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Numerical Solution for Conformable Fractional PDEs by Using a New Double Conformable Integral Transform-Decomposition Method

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Abstract
The conformable Sumudu-Elzaki Transform Decomposition Method CSETDM is used in this research to handle the approximate numerical solutions for the regular and singular one-dimensional conformable Fractional Coupled Burger's Equation CFCBsE and the Nonlinear Singular Conformable Pseudohyperbolic equation NSCPE. This method is generalized in the sense of conformable derivatives. Essential results and theorems concerning this method are discussed. Some numerical examples are given to exhibit the proposed technique's viability, applicability, and effortlessness.

Keywords: Adomin decomposition method, Burger’s Equation, Conformable Fractional Derivatives, Conformable Sumudu–Elzaki transform, pseudohyperbolic equation.

الحل العددي للمعادلات التفاضلية الجزئية ذات الرتب الكسرية التوافقية باستخدام تحويلات تكاملية مزدوجة جديدة توافقية مع طريقة الانحلال

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الخلاصة
تم استخدام تحويل Sumudu–Elzaki المتوافق وطريقة الانحلال في هذا البحث للتعامل مع الحلول العددية التقريبية لمعادلة Burger’s الجزئية المنتظمة والمفردة ذات البعد الواحد وذات رتبة كسرية متوافقة ومعادلة pseudohyperbolic المفردة غير الخطية المتوافقة. هذه الطريقة معممة بمعنى المشتقات تكون متوافقة. تمت مناقشة النتائج والنظريات الهامة المتعلقة بهذه الطريقة. يتم إعطاء بعض الأمثلة العددية لإظهار الجدوى والتطبيق والجهد للتقنية المقترحة.

1. Introduction

During the last years, many researchers found that the derivatives of non-integer order are very suited for the description of various physical phenomena such as dumping laws, diffusion processes, etc. Fractional calculus has been utilized as an excellent instrument to discover the hidden aspects of various material and physical processes that deal with derivatives and integrals of arbitrary orders [1-4]. Due to their significance in both the mathematical and physical sciences, many mathematicians studied and discussed the linear and nonlinear fractional differential equations (FDEs) that are encountered in a variety of

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engineering and physical research fields [5-7]. In 1915, Bateman [8] introduced Burger's equation which was later modified by Burger's in 1948 [9]. Nonlinear wave motion combined with linear diffusion is used to symbolize Burger's equation. These kinds of equations have been used in a wide range of fields for more_details see [10–15]. One of the very few nonlinear partial differential equations with an exact solution is Burger's equation [16–23].

A new definition of fractional derivatives was given by Khalil et al.[24,25] which is called the Conformable Fractional Derivatives CFDs. It has interested many researchers_because it includes several properties that correlate to the normal derivative, especially the Leibniz laws. Its description is more precise and welcoming than other fractional definitions since it has received extensive coverage. Many applications and phenomena may be represented using CFDs and there are many exciting benefits, for example, a local derivative that replaces the basic derivative since it is based on the limit of its formulation. The authors generalized the standard calculus notions to solve numerous fractional partial differential equations FPDE's in every circumstance [26]. In addition, the conformable Laplace transform technique is used by the authors in [27-30] to solve the fractional differential equations. The conformable double Laplace transform approach was used by the researchers to solve the FPDEs for more details, see [31]. The conformable double Laplace decomposition method [32] has been presented in order to get exact and approximate answers to the regular and singular one-dimensional CFCBsE. Moreover, conformable double Sumudu transform decomposition was used in [33] to solve the CFCB'sE. Therefore, the objective of this research is to examine a numerical analytic solution for the regular and singular one-dimensional conformable fractionally coupled Burger's equation and the NSCPE using the Conformable Sumudu-Elzaki transform decomposition method CSETDM. The entire build-up of the presented study involves the following: definitions and theorems of the CFDs and conformable Sumudu-Elzaki transform CSET are given in section 2. In section 3, the conformable Sumudu-Elzaki transform Decomposition method CSETDM is presented. Then, in section 4, numerical approximate solutions to the CFCB'sE with NSCPE are shown to be close to the exact solutions. Section 5 discusses the numerical results. Finally, the study is round-up with several conclusions.

2. Basic Definitions and Main Results Theorems of the Conformable Partial Derivatives and integral transform:

In this section, the definitions and theorems of CFDs with Sumudu and Elzaki transform that should be utilized in the current study are presented:

2.1 Basic Definitions

Definition 2.1.1 [33]

The conformable Sumudu transform of the real valued function $f:[0,\infty)\to\mathbb{R}$ of order $0<\alpha\leq 1$ over the following set of functions:

$$A=\{f(\frac{x^\alpha}{\alpha}):\exists\, M,\xi_1,\xi_2>0, Such\ that,\ |f(\frac{x^\alpha}{\alpha})|<M\,e^{(|\frac{x^\alpha}{\alpha}|)/\xi_i}, if\ x\in\ (-1)^i\times[0,\infty)\},$$

is given by:

$$S_x^\alpha\left[f\left(\frac{x^\alpha}{\alpha}\right)\right]=F_\alpha(u)=\frac{1}{u}\int_0^\infty f\left(\frac{x^\alpha}{\alpha}\right)e^{-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)}x^{\alpha-1}dx,\ \ u\in(-\xi_1,\xi_2). \quad (1)$$

Definition 2.1.2 [34]

The conformable Elzaki transform of the real valued function $g:[0,\infty)\to\mathbb{R}$ of order $0<\beta\leq 1$ over the following set of functions:

$$B=\{g(\frac{t^\beta}{\beta}):\exists\, N,\rho_1,\rho_2>0, Such\ that,\ |g(\frac{t^\beta}{\beta})|<N\,e^{(|\frac{t^\beta}{\beta}|)/\rho_j}, if\ t\in\ (-1)^j\times[0,\infty)\},$$

is given by:

$$E_t^\beta\left[g\left(\frac{t^\beta}{\beta}\right)\right]=G_\beta(v)=v\int_0^\infty g\left(\frac{t^\beta}{\beta}\right)e^{-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)}t^{\beta-1}dt,\quad v\in(-\rho_1,\rho_2). \tag{2}$$

Definition 2.1.3 [32]

Given a function $f:(0,\infty)\to\mathbb{R}$ is a real-valued function, then the conformable fractional derivative of f of order v is defined by

$$\frac{d^\beta}{dt^\beta}f\left(\frac{t^\beta}{\beta}\right)=\lim_{\varepsilon\to 0}\frac{f\left(\frac{t^\beta}{\beta}+\varepsilon t^{1-v}\right)-f(\frac{t^\beta}{\beta})}{\varepsilon},\quad \frac{t^\beta}{\beta}>0,\ 0<\beta\le 1. \tag{3}$$

Definition 2.1.4 [32]

Given a function $f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right):R\times(0,\infty)\to R$, then the conformable space and time fractional partial derivative for the function $f(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ of order α and β respectively is given as follows:

$$\frac{\partial^\alpha}{\partial x^\alpha}f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\lim_{\varepsilon\to 0}\frac{f\left(\frac{x^\alpha}{\alpha}+\varepsilon x^{1-\alpha},t\right)-f(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})}{\varepsilon},\ \frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}>0,0<\alpha,\beta\le 1. \tag{4}$$

$$\frac{\partial^\beta}{\partial t^\beta}f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\lim_{\delta\to 0}\frac{f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}+\delta t^{1-\beta}\right)-f(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})}{\delta},\ \frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}>0,0<\alpha,\beta\le 1. \tag{5}$$

We refer to the paper of H. Eltayeb et al. [32] for the conformable fractional derivative of the functions listed there.

Definition 2.1.5 [7]

Let $\mathrm{h}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ be a function of two variables, then the CSET of the function $\mathrm{h}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right),\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\in\mathbb{R}^+$ is denoted by:

$$S_x^\alpha E_t^\beta\left[\mathrm{h}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})\right]=H_{\alpha,\beta}(\mathrm{u},\mathrm{v})=\frac{v}{u}\int_0^\infty\int_0^\infty \mathrm{h}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)e^{-\left(\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)+\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}x^{\alpha-1}t^{\beta-1}dxdt, \tag{6}$$

Where $\mathrm{u},\mathrm{v}\in\mathbb{C},0<\alpha,\beta\le 1$ are variables of the Sumudu and Elzaki, respectively and $S_x^\alpha E_t^\beta$ represents the CSET. The inverse conformable Sumudu-Elzaki transform ICSET,

$S_x^\alpha E_x^\beta\ ^{-1}\left[H_{\alpha,\beta}(\mathrm{u},\mathrm{v})\right]=\mathrm{h}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$ is defined by:

$$S_x^\alpha E_x^\beta\ ^{-1}\left[H_{\alpha,\beta}(\mathrm{u},\mathrm{v})\right]=\mathrm{h}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=$$

$$=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{\alpha}e^{-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)}\left(\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\beta e^{-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)}H_{\alpha,\beta}(\mathrm{u},\mathrm{v})dv\right)du.$$

The relation between the usual and the CSET is given below:

$$S_x^\alpha E_t^\beta\left[(\frac{x^\alpha}{\alpha})^c(\frac{t^\beta}{\beta})^d\right]=S_x^\alpha E_t^\beta[(x)^c(t)^d]=\mathrm{c}!\,\rho^c\mathrm{d}!\,\tau^{d+2},$$

$$S_x^\alpha E_t^\beta\left[e^{c(\frac{x^\alpha}{\alpha})+d(\frac{t^\beta}{\beta})}\right]=S_xE_t[e^{cx+dt}]=\frac{\tau^2}{(1-d\tau)(1-\rho c)},$$

$$S_x^\alpha E_t^\beta\left[\sin\left(\rho(\frac{x^\alpha}{\alpha})\right)\sin\left(\tau(\frac{t^\beta}{\beta})\right)\right]=S_xE_t[\sin(ux)\sin(vt)]=\frac{\rho u}{(1+\rho^2u^2)}\frac{\tau v^3}{(1+\tau^2v^2)},$$

$$S_x^\alpha E_t^\beta\left[\cos\left(\rho(\frac{x^\alpha}{\alpha})\right)\cos\left(\tau(\frac{t^\beta}{\beta})\right)\right]=S_xE_t[\cos(cx)\cos(dt)]=\frac{1}{(1+c^2\rho^2)}\frac{s^2}{(1+d^2s^2)},$$

Definition 2.1.6 (Existence conditions)

A function h$\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$ is an of exponential order for $a>0$, $b>0$ on $0\le\frac{x^\alpha}{\alpha}<\infty$, $0\le\frac{t^\beta}{\beta}<\infty$, if there exists $K>0$, such that for all $\frac{x^\alpha}{\alpha}>X,\frac{t^\beta}{\beta}>T$,

$$\left|h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right|\le Ke^{a\left(\frac{x^\alpha}{\alpha}\right)+b\left(\frac{t^\beta}{\beta}\right)}, \tag{7}$$

Where K is constant. We write $h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=O\left(e^{a\left(\frac{x^\alpha}{\alpha}\right)+b\left(\frac{t^\beta}{\beta}\right)}\right)$ as $\left(\frac{x^\alpha}{\alpha}\right)\longrightarrow\infty$, $\left(\frac{t^\beta}{\beta}\right)\longrightarrow\infty$, or, equivalently,

$$\lim_{\substack{\frac{x^\alpha}{\alpha}\to\infty\\ \frac{t^\beta}{\beta}\to\infty}} e^{-\left(\left(\frac{1}{\mu}\right)\left(\frac{x^\alpha}{\alpha}\right)+\left(\frac{1}{\omega}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\left|\mathrm{h}\left(\left(\frac{x^\alpha}{\alpha}\right),\left(\frac{t^\beta}{\beta}\right)\right)\right| = K\lim_{\substack{\frac{x^\alpha}{\alpha}\to\infty\\ \frac{t^\beta}{\beta}\to\infty}} e^{-\left(\frac{1}{\mu}-a\right)\left(\frac{x^\alpha}{\alpha}\right)-\left(\frac{1}{\omega}-b\right)\left(\frac{t^\beta}{\beta}\right)}=0\ ,\left(\frac{1}{u}\right)>a,\ \left(\frac{1}{v}\right)>b.$$

Where $\left(\frac{1}{\mu}>a\right)$ and $\left(\frac{1}{\omega}>b\right)$. The function h$\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$ is called an exponential order as $\left(\frac{x^\alpha}{\alpha}\right)\to\infty,\left(\frac{t^\beta}{\beta}\right)\to\infty$. Obviously, it doesn't grow faster than $Ke^{a\left(\frac{x^\alpha}{\alpha}\right)+b\left(\frac{t^\beta}{\beta}\right)}$ as $\left(\frac{x^\alpha}{\alpha}\right)\to\infty,\left(\frac{t^\beta}{\beta}\right)\to\infty$.

2.2 Main Results Theorems

Theorem2.2.1

The function $h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$, where $0<\alpha,\beta\le1$ is defined on $(0,X)$ and $(0,T)$ and of exponential order $e^{a\left(\frac{x^\alpha}{\alpha}\right)+b\left(\frac{t^\beta}{\beta}\right)}$, then the CSET of $h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$ exists for all $Re\left[\frac{1}{u}\right]>a$ and $Re\left[\frac{1}{v}\right]>b$.

Proof: From (7), we have

$$\left|H_{\alpha,\beta}(\mathrm{u,v})\right| = \left|\frac{v}{u}\int_0^\infty\int_0^\infty e^{-\left(\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)+\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)} h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)x^{\alpha-1}t^{\beta-1}dxdt\right|,$$

$$\le K\left|\frac{v}{u}\int_0^\infty\int_0^\infty e^{-\left(\frac{x^\alpha}{\alpha}\right)\left(\frac{1}{u}-a\right)-\left(\frac{t^\beta}{\beta}\right)\left(\frac{1}{v}-b\right)}x^{\alpha-1}dx\,dt\right|,$$

$$=\frac{Kv^2}{(1-ua)(1-vb)},\ Re\left[\frac{1}{u}\right]>\frac{1}{\mu},\ Re\left[\frac{1}{v}\right]>\frac{1}{\omega}. \tag{8}$$

Then, from (8), we have:

$$\lim_{\substack{\frac{x^\alpha}{\alpha}\to\infty\\ \frac{t^\beta}{\beta}\to\infty}}\left|H_{\alpha,\beta}(\mathrm{u,v})\right|=0, \tag{9}$$

Or

$$\lim_{\substack{\frac{x^\alpha}{\alpha}\to\infty\\ \frac{t^\beta}{\beta}\to\infty}} H_{\alpha,\beta}(\mathrm{u,v})=0. \tag{10}$$

Now we will discuss the following result which deals with the multiplication of the function. $h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$ by $\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}$.

Theorem 2.2.2

If conformable CSET of the function $h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$ where $0<\alpha,\beta\leq 1$ is given by $H_{\alpha,\beta}(\mathrm{u},\mathrm{v})=S_x^\alpha E_t^\beta\left[h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right]$, then the CSET of the functions $\frac{x^\alpha}{\alpha}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right),\frac{t^\beta}{\beta}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right),\left(\frac{x^\alpha}{\alpha}\right)^2h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right),\left(\frac{t^\beta}{\beta}\right)^2h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right),\frac{x^\alpha}{\alpha}\frac{t^\beta}{\beta}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$, are given by

$$S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right]=u^2\frac{\partial H_{\alpha,\beta}(\mathrm{u},\mathrm{v})}{\partial u}+uH_{\alpha,\beta}(\mathrm{u},\mathrm{v}), \tag{11}$$

$$S_x^\alpha E_t^\beta\left[\frac{t^\beta}{\beta}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right]=v^2\frac{\partial H_{\alpha,\beta}(\mathrm{u},\mathrm{v})}{\partial v}-vH_{\alpha,\beta}(\mathrm{u},\mathrm{v}), \tag{12}$$

$$S_x^\alpha E_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^2h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right]=u^4\frac{\partial^2 H_{\alpha,\beta}(\mathrm{u},\mathrm{v})}{\partial u^2}+4u^3\frac{\partial H_{\alpha,\beta}(\mathrm{u},\mathrm{v})}{\partial u}+2u^3H_{\alpha,\beta}(\mathrm{u},\mathrm{v}), \tag{13}$$

$$S_x^\alpha E_t^\beta\left[\left(\frac{t^\beta}{\beta}\right)^2h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right]=v^4\frac{\partial^2 H_{\alpha,\beta}(\mathrm{u},\mathrm{v})}{\partial v^2}+4v^3\frac{\partial H_{\alpha,\beta}(\mathrm{u},\mathrm{v})}{\partial v}+2v^3H_{\alpha,\beta}(\mathrm{u},\mathrm{v}), \tag{14}$$

$$S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{t^\beta}{\beta}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right]= uv\frac{\partial^2 H_{\alpha,\beta}(\mathrm{u},\mathrm{v})}{\partial u\partial v}+uv^2\frac{\partial H_{\alpha,\beta}(\mathrm{u},\mathrm{v})}{\partial v}+vu^2\frac{\partial H_{\alpha,\beta}(\mathrm{u},\mathrm{v})}{\partial u}+uvH_{\alpha,\beta}(\mathrm{u},\mathrm{v}), \tag{15}$$

Proof: By using the definition of the CSET for (11), we get

$$\frac{\partial H_{\alpha,\beta}(\mathrm{u},\mathrm{v})}{\partial u}=\frac{\partial}{\partial u}\int_0^\infty\int_0^\infty\frac{v}{u}e^{-\left(\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)+\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)x^{\alpha-1}t^{\beta-1}dxdt,$$

$$=\int_0^\infty ve^{\left(-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\left(\int_0^\infty\frac{\partial}{\partial u}\frac{1}{u}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)x^{\alpha-1}dx\right)t^{\beta-1}dt, \tag{16}$$

We may easily obtain the partial derivative by doing the calculation inside the brackets.

$$\int_0^\infty\frac{\partial}{\partial u}\frac{1}{u}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)x^{\alpha-1}dx=\int_0^\infty\left(\frac{1}{u^3}\frac{x^\alpha}{\alpha}-\frac{1}{u^2}\right)e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)x^{\alpha-1}dx,$$

$$=\int_0^\infty\frac{1}{u^3}\frac{x^\alpha}{\alpha}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)x^{\alpha-1}dx-\int_0^\infty\frac{1}{u^2}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)x^{\alpha-1}dx \tag{17}$$

substituting (17) into (16), we obtain

$$\frac{\partial H_{\alpha,\beta}(\mathrm{u},\mathrm{v})}{\partial u}=\int_0^\infty ve^{\left(-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\left(\int_0^\infty\frac{1}{u^3}\frac{x^\alpha}{\alpha}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)x^{\alpha-1}dx\right)t^{\beta-1}dt-\int_0^\infty \mathrm{v}e^{\left(-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\left(\int_0^\infty\frac{1}{u^2}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}h\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)x^{\alpha-1}dx\right)t^{\beta-1}dt, \tag{18}$$

By applying (6), we get (11).

$$S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}h\left(\left(\frac{x^\alpha}{\alpha}\right),\left(\frac{t^\beta}{\beta}\right)\right)\right]=u^2\frac{\partial H_{\alpha,\beta}(\mathrm{u},\mathrm{v})}{\partial u}+uH_{\alpha,\beta}(\mathrm{u},\mathrm{v}).$$

For (13), by using a second partial derivative for $H_{\alpha,\beta}(\mathrm{u},\mathrm{v})$ with respect to u and the definition of CSET, we have

$$\frac{\partial^2 H_{\alpha,\beta}(\mathrm{u,v})}{\partial u^2} = \frac{\partial^2}{\partial u^2}\int_0^\infty\int_0^\infty \frac{v}{u} e^{-\left(\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)+\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)} h\left(\left(\frac{x^\alpha}{\alpha}\right),\left(\frac{t^\beta}{\beta}\right)\right) x^{\alpha-1}t^{\beta-1}dxdt$$

$$= \int_0^\infty ve^{\left(-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\left(\int_0^\infty \frac{\partial^2}{\partial u^2}\left(\frac{1}{u}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}\right)h\left(\left(\frac{x^\alpha}{\alpha}\right),\left(\frac{t^\beta}{\beta}\right)\right)x^{\alpha-1}dx\right)t^{\beta-1}dt,$$

$$= \int_0^\infty ve^{\left(-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\left(\int_0^\infty\left(\frac{2e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}}{u^3} - \frac{4\left(\frac{x^\alpha}{\alpha}\right)e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}}{u^4} + \frac{\left(\frac{x^\alpha}{\alpha}\right)^2 e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}}{u^5}\right)h\left(\left(\frac{x^\alpha}{\alpha}\right),\left(\frac{t^\beta}{\beta}\right)\right)x^{\alpha-1}dx\right)t^{\beta-1}dt,$$

$$= \int_0^\infty ve^{\left(-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\left(\int_0^\infty\left(\frac{1}{u^5}\left(\frac{x^\alpha}{\alpha}\right)^2 - \frac{3}{u^4}\left(\frac{x^\alpha}{\alpha}\right)\right)e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}h\left(\left(\frac{x^\alpha}{\alpha}\right),\left(\frac{t^\beta}{\beta}\right)\right)x^{\alpha-1}dx\right)t^{\beta-1}dt - \int_0^\infty \mathrm{v}e^{\left(-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\left(\int_0^\infty\left(\frac{1}{u^4}\left(\frac{x^\alpha}{\alpha}\right) - \frac{2}{u^3}\right)e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}h\left(\left(\frac{x^\alpha}{\alpha}\right),\left(\frac{t^\beta}{\beta}\right)\right)x^{\alpha-1}dx\right)t^{\beta-1}dt, \quad (19)$$

By applying (6), we get (13).

$$S_x^\alpha E_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^2 h\left(\left(\frac{x^\alpha}{\alpha}\right),\left(\frac{t^\beta}{\beta}\right)\right)\right] = u^4\frac{\partial^2 H_{\alpha,\beta}(\mathrm{u,v})}{\partial u^2} + 4u^3\frac{\partial H_{\alpha,\beta}(\mathrm{u,v})}{\partial u} + 2u^2 H_{\alpha,\beta}(\mathrm{u,v}),$$

In the same way, one can prove ((14)-(16)).

Theorem 2.2.3

For the CSET of the first and second partial derivatives**,**
if $H(\alpha,\beta) = S_x^\alpha E_t^\beta[\mathrm{h}(x,t)]$, then

$$S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha h_{\alpha,\beta}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})}{\partial x^\alpha}\right] = \frac{1}{u}H_{\alpha,\beta}(u,v) - \frac{1}{u}H_\beta(u,0), \quad (20)$$

$$S_x^\alpha E_t^\beta\left[\frac{\partial^\beta h_{\alpha,\beta}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})}{\partial t^\beta}\right] = \frac{1}{v}H_{\alpha,\beta}(u,v) - vH_\alpha(u,0), \quad (21)$$

$$S_x^\alpha E_t^\beta\left[\frac{\partial^{2\alpha} h_{\alpha,\beta}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})}{\partial x^{2\alpha}}\right] = \frac{1}{u^2}H_{\alpha,\beta}(u,v) - \frac{1}{u^2}H_\beta(0,v) - \frac{1}{u}E_t^\beta\left(\frac{\partial^\alpha h_{\alpha,\beta}\left(0,\frac{t^\beta}{\beta}\right)}{\partial x^\alpha}\right), \quad (22)$$

$$S_x^\alpha E_t^\beta\left[\frac{\partial^{2\beta} h_{\alpha,\beta}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})}{\partial t^{2\beta}}\right] = \frac{1}{v^2}H_{\alpha,\beta}(u,v) - H_\alpha(u,0) - vS_x^\alpha\left(\frac{\partial^\beta h_{\alpha,\beta}(\frac{x^\alpha}{\alpha},0)}{\partial t^\beta}\right), \quad (23)$$

Proof: see [3].

The following theorem gives the CSET of the fractional partial derivatives $\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}$ and $\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}$.

Theorem 2.2.4

The CSET of the fractional partial derivatives $\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}$and $\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}$ are given by:

$$S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\right]=\frac{u}{v}\frac{d}{du}\left[uH_{\alpha,\beta}(u,v)\right]-uv\frac{d}{du}\left[uH_\beta(u,0)\right],$$
(24)

$$S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}\right]=\frac{u}{v^2}\frac{d}{du}\left[uH_{\alpha,\beta}(u,v)\right]-u\frac{d}{du}\left[uH_\beta(u,0)\right]-uv\frac{d}{du}\left[uS_x^\alpha\left(\frac{\partial^\beta h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right)}{\partial t^\beta}\right)\right] \quad (25)$$

, respectively.

Proof: By using the definition CSET, we have

$$\frac{d}{du}\left(S_x^\alpha E_t^\beta\left[\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\right]\right)=\frac{d}{du}\int_0^\infty\int_0^\infty\frac{v}{u}e^{-\left(\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)+\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}x^{\alpha-1}t^{\beta-1}dxdt,$$

$$=\int_0^\infty ve^{\left(-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\left(\int_0^\infty\frac{d}{du}\left(\frac{1}{u}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}\right)x^{\alpha-1}dx\right)t^{\beta-1}dt \quad (26)$$

The partial derivative inside the brackets is as follows:

$$\int_0^\infty\frac{d}{du}\left(\frac{1}{u}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}\right)x^{\alpha-1}dx=\int_0^\infty\left(\frac{1}{u^3}\frac{x^\alpha}{\alpha}-\frac{1}{u^2}\right)e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}x^{\alpha-1}dx,$$

$$=\int_0^\infty\frac{1}{u^3}\frac{x^\alpha}{\alpha}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}x^{\alpha-1}dx-\int_0^\infty\frac{1}{u^2}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}x^{\alpha-1}dx \quad (27)$$

substituting (27) into (26), we obtain

$$\frac{d}{du}\left(S_x^\alpha E_t^\beta\left[\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\right]\right)=\int_0^\infty ve^{\left(-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\left(\int_0^\infty\frac{1}{u^3}\frac{x^\alpha}{\alpha}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}x^{\alpha-1}dx-\right.$$
$$\left.\int_0^\infty\frac{1}{u^2}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}x^{\alpha-1}dx\right)t^{\beta-1}dt,$$

$$u^2\frac{d}{du}\left(S_x^\alpha E_t^\beta\left[\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\right]\right)=\int_0^\infty\int_0^\infty\frac{v}{u}\frac{x^\alpha}{\alpha}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}t^{\beta-1}x^{\alpha-1}dxdt-$$
$$u\int_0^\infty\int_0^\infty\frac{v}{u}e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}t^{\beta-1}x^{\alpha-1}dxdt, \quad (28)$$

applying (6), we have:

$$S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\right]=u^2\frac{d}{du}\left(S_x^\alpha E_t^\beta\left[\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\right]\right)+uS_x^\alpha E_t^\beta\left[\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\right], \quad (29)$$

applying (21), we have:

$$S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\right]=u^2\frac{d}{du}\left(\frac{1}{v}H_{\alpha,\beta}(u,v)-vH_\alpha(u,0)\right)+u\left(\frac{1}{v}H_{\alpha,\beta}(u,v)-vH_\alpha(u,0)\right),$$

$$S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\right]=\frac{1}{v}u^2\frac{d}{du}H_{\alpha,\beta}(u,v)-u^2v\frac{d}{du}H_\alpha(u,0)+\frac{1}{v}uH_{\alpha,\beta}(u,v)-uvH_\alpha(u,0),$$

$$S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\right]=\frac{u}{v}\left(u\frac{d}{du}H_{\alpha,\beta}(u,v)+H_{\alpha,\beta}(u,v)\right)-uv\left(u\frac{d}{du}H_\alpha(u,0)+H_\alpha(u,0)\right),$$

$$S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\right]=\frac{u}{v}\frac{d}{du}\left[uH_{\alpha,\beta}(u,v)\right]-uv\frac{d}{du}\left[uH_\beta(u,0)\right], \quad (30)$$

by the same way, one can prove (25):

$$\frac{d}{du}\left(S_x^\alpha E_t^\beta\left[\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}\right]\right)=\frac{d}{du}\int_0^\infty\int_0^\infty\frac{v}{u}e^{-\left(\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)+\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)}\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}x^{\alpha-1}t^{\beta-1}dxdt,$$

$$= \int_0^\infty v e^{\left(-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)} \frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}} \left(\int_0^\infty \frac{d}{du}\left(\frac{1}{u} e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}\right) x^{\alpha-1} dx\right) t^{\beta-1} dt, \quad (31)$$

we calculate the partial derivative inside brackets as follows:

$$\int_0^\infty \frac{d}{du}\left(\frac{1}{u} e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)}\right) x^{\alpha-1} dx = \int_0^\infty \left(\frac{1}{u^3}\frac{x^\alpha}{\alpha} - \frac{1}{u^2}\right) e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)} x^{\alpha-1} dx,$$

$$= \int_0^\infty \frac{1}{u^3}\frac{x^\alpha}{\alpha} e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)} x^{\alpha-1} dx - \int_0^\infty \frac{1}{u^2} e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)} x^{\alpha-1} dx, \quad (32)$$

substituting (32) into (31), we obtain

$$\frac{d}{du}\left(S_x^\alpha E_t^\beta \left[\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}\right]\right) = \int_0^\infty v e^{\left(-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)} \frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}} \left(\int_0^\infty \frac{1}{u^3}\frac{x^\alpha}{\alpha} e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)} x^{\alpha-1} dx - \int_0^\infty \frac{1}{u^2} e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)\right)} x^{\alpha-1} dx\right) t^{\beta-1} dt,$$

$$u^2 \frac{d}{du}\left(S_x^\alpha E_t^\beta \left[\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}\right]\right) = \int_0^\infty \int_0^\infty \frac{v}{u}\frac{x^\alpha}{\alpha} e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)} \frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}} t^{\beta-1} x^{\alpha-1} dx dt - \mathrm{u} \int_0^\infty \int_0^\infty \frac{v}{u} e^{\left(-\left(\frac{1}{u}\right)\left(\frac{x^\alpha}{\alpha}\right)-\left(\frac{1}{v}\right)\left(\frac{t^\beta}{\beta}\right)\right)} \frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}} t^{\beta-1} x^{\alpha-1} dx dt, \quad (33)$$

applying (6), we have:

$$S_x^\alpha E_t^\beta \left[\frac{x^\alpha}{\alpha}\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}\right] = u^2 \frac{d}{du}\left(S_x^\alpha E_t^\beta \left[\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}\right]\right) + u S_x^\alpha E_t^\beta \left[\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}\right], \quad (34)$$

applying (22), we have:

$$S_x^\alpha E_t^\beta \left[\frac{x^\alpha}{\alpha}\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}\right] = u^2 \frac{d}{du}\left(\frac{1}{v^2} H_{\alpha,\beta}(u,v) - H_\alpha(u,0) - v S_x^\alpha \left(\frac{\partial^\beta h_{\alpha,\beta}(\frac{x^\alpha}{\alpha},0)}{\partial t^\beta}\right)\right) + u\left(\frac{1}{v^2} H_{\alpha,\beta}(u,v) - H_\alpha(u,0) - v S_x^\alpha \left(\frac{\partial^\beta h_{\alpha,\beta}(\frac{x^\alpha}{\alpha},0)}{\partial t^\beta}\right)\right),$$

$$S_x^\alpha E_t^\beta \left[\frac{x^\alpha}{\alpha}\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}\right] = \frac{1}{v^2} u^2 \frac{d}{du} H_{\alpha,\beta}(u,v) - u^2 \frac{d}{du} H_\alpha(u,0) - v u^2 \frac{d}{du} S_x^\alpha \left(\frac{\partial^\beta h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right)}{\partial t^\beta}\right) + \frac{1}{v^2} u H_{\alpha,\beta}(u,v) - u H_\alpha(u,0) - v u S_x^\alpha \left(\frac{\partial^\beta h_{\alpha,\beta}(\frac{x^\alpha}{\alpha},0)}{\partial t^\beta}\right),$$

$$S_x^\alpha E_t^\beta \left[\frac{x^\alpha}{\alpha}\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}\right] = \frac{u}{v^2}\left(u \frac{d}{du} H_{\alpha,\beta}(u,v) + H_{\alpha,\beta}(u,v)\right) - u\left(u \frac{d}{du} H_\alpha(u,0) + H_\alpha(u,0)\right) - uv\left(u \frac{d}{du} S_x^\alpha \left(\frac{\partial^\beta h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right)}{\partial t^\beta}\right) + S_x^\alpha \left(\frac{\partial^\beta h_{\alpha,\beta}(\frac{x^\alpha}{\alpha},0)}{\partial t^\beta}\right)\right),$$

$$S_x^\alpha E_t^\beta \left[\frac{x^\alpha}{\alpha}\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}\right] = \frac{u}{v^2}\frac{d}{du}\left[u H_{\alpha,\beta}(u,v)\right] - u \frac{d}{du}\left[u H_\beta(u,0)\right] - uv \frac{d}{du}\left[u S_x^\alpha \left(\frac{\partial^\beta h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right)}{\partial t^\beta}\right)\right] \quad (35)$$

3. Descriptions of the method

In this section, we will use the CSETDM to solve the regular and singular one-dimensional CFCB'sE and the NSCPE.

The first problem: One-dimensional CFCB'sE is given by:

$$\frac{\partial^{\beta} h_{\alpha,\beta}}{\partial t^{\beta}} - \frac{\partial^{2\alpha} h_{\alpha,\beta}}{\partial x^{2\alpha}} + \rho h_{\alpha,\beta} \frac{\partial^{\alpha}}{\partial x^{\alpha}} h_{\alpha,\beta} + \tau \frac{\partial^{\alpha}}{\partial x^{\alpha}} \left(h_{\alpha,\beta} k_{\alpha,\beta}\right) = f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)$$
$$\frac{\partial^{\beta} k_{\alpha,\beta}}{\partial t^{\beta}} - \frac{\partial^{2\alpha} k_{\alpha,\beta}}{\partial x^{2\alpha}} + \rho k_{\alpha,\beta} \frac{\partial^{\alpha}}{\partial x^{\alpha}} k_{\alpha,\beta} + \mu \frac{\partial^{\alpha}}{\partial x^{\alpha}} \left(h_{\alpha,\beta} k_{\ \alpha,\beta}\right) = g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right), \quad (36)$$

Subject to

$$h_{\alpha,\beta}\left(\frac{x^{\alpha}}{\alpha}, 0\right) = f_1\left(\frac{x^{\alpha}}{\alpha}\right), \quad k_{\alpha,\beta}\left(\frac{x^{\alpha}}{\alpha}, 0\right) = g_1\left(\frac{x^{\alpha}}{\alpha}\right), \quad (37)$$

For $t > 0$. Here, $f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right), g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right), f_1\left(\frac{x^{\alpha}}{\alpha}\right)$ and $g_1\left(\frac{x^{\alpha}}{\alpha}\right)$ are given functions where ρ, τ, μ are arbitrary constants. Both sides of (36) can be treated using CSET, while (37) can be treated using conformable Sumudu transformations.

$$H_{\alpha,\beta}(u, v) = v^2 H_{\alpha}(u, 0) + v S_x^{\alpha} E_t^{\beta}\left[f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) + \frac{\partial^{2\alpha} h_{\alpha,\beta}}{\partial x^{2\alpha}} - \rho h \frac{\partial^{\alpha}}{\partial x^{\alpha}} h_{\alpha,\beta} - \tau \frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(h_{\alpha,\beta} k_{\alpha,\beta}\right)\right] \quad (38)$$

$$K_{\alpha,\beta}(u, v) = v^2 K_{\alpha}(u, 0) + v S_x^{\alpha} E_t^{\beta}\left[g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) + \frac{\partial^{2\alpha} k_{\alpha,\beta}}{\partial x^{2\alpha}} - \rho k \frac{\partial^{\alpha}}{\partial x^{\alpha}} k_{\alpha,\beta} - \mu \frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(h_{\alpha,\beta} k_{\alpha,\beta}\right)\right] \quad (39)$$

The solution to the one-dimensional CFCB'sE is given by the following infinite series:

$$h_{\alpha,\beta}\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) = \sum_{n=0}^{\infty}\left(h_{\alpha,\beta,n}\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right), \quad (40)$$
$$k_{\alpha,\beta}\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) = \sum_{n=0}^{\infty}\left(k_{\alpha,\beta,n}\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right). \quad (41)$$

Adomian's polynomials describe the nonlinear operators A_n, B_n, and C_n in the following way:

$$A_n = \sum_{n=0}^{\infty} h_n h_{nx}$$
$$B_n = \sum_{n=0}^{\infty} k_n k_{nx}, \quad (42)$$
$$C_n = \sum_{n=0}^{\infty} h_n k_n$$

In particular, the Adomian polynomials for the nonlinear terms $h_n h_{xn}$, $k_n k_{xn}$ and $h_n v_{xn}$ can be computed by the following equations

$$A_0 = h_0 h_{0x},$$
$$A_1 = h_0 h_{1x} + h_1 h_{0x},$$
$$A_2 = h_0 h_{2x} + h_1 h_{1x} + h_2 h_{0x},$$
$$A_3 = h_0 h_{3x} + h_1 h_{2x} + h_2 h_{1x} + h_3 h_{0x},$$
$$A_4 = h_0 h_{4x} + h_1 h_{3x} + h_2 h_{2x} + h_3 h_{1x} + h_4 h_{0x},$$
$$B_0 = k_0 k_{0x},$$
$$B_1 = k_0 k_{1x} + k_1 k_{0x},$$
$$B_2 = k_0 k_{2x} + k_1 k_{1x} + k_2 k_{0x},$$
$$B_3 = k_0 k_{3x} + k_1 k_{2x} + k_2 k_{1x} + k_3 k_{0x},$$
$$B_4 = k_0 k_{4x} + k_1 k_{3x} + k_2 k_{2x} + k_3 k_{1x} + k_4 k_{0x},$$

And

$$C_0 = h_0 k_0,$$
$$C_1 = h_0 k_1 + h_1 k_0,$$
$$C_2 = h_0 k_2 + h_1 k_1 + h_2 k_0,$$
$$C_3 = h_0 k_3 + h_1 k_2 + h_2 k_1 + h_3 k_0,$$
$$C_4 = h_0 k_4 + h_1 k_3 + h_2 k_2 + h_3 k_1 + h_4 k_0.$$

By applying the ICSET on both sides of (38) and (39), making use of (42), we have:

$$\sum_{n=0}^{\infty} h_{\alpha,\beta,n}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = v^2 f_1\left(\frac{x^\alpha}{\alpha}\right)$$
$$+S_x^\alpha E_x^\beta\ ^{-1}\left[vF\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right] + S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{\partial^{2\alpha} h_{\alpha,\beta}}{\partial x^{2\alpha}}\right]\right]$$
$$-S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta[\rho A_n]\right] - S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta[\tau C_n]\right], \quad (43)$$
$$\sum_{n=0}^{\infty} k_{\alpha,\beta,n}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = v^2 g_1\left(\frac{x^\alpha}{\alpha}\right)$$
$$+S_x^\alpha E_x^\beta\ ^{-1}\left[vG\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right] + S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{\partial^{2\alpha} k_{\alpha,\beta}}{\partial x^{2\alpha}}\right]\right]$$
$$-S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta[\rho B_n]\right] - S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta[\mu C_n]\right]$$

for the case $n = 0$, we set:

$$h_{\alpha,\beta,0} = v^2 f_1\left(\frac{x^\alpha}{\alpha}\right) + S_x^\alpha E_x^\beta\ ^{-1}[vF(u,v)]$$
$$k_{\alpha,\beta,0} = v^2 g_1\left(\frac{x^\alpha}{\alpha}\right) + S_x^\alpha E_x^\beta\ ^{-1}[vG(u,v)], \quad (44)$$

Now, we can obtain the following general form:

$$h_{\alpha,\beta,n+1} = S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{\partial^{2\alpha} h_{\alpha,\beta}}{\partial x^{2\alpha}}\right]\right]$$
$$-S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta[\rho A_n]\right] - S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta[\tau C_n]\right], \quad (45)$$
$$k_{\alpha,\beta,n+1} = S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{\partial^{2\alpha} k_{\alpha,\beta}}{\partial x^{2\alpha}}\right]\right]$$
$$-S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta[\rho B_1]\right] - S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta[\mu C_n]\right]$$

This provides that the ICSET with respect to u and v exists in the above equations.

The second problem: Now we consider the NSCPE in one dimension of the form:

$$\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}} - \frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) - \frac{\alpha}{x^\alpha}\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) - a\left(\frac{x^\alpha}{\alpha}\right)h_{\alpha,\beta}\frac{\partial^\alpha h_{\alpha,\beta}}{\partial x^\alpha} + \left(h_{\alpha,\beta}\right)^2 = f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right), \quad (46)$$

subject to the initial condition

$$h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right) = g_1\left(\frac{x^\alpha}{\alpha}\right), \quad \frac{\partial^{\beta} h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right)}{\partial t^\beta} = g_2\left(\frac{x^\alpha}{\alpha}\right), \quad (47)$$

Where the functions $a\left(\frac{x^\alpha}{\alpha}\right)$ are arbitrary. To find the answer to problem (46), multiply (46) by $\frac{x^\alpha}{\alpha}$ and taking CSET, we have:

$$S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}}\right] = S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right)\right] +$$
$$S_x^\alpha E_t^\beta\left[a\left(\frac{x^\alpha}{\alpha}\right)\left(\frac{x^\alpha}{\alpha}\right)h_{\alpha,\beta}\frac{\partial^\alpha h_{\alpha,\beta}}{\partial x^\alpha} - \frac{x^\alpha}{\alpha}\left(h_{\alpha,\beta}\right)^2 = S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right]\right], \quad (48)$$

Where conformable Sumudu transform for the initial condition $h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right)$, $\frac{\partial^{\beta} h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right)}{\partial t^{\beta}}$, given by

$$H_\alpha(u,0) = G_1(u,0), \ S_x^\alpha\left[\frac{\partial^{\beta} h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right)}{\partial t^{\beta}}\right] = G_2(u,0). \quad (49)$$

Applying (23) and (49) into (48), we get the following:

$\frac{u}{v^2}\frac{d}{du}\left[uH_{\alpha,\beta}(u,v)\right] = u\frac{d}{du}[uG_1(u,0)] + uv\frac{d}{du}[uG_2(u,0)] + u^2\frac{\partial F_{\alpha,\beta}(\mathrm{u,v})}{\partial u} + uF_{\alpha,\beta}(\mathrm{u,v}) + S_x^\alpha E_t^\beta[\emptyset]$, (50)

$\emptyset = \frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + a\left(\frac{x^\alpha}{\alpha}\right)\left(\frac{x^\alpha}{\alpha}\right)h_{\alpha,\beta}\frac{\partial^\alpha h_{\alpha,\beta}}{\partial x^\alpha} - \frac{x^\alpha}{\alpha}\left(h_{\alpha,\beta}\right)^2.$

By multiplying both sides of (50) by $\frac{v^2}{u}$, we have:

$\frac{d}{du}\left[uH_{\alpha,\beta}(u,v)\right] = v^2\frac{d}{du}[uG_1(u,0)] + v^3\frac{d}{du}[uG_2(u,0)] + \frac{v^2}{u}S_x^\alpha E_t^\beta[\emptyset] + uv^2\frac{\partial F_{\alpha,\beta}(\mathrm{u,v})}{\partial u} + v^2F_{\alpha,\beta}(\mathrm{u,v})$, (51)

by applying the integral for both sides of (51) from 0 to u with respect to u, we have

$H_{\alpha,\beta}(u,v) = \frac{1}{u}\int_0^u\left(v^2\frac{d}{du}[uG_1(u,0)] + v^3\frac{d}{du}[uG_2(u,0)] + \frac{v^2}{u}S_x^\alpha E_t^\beta[\emptyset] + uv^2\frac{\partial F_{\alpha,\beta}(\mathrm{u,v})}{\partial u} + v^2F_{\alpha,\beta}(\mathrm{u,v})\right)du$, (52)

Applying the ICSET, to both sides of (62) yields the following:

$H_{\alpha,\beta}(u,v) = v^2G_1(u,0) + v^3G_2(u,0) + S_x^\alpha E_x^\beta{}^{-1}\left[\frac{1}{u}\int_0^u\left(uv^2\frac{\partial F_{\alpha,\beta}(\mathrm{u,v})}{\partial u} + v^2F_{\alpha,\beta}(\mathrm{u,v})\right)du\right] + S_x^\alpha E_x^\beta{}^{-1}\left[\frac{1}{u}\int_0^u\frac{v^2}{u}S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + a\left(\frac{x^\alpha}{\alpha}\right)\left(\frac{x^\alpha}{\alpha}\right)h_{\alpha,\beta}\frac{\partial^\alpha h_{\alpha,\beta}}{\partial x^\alpha} - \frac{x^\alpha}{\alpha}\left(h_{\alpha,\beta}\right)^2\right]du\right]$,
(53)

The CSET proposes an infinite series solution for the functions $h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$

$h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = \sum_{n=0}^\infty h_{\alpha,\beta,n}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$, also, the nonlinear terms $h_{\alpha,\beta}\frac{\partial^\alpha h_{\alpha,\beta}}{\partial x^\alpha}$ and $\left(h_{\alpha,\beta}\right)^2$ can be defined by:

$\left(h_{\alpha,\beta}\right)^2 = N_1 = \sum_{n=0}^\infty A_n,\ h_{\alpha,\beta}\frac{\partial^\alpha h_{\alpha,\beta}}{\partial x^\alpha} = N_1 = \sum_{n=0}^\infty B_n.$ (54)

We have a few terms of the Adomian polynomials for A_n and B_n that are denoted by

$A_n = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}[N_1\sum_{n=0}^\infty(\lambda^n u_n)]\right)_{\lambda=0}$, and $B_n = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}[N_2\sum_{n=0}^\infty(\lambda^n u_n)]\right)_{\lambda=0}$, (55)

where $n = 0,1,2,....$ We put ((54)-(55)) into (53), then we get

$\sum_{n=0}^\infty h_{\alpha,\beta,n}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) = v^2G_1(u,0) + v^3G_2(u,0) + S_x^\alpha E_x^\beta{}^{-1}\left[\frac{1}{u}\int_0^u\left(uv^2\frac{\partial F_{\alpha,\beta}(\mathrm{u,v})}{\partial u} + v^2F_{\alpha,\beta}(\mathrm{u,v})\right)du\right] + S_x^\alpha E_x^\beta{}^{-1}\left[\frac{1}{u}\int_0^u\left(\frac{v^2}{u}S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^\infty h_{\alpha,\beta,n}\right)\right)\right]\right)du\right] + S_x^\alpha E_x^\beta{}^{-1}\left[\frac{1}{u}\int_0^u\left(\frac{v^2}{u}S_x^\alpha E_t^\beta\left[\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^\infty h_{\alpha,\beta,n}\right)\right)\right]\right)du\right] + S_x^\alpha E_x^\beta{}^{-1}\left[\frac{1}{u}\int_0^u\left(\frac{v^2}{u}S_x^\alpha E_t^\beta\left[a\left(\frac{x^\alpha}{\alpha}\right)\left(\frac{x^\alpha}{\alpha}\right)\sum_{n=0}^\infty A_{\alpha,\beta,n}\right]\right)du\right] - S_x^\alpha E_x^\beta{}^{-1}\left[\frac{1}{u}\int_0^u\left(\frac{v^2}{u}S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}\left(\sum_{n=0}^\infty B_{\alpha,\beta,n}\right)^2\right]\right)du\right]$, (56)

the few components of the Adomian polynomials of ((54)-(55)) are given as follows:

$A_{\alpha,\beta,0} = h_{\alpha,\beta,0}\frac{\partial^\alpha h_{\alpha,\beta,0}}{\partial x^\alpha}$, $A_{\alpha,\beta,1} = h_{\alpha,\beta,0}\frac{\partial^\alpha h_{\alpha,\beta,1}}{\partial x^\alpha} + h_{\alpha,\beta,1}\frac{\partial^\alpha h_{\alpha,\beta,0}}{\partial x^\alpha}$, $A_{\alpha,\beta,2} = h_{\alpha,\beta,0}\frac{\partial^\alpha h_{\alpha,\beta,2}}{\partial x^\alpha} + h_{\alpha,\beta,1}\frac{\partial^\alpha h_{\alpha,\beta,1}}{\partial x^\alpha} + h_{\alpha,\beta,2}\frac{\partial^\alpha h_{\alpha,\beta,0}}{\partial x^\alpha}$, $A_{\alpha,\beta,3} = h_{\alpha,\beta,0}\frac{\partial^\alpha h_{\alpha,\beta,3}}{\partial x^\alpha} + h_{\alpha,\beta,1}\frac{\partial^\alpha h_{\alpha,\beta,2}}{\partial x^\alpha} + h_{\alpha,\beta,2}\frac{\partial^\alpha h_{\alpha,\beta,1}}{\partial x^\alpha} + h_{\alpha,\beta,3}\frac{\partial^\alpha h_{\alpha,\beta,0}}{\partial x^\alpha}$, (57)

and

$B_{\alpha,\beta,0} = \left(h_{\alpha,\beta,0}\right)^2$, $B_{\alpha,\beta,1} = 2h_{\alpha,\beta,0}h_{\alpha,\beta,1}$, $B_{\alpha,\beta,2} = 2h_{\alpha,\beta,0}h_{\alpha,\beta,2} + \left(h_{\alpha,\beta,1}\right)^2$, $B_{\alpha,\beta,3} = 2h_{\alpha,\beta,0}h_{\alpha,\beta,3} + 2h_{\alpha,\beta,1}h_{\alpha,\beta,2}$, (58)

Hence, the zeroth component $h_{\alpha,\beta,0}$ from (56) is given by

$$h_{\alpha,\beta,0}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) = v^2 G_1(u,0) + v^3 G_2(u,0) + S_x^\alpha E_x^\beta{}^{-1}\left[\frac{1}{u}\int_0^u \left(uv^2\frac{\partial F_{\alpha,\beta}(\mathrm{u,v})}{\partial u} + v^2 F_{\alpha,\beta}(\mathrm{u,v})\right)du\right],$$
(59)

Now, we can obtain the following general form:

$$h_{\alpha,\beta,n+1}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) = S_x^\alpha E_x^\beta{}^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v^2}{u} S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha h_{\alpha,\beta,k}}{\partial x^\alpha}\right)\right]\right)du\right] + S_x^\alpha E_x^\beta{}^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v^2}{u} S_x^\alpha E_t^\beta\left[\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha h_{\alpha,\beta,k}}{\partial x^\alpha}\right)\right]\right)du\right] + S_x^\alpha E_x^\beta{}^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v^2}{u} S_x^\alpha E_t^\beta\left[a\left(\frac{x^\alpha}{\alpha}\right)\left(\frac{x^\alpha}{\alpha}\right)A_{\alpha,\beta,n} - \frac{x^\alpha}{\alpha}B_{\alpha,\beta,k}{}^2\right]\right)du\right], \quad (60)$$

Where $k \geq 0$.

The third problem: Now consider the singular one dimensional CFCB'sE with Bessel operator

$$\begin{aligned}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta} - \frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + \rho h_{\alpha,\beta}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta} + \tau\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right) = f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\\ \frac{\partial^\beta k_{\alpha,\beta}}{\partial t^\beta} - \frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta}\right) + \rho k_{\alpha,\beta}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta} + \sigma\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right) = g\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\end{aligned}, \quad (61)$$

Subject to the initial condition

$$h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right) = f_1\left(\frac{x^\alpha}{\alpha}\right), \quad k_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right) = g_1\left(\frac{x^\alpha}{\alpha}\right), \quad (62)$$

where the linear terms $\frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\right)$ is known as Conformable Bessel operator where τ, ρ, and σ are real constants. Now to obtain the solution of (61), First, we multiply both sides of (61) by $\frac{x^\alpha}{\alpha}$ and obtain:

$$\begin{aligned}\frac{x^\alpha}{\alpha}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta} - \frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + \rho\frac{x^\alpha}{\alpha}h_{\alpha,\beta}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta} + \tau\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right) = \frac{x^\alpha}{\alpha}f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\\ \frac{x^\alpha}{\alpha}\frac{\partial^\beta k_{\alpha,\beta}}{\partial t^\beta} - \frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta}\right) + \rho\frac{x^\alpha}{\alpha}k_{\alpha,\beta}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta} + \sigma\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right) = \frac{x^\alpha}{\alpha}g\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\end{aligned}, \quad (63)$$

Now apply CSET to both sides of (63) and a single conformable Sumudu transform for the initial condition, we get

$$\begin{aligned}&S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}\right] =\\ &S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) - \rho\frac{x^\alpha}{\alpha}h_{\alpha,\beta}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta} - \tau\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right) + \frac{x^\alpha}{\alpha}f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right]\\ &S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^\beta k_{\alpha,\beta}}{\partial t^\beta}\right] =\\ &S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta}\right) + \rho\frac{x^\alpha}{\alpha}k_{\alpha,\beta}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta} + \sigma\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right) = \frac{x^\alpha}{\alpha}g\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right]\end{aligned}, \quad (64)$$

by applying (12) and (21), we have:

$$\frac{u}{v}\frac{d}{du}\left[uH_{\alpha,\beta}(u,v)\right]-uv\frac{d}{du}\left[uH_{\beta}(u,0)\right]$$
$$=u^{2}\frac{\partial F_{\alpha,\beta}(\mathrm{u,v})}{\partial u}+uF_{\alpha,\beta}(\mathrm{u,v})$$
$$+S_{x}^{\alpha}E_{t}^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}h_{\alpha,\beta}\right)-\rho\frac{x^{\alpha}}{\alpha}h_{\alpha,\beta}\frac{\partial^{\alpha}}{\partial x^{\alpha}}h_{\alpha,\beta}-\tau\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right)\right],\tag{65}$$
$$\frac{u}{v}\frac{d}{du}\left[uK_{\alpha,\beta}(u,v)\right]-uv\frac{d}{du}\left[uK_{\beta}(u,0)\right]$$
$$=v^{2}\frac{\partial G_{\alpha,\beta}(\mathrm{u,v})}{\partial u}+vG_{\alpha,\beta}(\mathrm{u,v})$$
$$+S_{x}^{\alpha}E_{t}^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}k_{\alpha,\beta}\right)-\rho\frac{x^{\alpha}}{\alpha}k_{\alpha,\beta}\frac{\partial^{\alpha}}{\partial x^{\alpha}}k_{\alpha,\beta}-\sigma\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right)\right]$$

simplifying (65), we obtain:

$$\frac{u}{v}\frac{d}{du}\left[uH_{\alpha,\beta}(u,v)\right]$$
$$=uv\frac{d}{du}\left[uH_{\beta}(u,0)\right]+u^{2}\frac{\partial F_{\alpha,\beta}(\mathrm{u,v})}{\partial u}$$
$$+uF_{\alpha,\beta}(\mathrm{u,v})+S_{x}^{\alpha}E_{t}^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}h_{\alpha,\beta}\right)-\rho\frac{x^{\alpha}}{\alpha}h_{\alpha,\beta}\frac{\partial^{\alpha}}{\partial x^{\alpha}}h_{\alpha,\beta}-\tau\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right)\right],\tag{66}$$
$$\frac{u}{v}\frac{d}{du}\left[uK_{\alpha,\beta}(u,v)\right]$$
$$=uv\frac{d}{du}\left[uK_{\beta}(u,0)\right]+v^{2}\frac{\partial G_{\alpha,\beta}(\mathrm{u,v})}{\partial u}$$
$$+vG_{\alpha,\beta}(\mathrm{u,v})+S_{x}^{\alpha}E_{t}^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}k_{\alpha,\beta}\right)-\rho\frac{x^{\alpha}}{\alpha}k_{\alpha,\beta}\frac{\partial^{\alpha}}{\partial x^{\alpha}}k_{\alpha,\beta}-\sigma\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right)\right]$$

When we integrate both sides of (66) from 0 to p with respect to p, we get:

$$H_{\alpha,\beta}(u,v)$$
$$=\frac{1}{u}\int_{0}^{u}\left[v^{2}\frac{d}{du}\left[uH_{\beta}(u,0)\right]+u^{2}\frac{\partial F_{\alpha,\beta}(\mathrm{u,v})}{\partial u}+uF_{\alpha,\beta}(\mathrm{u,v})\right]du$$
$$+\frac{1}{u}\int_{0}^{u}\left[S_{x}^{\alpha}E_{t}^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}h_{\alpha,\beta}\right)-\rho\frac{x^{\alpha}}{\alpha}\mathrm{N}_{1}-\tau\frac{x^{\alpha}}{\alpha}\mathrm{N}_{2}\right]\right]du,\tag{67}$$
$$K_{\alpha,\beta}(u,v)$$
$$=\frac{1}{v}\int_{0}^{u}\left[v^{2}\frac{d}{du}\left[uK_{\beta}(u,0)\right]+v^{2}\frac{\partial G_{\alpha,\beta}(\mathrm{u,v})}{\partial u}+vG_{\alpha,\beta}(\mathrm{u,v})\right]du$$
$$+\frac{1}{u}\int_{0}^{u}\left[S_{x}^{\alpha}E_{t}^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}k_{\alpha,\beta}\right)-\rho\frac{x^{\alpha}}{\alpha}\mathrm{N}_{3}-\sigma\frac{x^{\alpha}}{\alpha}\mathrm{N}_{2}\right]\right]du,$$

where

$\mathrm{N}_1=h_{\alpha,\beta}\frac{\partial^{\alpha}}{\partial x^{\alpha}}h_{\alpha,\beta}$, $\mathrm{N}_2=\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right)$, $\mathrm{N}_3=k_{\alpha,\beta}\frac{\partial^{\alpha}}{\partial x^{\alpha}}k_{\alpha,\beta}$, and $\mathrm{N}_4=\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right)$, The CSET proposes a series solution of the functions $h_{\alpha,\beta}\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)$ and $k_{\alpha,\beta}\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)$ by infinite series

$$h_{\alpha,\beta}\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=\sum_{n=0}^{\infty}h_{\alpha,\beta,n}\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right),\ k_{\alpha,\beta}\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=\sum_{n=0}^{\infty}k_{\alpha,\beta,n}\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right),\tag{68}$$

Here, the nonlinear operators can be defined as follows

$$\mathrm{N}_1=\sum_{0}^{\infty}A_n,\ \mathrm{N}_2=\sum_{0}^{\infty}C_n,\ \mathrm{N}_3=\sum_{0}^{\infty}B_n.\tag{69}$$

And applying the ICSET to both sides of (67)

$$\sum_{n=0}^{\infty} h_{\alpha,\beta,n}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = f_1\left(\frac{x^\alpha}{\alpha}\right)$$
$$+S_x^\alpha E_t^{\beta^{-1}}\left[\frac{1}{u}\int_0^u\left[v^2\frac{d}{du}\left[uH_\beta(u,0)\right]+u^2\frac{\partial F_{\alpha,\beta}(\mathrm{u,v})}{\partial u}+uF_{\alpha,\beta}(\mathrm{u,v})\right]du\right]$$
$$+S_x^\alpha E_t^{\beta^{-1}}\left[\frac{1}{u}\int_0^u\left[S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_0^\infty h_{\alpha,\beta}\right)\right)-\left[\rho\frac{x^\alpha}{\alpha}\sum_0^\infty A_n\right]-\left[\tau\frac{x^\alpha}{\alpha}\sum_0^\infty C_n\right]\right]\right]du\right], \quad (70)$$
$$\sum_{n=0}^{\infty} k_{\alpha,\beta,n}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = g_1\left(\frac{x^\alpha}{\alpha}\right)$$
$$+S_x^\alpha E_t^{\beta^{-1}}\left[\frac{1}{u}\int_0^u\left[v^2\frac{d}{du}\left[uK_\beta(u,0)\right]+u^2\frac{\partial G_{\alpha,\beta}(\mathrm{u,v})}{\partial u}+uG_{\alpha,\beta}(\mathrm{u,v})\right]du\right]$$
$$+S_x^\alpha E_t^{\beta^{-1}}\left[\frac{1}{u}\int_0^u\left[S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_0^\infty k_{\alpha,\beta}\right)\right)-\left[\rho\frac{x^\alpha}{\alpha}\sum_0^\infty B_n\right]-\left[\tau\frac{x^\alpha}{\alpha}\sum_0^\infty C_n\right]\right]\right]du\right]$$

The first few components can be written as follows

$$h_{\alpha,\beta,0}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = f_1\left(\frac{x^\alpha}{\alpha}\right)$$
$$+S_x^\alpha E_t^{\beta^{-1}}\left[\frac{1}{u}\int_0^u\left[v^2\frac{d}{du}\left[uH_\beta(u,0)\right]+u^2\frac{\partial F_{\alpha,\beta}(\mathrm{u,v})}{\partial u}+uF_{\alpha,\beta}(\mathrm{u,v})\right]du\right], \quad (71)$$
$$k_{\alpha,\beta,0}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = g_1\left(\frac{x^\alpha}{\alpha}\right)$$
$$+S_x^\alpha E_t^{\beta^{-1}}\left[\frac{1}{u}\int_0^u\left[v^2\frac{d}{du}\left[uK_\beta(u,0)\right]+u^2\frac{\partial G_{\alpha,\beta}(\mathrm{u,v})}{\partial u}+uG_{\alpha,\beta}(\mathrm{u,v})\right]du\right]$$

and

$$h_{\alpha,\beta,n+1}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) =$$
$$S_x^\alpha E_t^{\beta^{-1}}\left[\frac{1}{u}\int_0^u\left[S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_0^\infty h_{\alpha,\beta}\right)\right)-\left[\rho\frac{x^\alpha}{\alpha}\sum_0^\infty A_n\right]-\left[\tau\frac{x^\alpha}{\alpha}\sum_0^\infty C_n\right]\right]\right]du\right], \quad (72)$$
$$k_{\alpha,\beta,n+1}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) =$$
$$S_x^\alpha E_t^{\beta^{-1}}\left[\frac{1}{u}\int_0^u\left[S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_0^\infty k_{\alpha,\beta}\right)\right)-\left[\rho\frac{x^\alpha}{\alpha}\sum_0^\infty B_n\right]-\left[\tau\frac{x^\alpha}{\alpha}\sum_0^\infty C_n\right]\right]\right]du\right]$$

This provides the inverse of CSET with respect to p and s exist for (70)–(72).

4. Illustrative examples

Example 1: The one-dimensional CFCB'sEs in the homogeneous form [33] are as follows:

$$\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta}-\frac{\partial^{2\alpha} h_{\alpha,\beta}}{\partial x^{2\alpha}}-2h\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}+\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right)=0$$
$$\frac{\partial^\beta k_{\alpha,\beta}}{\partial t^\beta}-\frac{\partial^{2\alpha} k_{\alpha,\beta}}{\partial x^{2\alpha}}-2k\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta}+\mu\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right)=0. \quad (73)$$

Subject to

$$h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right)=sin\frac{x^\alpha}{\alpha},\quad k_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right)=sin\frac{x^\alpha}{\alpha}, \quad (74)$$

We can use the CSET for both sides of (73) and conformable single Sumudu transforms for (74):

$$H_{\alpha,\beta}\left(u_{\alpha,\beta}v_{\alpha,\beta}\right)=$$
$$v^2H_\alpha\left(u_{\alpha,\beta},0\right)+vS_x^\alpha E_t^\beta\left[f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)+\frac{\partial^{2\alpha}h_{\alpha,\beta}}{\partial x^{2\alpha}}-\rho h\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}-\tau\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right)\right], \quad (75)$$
$$K_{\alpha,\beta}\left(u_{\alpha,\beta}v_{\alpha,\beta}\right)=$$
$$v^2K_\alpha\left(u_{\alpha,\beta},0\right)+vS_x^\alpha E_t^\beta\left[g\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)+\frac{\partial^{2\alpha}k_{\alpha,\beta}}{\partial x^{2\alpha}}-\rho h\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta}-\mu\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right)\right]$$

According to ((44)-(45)), the first three terms are derived as follows:

$h_{\alpha,\beta,0} = sin\frac{x^\alpha}{\alpha}, k_{\alpha,\beta,0} = sin\frac{x^\alpha}{\alpha},$

$$h_{\alpha,\beta,1} = S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{\partial^{2\alpha}h_{\alpha,\beta,0}}{\partial x^{2\alpha}} + 2h_{\alpha,\beta,0}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta,0} - \frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta,0}k_{\alpha,\beta,0}\right)\right]\right]$$

$$= S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[-sin\frac{x^\alpha}{\alpha}\right]\right] = S_x^\alpha E_x^\beta\ ^{-1}\left[-\frac{uv^3}{(1+u^2)}\right] = -\frac{t^\beta}{\beta}sin\frac{x^\alpha}{\alpha}$$

$$k_{\alpha,\beta,1} = S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{\partial^{2\alpha}k_{\alpha,\beta,0}}{\partial x^{2\alpha}} + 2k_{\alpha,\beta,0}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta,0} - \frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta,0}k_{\alpha,\beta,0}\right)\right]\right],$$

$$= S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[-sin\frac{x^\alpha}{\alpha}\right]\right] = S_x^\alpha E_x^\beta\ ^{-1}\left[-\frac{uv^3}{(1+u^2)}\right] = -\frac{t^\beta}{\beta}sin\frac{x^\alpha}{\alpha},$$

$$h_{\alpha,\beta,2} =$$

$$S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{\partial^{2\alpha}h_{\alpha,\beta,1}}{\partial x^{2\alpha}} + 2(h_{\alpha,\beta,0}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta,1} + h_{\alpha,\beta,1}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta,0})\right]\right]$$

$$-S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[-\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta,0}k_{\alpha,\beta,1} + h_{\alpha,\beta,1}k_{\alpha,\beta,0}\right)\right]\right]$$

$$h_{\alpha,\beta,2} = S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{t^\beta}{\beta}sin\frac{x^\alpha}{\alpha}\right]\right] = S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{uv^4}{(1+u^2)}\right] = \frac{(\frac{t^\beta}{\beta})^2}{2}sin\frac{x^\alpha}{\alpha},$$

$$k_{\alpha,\beta,2} =$$

$$S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{\partial^{2\alpha}k_{\alpha,\beta,1}}{\partial x^{2\alpha}} + 2(k_{\alpha,\beta,0}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta,1} + k_{\alpha,\beta,1}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta,0})\right]\right]$$

$$-S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta,0}k_{\alpha,\beta,1} + h_{\alpha,\beta,1}k_{\alpha,\beta,0}\right)\right]\right]$$

$$k_{\alpha,\beta,2} = S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{uv^4}{(1+u^2)}\right] = \frac{(\frac{t^\beta}{\beta})^2}{2}sin\frac{x^\alpha}{\alpha}$$

$$h_{\alpha,\beta,3} =$$

$$S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{\partial^{2\alpha}h_{\alpha,\beta,2}}{\partial x^{2\alpha}} + 2(h_{\alpha,\beta,0}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta,2} + h_{\alpha,\beta,1}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta,1} + h_{\alpha,\beta,2}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta,0})\right]\right]$$

$$-S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta,0}k_{\alpha,\beta,2} + h_{\alpha,\beta,1}k_{\alpha,\beta,1} + h_{\alpha,\beta,2}k_{\alpha,\beta,0}\right)\right]\right] \quad ,(76)$$

$$k_{\alpha,\beta,3} =$$

$$S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{\partial^{2\alpha}k_{\alpha,\beta,2}}{\partial x^{2\alpha}} + 2(k_{\alpha,\beta,0}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta,2} + k_{\alpha,\beta,1}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta,1} + k_{\alpha,\beta,2}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta,0})\right]\right]$$

$$S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta,0}k_{\alpha,\beta,2} + h_{\alpha,\beta,1}k_{\alpha,\beta,1} + h_{\alpha,\beta,2}k_{\alpha,\beta,0}\right)\right]\right]$$

$$h_{\alpha,\beta,3} = S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{(\frac{t^\beta}{\beta})^2}{2}sin\frac{x^\alpha}{\alpha}\right]\right] = S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{uv^5}{(1+u^2)}\right] = -\frac{(\frac{t^\beta}{\beta})^3}{6}sin\frac{x^\alpha}{\alpha}, \quad (77)$$

$$k_{\alpha,\beta,3} = S_x^\alpha E_x^\beta\ ^{-1}\left[vS_x^\alpha E_t^\beta\left[\frac{(\frac{t^\beta}{\beta})^2}{2}sin\frac{x^\alpha}{\alpha}\right]\right] = S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{uv^5}{(1+u^2)}\right] = -\frac{(\frac{t^\beta}{\beta})^3}{6}sin\frac{x^\alpha}{\alpha}$$

The remaining components are likewise by using (40) and (41), the series solutions are deduced as follows:

$$h_{\alpha,\beta}\left(\frac{t^\beta}{\beta},\frac{x^\alpha}{\alpha}\right)=h_{\alpha,\beta,0}+h_{\alpha,\beta,1}+h_{\alpha,\beta,2}+\cdots$$
$$=\left(1-\left(\frac{t^\beta}{\beta}\right)+\left(\frac{\left(\frac{t^\beta}{\beta}\right)^2}{2}\right)-\left(\frac{\left(\frac{t^\beta}{\beta}\right)^3}{6}\right)+\cdots\right)sin\frac{x^\alpha}{\alpha},$$
$$k_{\alpha,\beta}\left(\frac{t^\beta}{\beta},\frac{x^\alpha}{\alpha}\right)=k_{\alpha,\beta,0}+k_{\alpha,\beta,1}+k_{\alpha,\beta,2}+\cdots$$
$$=\left(1-\left(\frac{t^\beta}{\beta}\right)+\left(\frac{\left(\frac{t^\beta}{\beta}\right)^2}{2}\right)-\left(\frac{\left(\frac{t^\beta}{\beta}\right)^3}{6}\right)+\cdots\right)sin\frac{x^\alpha}{\alpha} \tag{78}$$

Therefore, the exact solution takes the following form

$$h_{\alpha,\beta}\left(\frac{t^\beta}{\beta},\frac{x^\alpha}{\alpha}\right)=e^{-\frac{t^\beta}{\beta}}sin\frac{x^\alpha}{\alpha},$$
$$k_{\alpha,\beta}\left(\frac{t^\beta}{\beta},\frac{x^\alpha}{\alpha}\right)=e^{-\frac{t^\beta}{\beta}}sin\frac{x^\alpha}{\alpha} \tag{79}$$

by taking $\alpha=1$ and $\beta=1$, the fractional solution becomes:

$$h(t,x)=e^{-t}sinx$$
$$\mathrm{k}(t,x)=e^{-t}sinx, \tag{80}$$

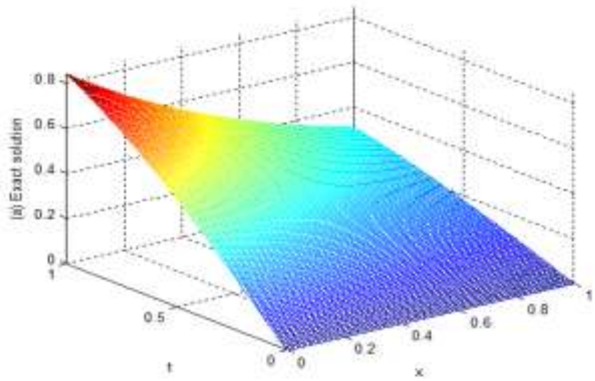
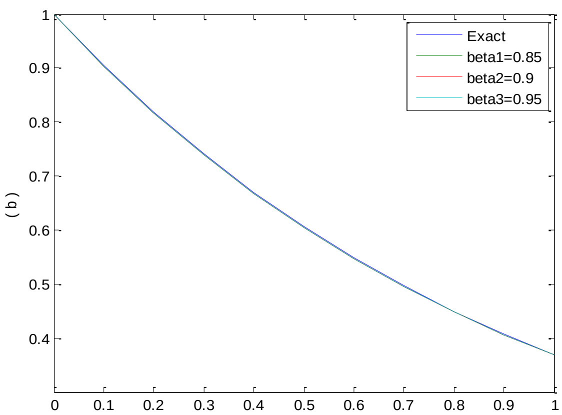
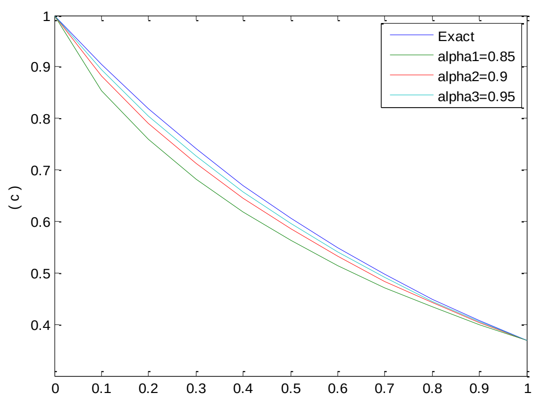


Figure 1: The exact solutions $h_{\alpha,\beta}\left(\frac{t^\beta}{\beta},\frac{x^\alpha}{\alpha}\right)=k_{\alpha,\beta}\left(\frac{t^\beta}{\beta},\frac{x^\alpha}{\alpha}\right)$ for (50) when $\alpha=\beta=1$.





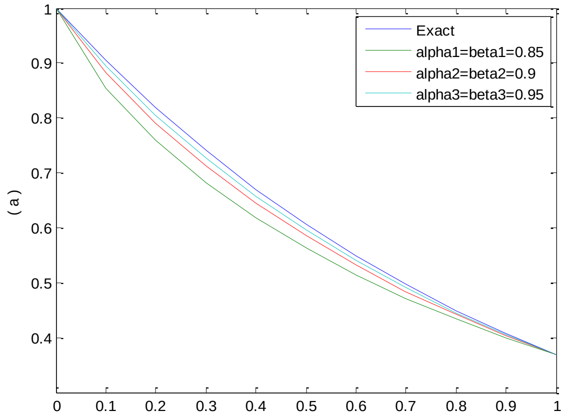


Figure 2: The answers $h(t,x), k(t,x)$ for (73) for various values of α and β when $t = 1$ are shown as: (a) Display solutions $h(t,x), k(t,x)$ for (88) at $\alpha = \beta = 0.85, 0.9, 0.95$. (b) Display the solutions $h(t,x), k(t,x)$ for (73) when $\alpha = 0.99$ and different values of β. (c) Display the solutions $h(t,x), k(t,x)$ for (73) for various values of α at $\beta = 0.99$.

*Example 2[32]***:** Now consider the singular one dimensional CFCB'sE with the Bessel operator:

$$\begin{aligned}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta} - \frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) - 2h\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta} + \frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4e^{\frac{t^\beta}{\beta}} \\ \frac{\partial^\beta k_{\alpha,\beta}}{\partial t^\beta} - \frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta}\right) - 2k\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta} + \frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\ \alpha,\beta}\right) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4e^{\frac{t^\beta}{\beta}}\end{aligned} \tag{81}$$

Subject to

$$h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right) = \left(\frac{x^\alpha}{\alpha}\right)^2, \quad k_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right) = \left(\frac{x^\alpha}{\alpha}\right)^2, \tag{82}$$

when we multiply (81) by the term $\frac{x^\alpha}{\alpha}$, we get the following result:

$$\begin{aligned}\frac{x^\alpha}{\alpha}\frac{\partial^\beta h_{\alpha,\beta}}{\partial t^\beta} - \frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) - 2\frac{x^\alpha}{\alpha}h\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta} + \frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\alpha,\beta}\right) \\ = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4e^{\frac{t^\beta}{\beta}} \\ \frac{x^\alpha}{\alpha}\frac{\partial^\beta k_{\alpha,\beta}}{\partial t^\beta} - \frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta}\right) - 2\frac{x^\alpha}{\alpha}k\frac{\partial^\alpha}{\partial x^\alpha}k_{\alpha,\beta} + \frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(h_{\alpha,\beta}k_{\ \alpha,\beta}\right), \\ = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4e^{\frac{t^\beta}{\beta}}\end{aligned} \tag{83}$$

By using the steps of the third problem , we get:

$$\sum_{n=0}^{\infty} h_{\alpha,\beta,n}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4e^{\frac{t^\beta}{\beta}} + 4$$
$$+S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v}{u} S_x^\alpha E_t^\beta \left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^{\infty} h_{\alpha,\beta,n}\right)\right)\right]\right) du\right]$$
$$+S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v}{u} S_x^\alpha E_t^\beta \left[2\frac{x^\alpha}{\alpha}\sum_{n=0}^{\infty} \mathrm{A}_n\right]\right) du\right]$$
$$-S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v}{u} S_x^\alpha E_t^\beta \left[\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\left(\sum_{n=0}^{\infty} \mathrm{C}_n\right)\right)\right]\right) du\right], \quad (84)$$
$$\sum_{n=0}^{\infty} k_{\alpha,\beta,n}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4e^{\frac{t^\beta}{\beta}} + 4$$
$$+S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v}{u} S_x^\alpha E_t^\beta \left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^{\infty} k_{\alpha,\beta,n}\right)\right)\right]\right) du\right]$$
$$+S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v}{u} S_x^\alpha E_t^\beta \left[2\frac{x^\alpha}{\alpha}\sum_{n=0}^{\infty} \mathrm{B}_n\right]\right) du\right]$$
$$-S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v}{u} S_x^\alpha E_t^\beta \left[\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^{\infty} \mathrm{C}_n\right)\right]\right) du\right]$$

where $A_{\alpha,\beta,n}$, $B_{\alpha,\beta,n}$, and $C_{\alpha,\beta,n}$ are given by (69).
In view of the recursive relation given in ((71)-(72)), we obtain other components as follows:

$$h_{\alpha,\beta,0}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4e^{\frac{t^\beta}{\beta}} + 4, \quad (85)$$
$$k_{\alpha,\beta,0}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4e^{\frac{t^\beta}{\beta}} + 4$$

Now, we can obtain the following general form:

$$h_{\alpha,\beta,1}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) =$$
$$-S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v}{u} S_x^\alpha E_t^\beta \left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha H_0}{\partial x^\alpha}\right) + 2\frac{x^\alpha}{\alpha}H_0\frac{\partial^\alpha H_0}{\partial x^\alpha} - \frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}(H_0K_0)\right]\right) du\right], \quad (86)$$
$$h_{\alpha,\beta,1}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) =$$
$$-S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v}{u} S_x^\alpha E_t^\beta \left[4\frac{x^\alpha}{\alpha}e^{\frac{t^\beta}{\beta}}\right]\right) du\right] = -S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{4v^3}{1-v}\right] = 4e^{\frac{t^\beta}{\beta}} - 4$$
$$k_{\alpha,\beta,1}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) =$$
$$-S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v}{u} S_x^\alpha E_t^\beta \left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha K_0}{\partial x^\alpha}\right) + 2\frac{x^\alpha}{\alpha}K_0\frac{\partial^\alpha K_0}{\partial x^\alpha} - \frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}(H_0K_0)\right]\right) du\right], \quad (87)$$
$$k_{\alpha,\beta,1}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) =$$
$$-S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v}{u} S_x^\alpha E_t^\beta \left[4\frac{x^\alpha}{\alpha}e^{\frac{t^\beta}{\beta}}\right]\right) du\right] = -S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{4v^3}{1-v}\right] = 4e^{\frac{t^\beta}{\beta}} - 4,$$

By the same way, we get:

$$h_{\alpha,\beta,2}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) = -S_x^\alpha E_x^\beta\ ^{-1}\left[\frac{1}{u}\int_0^u \left(\frac{v}{u} S_x^\alpha E_t^\beta \left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha H_0}{\partial x^\alpha}\right) + 2\frac{x^\alpha}{\alpha}\left(H_0\frac{\partial^\alpha H_1}{\partial x^\alpha} + H_1\frac{\partial^\alpha H_0}{\partial x^\alpha}\right) - \frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}(H_0K_1 + H_1K_0)\right]\right) du\right] = 0,$$
$$k_{\alpha,\beta,2}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) = 0,$$

So the solution to the singular one-dimensional CFCB'sE in the series form is given by:

$$h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\sum_{n=0}^{\infty}h_{\alpha,\beta,n}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=$$
$$h_{\alpha,\beta,0}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)+h_{\alpha,\beta,1}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)+h_{\alpha,\beta,2}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)+\cdots,$$
$$k_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\sum_{n=0}^{\infty}h_{\alpha,\beta,n}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=$$
$$h_{\alpha,\beta,0}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)+h_{\alpha,\beta,1}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)+h_{\alpha,\beta,2}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)+\cdots$$

Therefore, the exact solution is obtained as follows:
$h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}}$ and $k_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}}$,
the exact solution to (81) for $\alpha=\beta=1$ is
$h(x,t)=x^2e^t$, $k(x,t)=x^2e^t$.

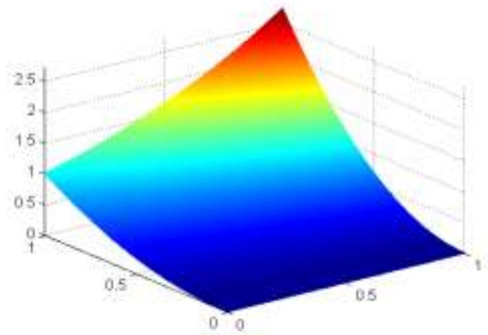


Figure 3: The exact solutions $h_{\alpha,\beta}\left(\frac{t^\beta}{\beta},\frac{x^\alpha}{\alpha}\right)=k_{\alpha,\beta}\left(\frac{t^\beta}{\beta},\frac{x^\alpha}{\alpha}\right)$ for (81) when $\alpha=\beta=1$.

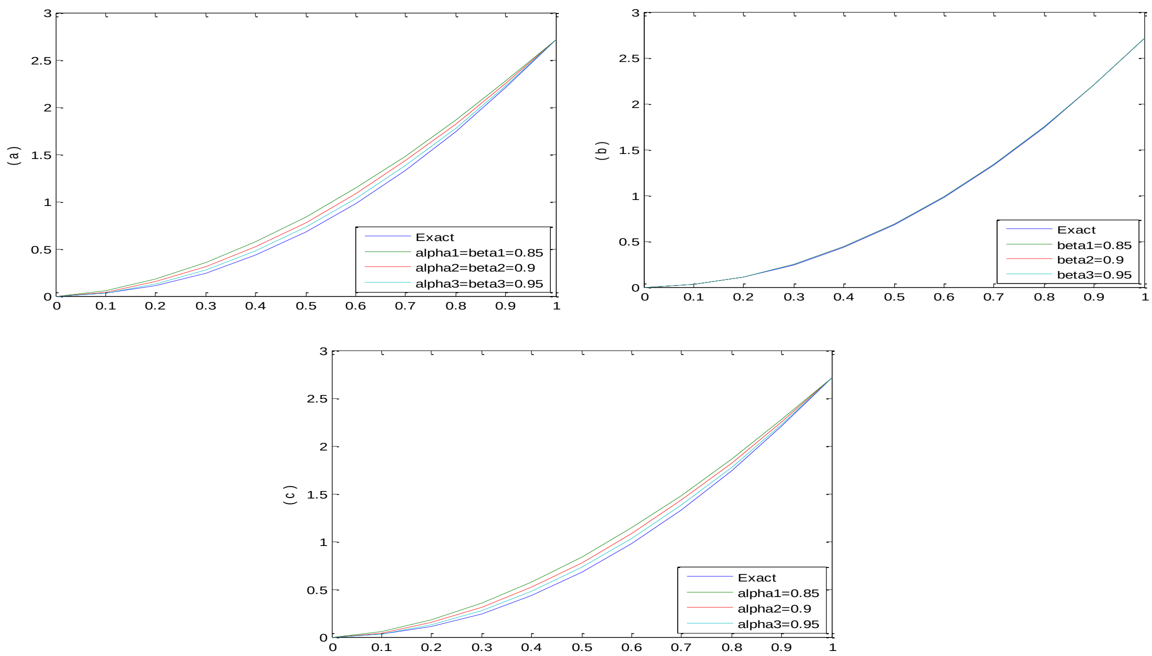


Figure 4: The answers $h(x,t),k(x,t)$ for (81) for various values of α and β when $t=1$ are shown as follows: (a) Display solutions $h(x,t),k(x,t)$ for (81) at $\alpha=\beta=0.85,0.9,0.95$. (b) Display solutions $h(x,t),k(x,t)$ for (81) when $\alpha=0.99$ and different values of β. (c) Display solutions $h(x,t),k(x,t)$ for (81) for various values of α at $\beta=0.99$.

Example 3[35]: Consider the NSCPE in one dimensional is governed by

$$\frac{\partial^{2\beta} h_{\alpha,\beta}}{\partial t^{2\beta}} - \frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) - \frac{\alpha}{x^\alpha}\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) - \frac{1}{2}\left(\frac{x^\alpha}{\alpha}\right)h_{\alpha,\beta}\frac{\partial^\alpha h_{\alpha,\beta}}{\partial x^\alpha} + \left(h_{\alpha,\beta}\right)^2 = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{-\frac{t^\beta}{\beta}}, \quad (88)$$

Subject to

$$h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right) = g_1\left(\frac{x^\alpha}{\alpha}\right) = \left(\frac{x^\alpha}{\alpha}\right)^2, \quad \frac{\partial^{,\beta} h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},0\right)}{\partial t^{,\beta}} = g_2\left(\frac{x^\alpha}{\alpha}\right) = -\left(\frac{x^\alpha}{\alpha}\right)^2, \quad (89)$$

By using the second problem steps, we get:

$$H_{\alpha,\beta}(u,v) = v^2 G_1(u,0) + v^3 G_2(u,0) + \frac{1}{u}\int_0^u \left(uv^2\frac{\partial F_{\alpha,\beta}(\mathrm{u,v})}{\partial u} + v^2 F_{\alpha,\beta}(\mathrm{u,v})\right)du + \frac{1}{u}\int_0^u \frac{v^2}{u} S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + a\left(\frac{x^\alpha}{\alpha}\right)\left(\frac{x^\alpha}{\alpha}\right)h_{\alpha,\beta}\frac{\partial^\alpha h_{\alpha,\beta}}{\partial x^\alpha} - \frac{x^\alpha}{\alpha}\left(h_{\alpha,\beta}\right)^2\right]du,$$

$$H_{\alpha,\beta}(u,v) = 2u^2v^2 - 2u^2v^3 + \frac{2u^2v^4}{1+v} + \frac{1}{u}\int_0^u \frac{v^2}{u} S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + a\left(\frac{x^\alpha}{\alpha}\right)\left(\frac{x^\alpha}{\alpha}\right)h_{\alpha,\beta}\frac{\partial^\alpha h_{\alpha,\beta}}{\partial x^\alpha} - \frac{x^\alpha}{\alpha}\left(h_{\alpha,\beta}\right)^2\right]du,$$

$$H_{\alpha,\beta}(u,v) = \frac{2u^2v^2}{1+v} + \frac{1}{u}\int_0^u \frac{v^2}{u} S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + a\left(\frac{x^\alpha}{\alpha}\right)\left(\frac{x^\alpha}{\alpha}\right)h_{\alpha,\beta}\frac{\partial^\alpha h_{\alpha,\beta}}{\partial x^\alpha} - \frac{x^\alpha}{\alpha}\left(h_{\alpha,\beta}\right)^2\right]du,$$

$$h_{\alpha,\beta}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{-\frac{t^\beta}{\beta}} + S_x^\alpha E_x^{\beta\ \ -1}\left[\frac{1}{u}\int_0^u \frac{v^2}{u} S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}h_{\alpha,\beta}\right) + a\left(\frac{x^\alpha}{\alpha}\right)\left(\frac{x^\alpha}{\alpha}\right)h_{\alpha,\beta}\frac{\partial^\alpha h_{\alpha,\beta}}{\partial x^\alpha} - \frac{x^\alpha}{\alpha}\left(h_{\alpha,\beta}\right)^2\right]du\right],$$

(90)

The CSETDM leads to the following scheme:

$$\sum_{n=0}^\infty h_{\alpha,\beta,n}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{-\frac{t^\beta}{\beta}} + S_x^\alpha E_x^{\beta\ \ -1}\left[\frac{1}{u}\int_0^u\left(\frac{v^2}{u}S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^\infty h_{\alpha,\beta,n}\right)\right)\right]\right)du\right] + S_x^\alpha E_x^{\beta\ \ -1}\left[\frac{1}{u}\int_0^u\left(\frac{v^2}{u}S_x^\alpha E_t^\beta\left[\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^\infty h_{\alpha,\beta,n}\right)\right)\right]\right)du\right] + S_x^\alpha E_x^{\beta\ \ -1}\left[\frac{1}{u}\int_0^u\left(\frac{v^2}{u}S_x^\alpha E_t^\beta\left[a\left(\frac{x^\alpha}{\alpha}\right)\left(\frac{x^\alpha}{\alpha}\right)\sum_{n=0}^\infty A_{\alpha,\beta,n}\right]\right)du\right] - S_x^\alpha E_x^{\beta\ \ -1}\left[\frac{1}{u}\int_0^u\left(\frac{v^2}{u}S_x^\alpha E_t^\beta\left[\frac{x^\alpha}{\alpha}\left(\sum_{n=0}^\infty B_{\alpha,\beta,n}\right)^2\right]\right)du\right], \quad (91)$$

Where $k \geq 0$.

$$h_{\alpha,\beta,0}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{-\frac{t^\beta}{\beta}},$$

$$h_{\alpha,\beta,1}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = S_x^\alpha E_x^{\beta\ \ -1}\left[\frac{1}{u}\int_0^u\left(\frac{v^2}{u}S_x^\alpha E_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha h_{\alpha,\beta,0}}{\partial x^\alpha}\right) + \frac{x^\alpha}{\alpha}\frac{\partial^{\alpha+\beta}h_{\alpha,\beta,0}}{\partial x^\alpha \partial t^\beta}\right]\right)du\right] + S_x^\alpha E_x^{\beta\ \ -1}\left[\frac{1}{u}\int_0^u\left(\frac{v^2}{u}S_x^\alpha E_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)h_{\alpha,\beta,0}\frac{\partial^\alpha h_{\alpha,\beta,0}}{\partial x^\alpha} - \frac{x^\alpha}{\alpha}\left(h_{\alpha,\beta,0}\right)^2\right]\right)du\right],$$

$$= S_x^\alpha E_x^{\beta\ \ -1}\left[\frac{1}{u}\int_0^u\left(\frac{v^2}{u}S_x^\alpha E_t^\beta\left[4\left(\frac{x^\alpha}{\alpha}\right)e^{-\frac{t^\beta}{\beta}} - 4\left(\frac{x^\alpha}{\alpha}\right)e^{-\frac{t^\beta}{\beta}}\right]\right)du\right] - S_x^\alpha E_x^{\beta\ \ -1}\left[\frac{1}{u}\int_0^u\left(\frac{v^2}{u}S_x^\alpha E_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^5 e^{-\frac{t^\beta}{\beta}} - \left(\frac{x^\alpha}{\alpha}\right)^5 e^{-\frac{t^\beta}{\beta}}\right]\right)du\right] = 0,$$

Continues the same procedure, we have

$$h_{\alpha,\beta,2}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = 0,\ h_{\alpha,\beta,3}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = 0,\ h_{\alpha,\beta,4}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = 0, \ldots$$

So that the solution $h_{\alpha,\beta}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ is given by

$$h_{\alpha,\beta}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) = \sum_{n=0}^{\infty} h_{\alpha,\beta,n}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) = h_{\alpha,\beta,0}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) + h_{\alpha,\beta,1}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) + h_{\alpha,\beta,2}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) + \cdots$$

$$h_{\alpha,\beta}(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{-\frac{t^\beta}{\beta}}. \quad (92)$$

By substituting $\alpha = 1$ and $\beta = 1$ into (92), the solution becomes:

$h(x,t) = x^2 e^{-t}$.

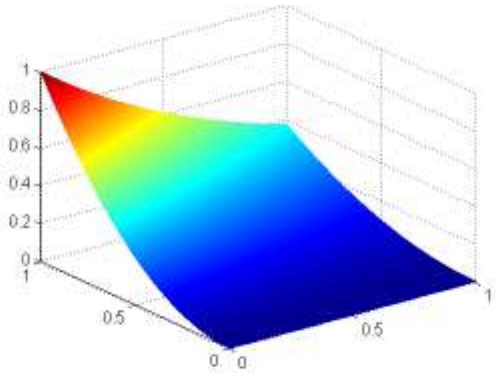


Figure 5: The exact solutions $h(x,t)$ for (88) when $\alpha = \beta = 1$.

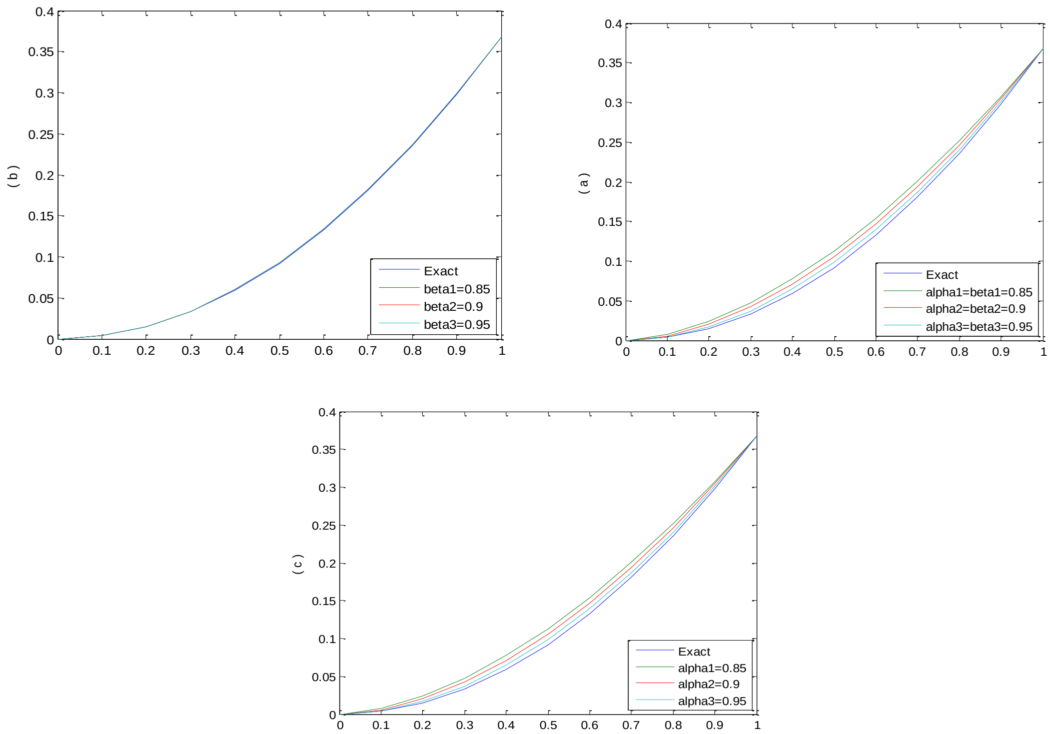


Figure 6: The answers $h(x,t)$ for (88) for various values of α and β when$t = 1$ are shown as follows: (a) Display solutions $h(x,t)$ for (88) at $\alpha = \beta = 0.8, 0.90, 0.95$. (b) Display solutions $h(x,t)$ for (88) when $\alpha = 0.99$ and different values of β. (c) Display solutions $h(x,t)$ for (88) for various values of α at $\beta = 0.99$.

5. Numerical Result

In this section, the numerical results of $\mathrm{h(x,t)}$ and $\mathrm{k(x,t)}$ will be used to demonstrate the accuracy and effectiveness of the CSETDM. For the exact solution, we take $\alpha = \beta$ while for the approximate solutions, different values of α and β are taken in (73), (81) and (88) which are depicted in Figures 1-6, respectively.

In Figure 1, the three-dimensional surface displays the exact solution to the one-dimensional CFCB'sEs (73) in the standard form at the value of $\alpha = \beta = 1$. Figure 2 contrasts the roughly approximate solutions to the (73) for $t = \frac{\pi}{2}$. The numerical solution at $0 < \alpha = \beta \leq 1$ in Figure 2a, in this case $h(x,t), k(x,t)$ converges gradually towards the exact solution as the fractional derivative decreases. In Figure 2b, the solution at $\alpha = 0.99$ and $\beta = 0.95, 0.90, 0.85$ is shown, and the solution $h(x,t), k(x,t)$ becomes close to the exact solution as β decreases. In Figure 2c, the solution $h(x,t), k(x,t)$ converges gradually towards the exact solution at $\alpha = 0.8, 0.90, 0.95$ and $\beta = 0.99$. In the similar way, Figures 3-4 show the exact and approximate answer of the solution to equation (81) for $t = 1$. Figure 3 illustrates the situation $\alpha = \beta = 1$, we obtain the exact solution to the singular one dimensional CFCB'sE with the Bessel operator. Figure 4 illustrates an approximation of the answer to the equation (81) for various values of α and β. Figure 4a shows the behavior of (81) at $0 < \alpha = \beta \leq 1$, where the solution $h(x,t), k(x,t)$ converges gradually towards the exact solution. Figure 4b shows the solution for the case where $\alpha = 0.99$ and various values of $\beta = 0.8, 0.9, 0.95$ where the solution $h(x,t), k(x,t)$ become close to the exact solution. Figure 4c shows the behavior of (81) at $\beta = 0.99$ and various values of α, where the solution $h(x,t), k(x,t)$ converges gradually towards an exact solution. Figures 5-6 show the exact and approximate answer of the solution to equation (88) for $t = 1$. Figure 5 illustrates the situation $\alpha = \beta = 1$, we obtain the exact solution to NSCPE in one dimensional. Figure 6 illustrates an approximation of the answer to the (88) for various values of α and β. Figure 6a shows the behavior of (88) at $0 < \alpha = \beta \leq 1$, where the solution $h(x,t)$ converges gradually towards the exact solution. Figure 6b shows the solution for the case where $\alpha = 0.99$ and various values of $\beta = 0.8, 0.9, 0.95$, where the solution $h(x,t)$ becomes close to the exact solution. Figure 6c shows the behavior of (88) at $\beta = 0.99$ and various values of α, where the solution $h(x,t)$ converges gradually towards the exact solution. The answers of (73), (81) and (88) demonstrate that the CESTDM and the exact answers to the issues have a good level of agreement.

6. Conclusion

In the present work, we have studied the regular and singular one dimensional CFCB'sE and the NSCPE by employing the conformable Sumudu and Elzaki transform decomposition method CSETDM. In addition, we obtain the analytic solutions when $\alpha = \beta = 1$. The numerical solutions for different fractional values are obtained as infinite series by using the CSETDM. They are in good agreement with the exact solutions to the problems. We have provided three different examples to demonstrate the efficiency, high accuracy, and simplicity of the present method. Furthermore, we plot the exact solutions, as well as the numerical solutions in Figures 1–6. We can easily see the efficiency of an agreement among the solutions.

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