the remainder of parameters in eq.(6) which satisfied the conditions of theorem (5.1) in addition to conditions (4.1be)-(4.1d), the solution close to Q_0 , see Figure (6.3)(a,b).

7. Conclusions:-

Three non-linear differential equations have been used to create a mathematical model that examines how the protected region and the harmful area affect an environmental model in the presence of toxic substances. When studying this model analytically and numerically, the following results were reached within a hypothetical set of parameters given in eq.(6),and the following results have been summarized as follows:

1-There is no periodic solution of system (2.2) in Int. R_+^3
2-The parameters $k_{i,} i = 1,3,4,6,11$, played an important role on dynamics of system (2.2).
3-It is noticed that the behavior of the system (2.2) does not change if $k_i, i = 2,5,7,8$ *and* 10 is varied and the solution still close to Q_2 .

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