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Some New Certain Properties of Modified Mittag-Leffler Function

Hussein Sami Shihab
Al Karkh University of Science, Baghdad, Iraq

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Abstract

A brand-new generalized derivative operator $\mathbb{H}_{\vartheta,\delta}^{\tau} f(z)$ is presented in this study. This operator was created by convolution or (Hadamard product) the generalized Mittag-Leffler function's linear operator with the homologue of the Ruscheweyh operator defined by the widely studied ${}_p\Psi_q$ function.

$$\mathbb{H}_{\vartheta,\delta}^{\tau} f(z) = \mathfrak{I}_{\vartheta,\delta} f(z) * \mathfrak{R}^{\tau} f(z) = z + \sum_{m=2}^{\infty} \frac{\Gamma(\delta)(m+\vartheta-1)!}{\Gamma[\vartheta(m-1)+\delta]\, \tau!\,(m-1)!} a_m z^m,$$

where $\tau \in \mathbb{N}_0$ and $min\{Re(\vartheta), Re(\delta)\} > 0$. Through utilization of $\mathbb{H}_{\vartheta,\delta}^{\tau} f(z)$, a class of analytical functions is introduced through their use. It is possible to obtain the nonlinear $|a_2 a_4 - a_3^2|$, (commonly known as the Second Hankel functional). There are also clear parallels between the findings provided here and those in earlier research.

Keywords: Modified Mittag-Leffler function, Analogue of Ruscheweyh operator, Hankel determinant, Hadamard product.

بعض الخصائص الجديدة لدالة Mittag –Leffler المعدلة

حسين سامي شهاب
جامعة الكرخ للعلوم، بغداد، العراق

الخلاصة

تم تقديم عامل مشتق معمم جديد تمامًا $\mathbb{H}_{\vartheta,\delta}^{\tau} f(z)$ في هذه الدراسة. تم إنشاء هذا العامل عن طريق التلافيف او (ضرب هادامارد) العامل الخطي المعمم لدالة Mittag–Leffler مع تماثل عامل Ruscheweyhالمحدد بواسطة وظيفة ${}_p\Psi_q$ التي تمت دراستها على نطاق واسع.

$$\mathbb{H}_{\vartheta,\delta}^{\tau} f(z) = \mathfrak{I}_{\vartheta,\delta} f(z) * \mathfrak{R}^{\tau} f(z) = z + \sum_{m=2}^{\infty} \frac{\Gamma(\delta)(m+\vartheta-1)!}{\Gamma[\vartheta(m-1)+\delta]\, \tau!\,(m-1)!} a_m z^m$$

حيث $\tau \in \mathbb{N}_0$ و $min\{Re(\vartheta), Re(\delta)\} > 0$. من خلال استخدام $\mathbb{H}_{\vartheta,\delta}^{\tau} f(z)$ ، يتم تقديم فئة من الوظائف التحليلية من خلال استخدامها. من الممكن الحصول على $|a_2 a_4 - a_3^2|$ (المعروف باسم دالة هانكل الثانية). هناك أيضًا أوجه تشابه واضحة بين النتائج المقدمة هنا وتلك الموجودة في الأبحاث السابقة.

Email: hussein.s.shihab2021@kus.edu.iq

1. Introduction

If the disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ in the open unit. Let $\mathcal{A}$ be a symbol for the type of family of analytical functions.

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m. \tag{1}$$

If an equation f in a domain $\mathbb{U}$ is one-to-one, it is said to be univalent in that domain. Let $\mathcal{S}$ stand for the $\mathcal{A}$ subclass that contains only functions that have univalent values in $\mathbb{U}$.

Geometric function theory (GFT) places a high priority on studies of convolution. The Hadamard product or (convolution) has been used in order to and study a number of new and intriguing subclasses of analytical and univalent functions within the context the subordination and suberordination partial sums, the integral mean, and inequalities, the Hankel determinant, and other well-known ideas.

For the functions $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ and $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$ provided in Equation (1), the definition of the Hadamard product or (convolution) of f and g is

$$(f * g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m, \;\; z \in \mathbb{U}. \tag{2}$$

In the current work, we present a brand-newly developed generalized derivative operator $\mathbb{H}_{\vartheta,\delta}^{\tau} f(z)$, which is produced by convolution of the generalized Mittag-Leffler function's linear operator in terms of the well-known Fox-Wright $_p\Psi_q$ function and analogue of the Ruscheweyh operator defined as follows.

Definition 1.1. [1] For $\mathfrak{J}_{\vartheta,\delta} f(z): \mathcal{A} \to \mathcal{A}$, expresses the Fox-Wright $_p\Psi_q$ function in terms with respect to the generalized Mittag-Leffler function's linear operator.

$$\mathfrak{J}_{\vartheta,\delta} f(z) = z + \sum_{m=2}^{\infty} \frac{\Gamma(\delta)}{\Gamma[\vartheta(m-1)+\delta]} a_m z^m,$$

where $min\{Re(\vartheta), Re(\delta)\} > 0; z \in \mathbb{U}$.

Definition 1.2. [2] Let $f \in \mathcal{A}$. The q-analogue of the Ruscheweyh operator as indicated by

$$\mathfrak{R}_q^{\tau} f(z) = z + \sum_{m=2}^{\infty} \frac{[m+\tau-1]_q!}{[\tau]_q! \; [m-1]_q!} a_m z^m,$$

where $[m]_q!$ given by

$$[m]_q! = \begin{cases} [m]_q \, [m-1]_q \dots [1]_q & m = 1, 2, 3, \dots; \\ & m = 0. \end{cases}$$

We can see from the definition that if $q \to 1$, we have

$$\lim_{q \to 1} \mathfrak{R}_q^{\tau} f(z) = z + \lim_{q \to 1} \left[\sum_{m=2}^{\infty} \frac{[m+\tau-1]_q!}{[\tau]_q! \; [m-1]_q!} a_m z^m \right],$$

$$\mathfrak{R}^{\tau} f(z) = z + \sum_{m=2}^{\infty} \frac{(m+\tau-1)!}{\tau! \; (m-1)!} a_m z^m.$$

Where $\mathfrak{R}_q^{\tau}$ is Ruscheweyh differential operator defined in [2].

By way of the convolution or Hadamard product provided by Equation (2), this is the case. We specified the next operators for convolution.

Definition 1.3. The Mittag-Leffler function in its generic form for $f \in \mathcal{A}$, expressed by way of the well-studied Fox-Wright ${}_p\Psi_q$ function $\mathfrak{J}_{\vartheta,\delta} f(z)$ and the analogue of Ruscheweyh operator $\mathfrak{R}^{\tau} f(z)$ is defined by $\mathbb{H}^{\tau}_{\vartheta,\delta} f(z): \mathcal{A} \to \mathcal{A}$,

$$\mathbb{H}^{\tau}_{\vartheta,\delta} f(z) = \mathfrak{J}_{\vartheta,\delta} f(z) * \mathfrak{R}^{\tau} f(z) = z + \sum_{m=2}^{\infty} \frac{\Gamma(\delta)\,(m+\tau-1)!}{\Gamma[\vartheta(m-1)+\delta]\,\tau!\,(m-1)!} a_m z^m,$$

where $\tau \in \mathbb{N}_0$, and $min\{Re(\vartheta), Re(\delta)\} > 0$; $z \in \mathbb{U}$.

In [3], if $q \geq 1$ and $m \geq 1$ then q^{th} Hankel determinant according to Noonan and Thomas is defined as

$$\mathcal{H}_q(m) = \begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+q+1} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+q+2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m+q-1} & a_{m+q} & \cdots & a_{m+2q-2} \end{vmatrix}.$$

In Cantor [4], the Hankel determinant is used to show the rationality of a function with bounded characteristic in $\mathbb{U}$, that is, a function that is the ratio of two bounded analytic functions with integral coefficients in their Laurent series around the origin. Wilson [5] investigated how the Hankel determinant is used in meromorphic functions. Noor in [6] has a determinant for its rate of growth with a constrained boundary, and Hankel's determinant for the Bazilevic function was also investigated in [7]. Ehrenborg [8] examined the exponential polynomials' Hankel determinant, and Layman [9] covered some of their characteristics.

It is simple to see that for $q = 2$ and $m = 1$, we will get a Fekete and Szegö classical theorem [10].

$$\mathcal{H}_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}.$$

They conducted an early investigation to estimate $|a_3 - \mu a_2^2|$ when $a_1 = 1$ and μ real. If $f \in \mathcal{S}$, then according to the well-known finding owing to then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3, & if\ \mu \geq 1, \\ 1 + 2e^{\left(\frac{-2\mu}{1-\mu}\right)}, & if\ 0 \leq \mu \leq 1, \\ 3 - 4\mu, & if\ \mu \leq 0. \end{cases}$$

When f in $\mathbb{U}$ is nearly convex and starlike, Keogh and Merkes [11] explored the sharp estimates for $|a_3 - \mu a_2^2|$.

For the purposes of our discussion, we take into account with $q = 2$ and $m = 2$, the Hankel determinant of $f \in \mathcal{S}$ for, as given by this paper.

$$\mathcal{H}_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

The second Hankel determinant is referred to as. Several scholars have looked into the determinant $\mathcal{H}_2(2)$. For instance, Janteng et al. [12] and [13] analyzed the sharp constraint in Equation (1) for the function f, and found that it consists the derivatives of functions whose real portions are positive and the result $|a_2a_4 - a_3^2| \leq {}^4/_9$. The conclusion starlike and convex functions for the sharp upper bounds was determined by the same author [14] as $|a_2a_4 - a_3^2| \leq 1$ and $|a_2a_4 - a_3^2| \leq {}^1/_8$, respectively.

The definition of the subclass $\mathbb{H}_{\vartheta,\delta}^{\tau}f(z)$ is as follows.

Definition 1.4. Let Equation (1) be the source of f. If f fulfills the inequality, it is said to be the class $\mathbb{H}_{\vartheta,\delta}^{\tau}f(z)$

$$Re\{[\mathbb{H}_{\vartheta,\delta}^{\tau}f(z)]'\} > 0, \quad (z \in \mathbb{U}),$$

where $\mathbb{H}_{\vartheta,\delta}^{\tau}f(z)$ is a brand-new general operator produced by Hadamard product of the linear Mittag-Leffler function operator in terms of the extensively researched Fox-Wright $p\Psi q$ function created by Srivastava et al. In [1] and the analogue of the Ruscheweyh operator in [2].

For a certain group of analytical operations, several authors, including Al-Refai and Darus [15], Abubaker and Darus [16], Al-abbadi and Darus [17], and Bansal [18], have explored and investigated the second determinant of Hankel.

In the current study, we created an entirely new generalized derivative operator $\mathbb{H}_{\vartheta,\delta}^{\tau}f(z)$ and investigated the maximum value of functional $|a_2a_4 - a_3^2|$ or the function f belonging to the class $\mathbb{H}_{\vartheta,\delta}^{\tau}f(z)$. This was motivated by the results achieved by several writers in this area as described above.

We start by stating some preliminary lemmas that are necessary to support the results we have obtained.

2. Preliminary Results

Let $\mathcal{P}$ be the collection of all p-analytical functions in $\mathbb{U}$ for which $Re\{p(z)\} > 0$ and

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots \tag{3}$$

for $z \in \mathbb{U}$.

Lemma 2.1. [3] If $p \in \mathcal{P}$, then $|c_k| \leq 2$ for each k.

Lemma 2.2. [3] If and only if the Hankel determinants are true, in Equation (3), the $p(z)$ power series converges in $\mathbb{U}$ to operate in $\mathcal{P}$.

$$\mathbb{D}_m = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_m \\ c_{-1} & 2 & c_1 & \cdots & c_{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-m} & c_{-m+1} & c_{-m+2} & \cdots & 2 \end{vmatrix}, \quad m = 1, 2, 3, \ldots$$

and $c_k = \overline{c_k}$, are all nonnegative. They are only positive, $p(z) = \sum_{k=1}^{l} p_k p_0(e^{it_k}z)$, $p_k > 0$, t_k real and $t_k \neq t_j$, is the only exception, in this instance $\mathbb{D}_m > 0$ for $m < l - 1$ and $\mathbb{D}_m = 0$ for $m \geq l$.

This condition is both necessary and sufficient, and it can be found in [19] thanks to Grenander and Szego.

In light of the approach established by R. Ehrenborg [8], we can freely assume that $c_1 > 0$ and rewrite Lemma 2.2 for the situations $m = 2$, and $m = 3$.

$$\mathbb{D}_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ c_1 & 2 & c_1 \\ c_2 & c_1 & 2 \end{vmatrix} = 8 + 2Re\{c_1^2 c_2\} - 2|c_2| - 4c_1 \geq 0,$$

it is comparable to

$$2c_2 = c_1^2 + (4 - c_1^2)x \tag{4}$$

$\exists\, x$, $|x| \leq 1$. Then $\mathbb{D}_3 \geq 0$, is then comparable to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|2c_2 - c_1^2|^2,$$

and this establishes the relationship with (4).

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{5}$$

$|z| \leq 1$, for some value of z.

3. Main Result

The following is the main outcome.

Theorem 3.1. Assume that the function f in Equation (1) belongs to the class $\mathbb{H}_{\vartheta,\delta}^{\tau} f(z)$. Then

$$|a_2 a_4 - a_3^2| \leq \frac{1}{(\tau+2)(\tau+1)^2[\Gamma(\delta)]^2} \left| \frac{3\Gamma(\vartheta+\delta)\,\Gamma(3\vartheta+\delta)}{4\,(\tau+3)} - \frac{16\,[\Gamma(2\vartheta+\delta)]^2}{9\,(\tau+2)} \right|.$$

The final result is sharp.

Proof. Since $f \in \mathbb{H}_{\vartheta,\delta}^{\tau} f(z)$, Due to Definition (4), the unit disk $\mathbb{U}$ has an analytical function $\wp \in \mathcal{P}$. That has the properties $[Re\, \wp(z)] > 0$ and $\wp(0) = 1$ so that

$$\left[\mathbb{H}_{\vartheta,\delta}^{\tau} f(z)\right]' = \wp(z). \tag{6}$$

We have for some $z \in \mathbb{U}$ by swapping out $\left[\mathbb{H}_{\vartheta,\delta}^{\tau} f(z)\right]'$ and $\wp(z)$ as well as the series expressions for each of them in Equation (6).

$$1 + \sum_{m=2}^{\infty} \frac{m\,\Gamma(\delta)\,(m+\tau-1)!}{\Gamma[\vartheta(m-1)+\delta]\,\tau!\,(m-1)!} a_m z^{m-1} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots.$$

Upon simplification, we have

$$1 + \frac{2\Gamma(\delta)(\tau+1)\tau!}{\Gamma(\vartheta+\delta)\,\tau!\,(1)!} a_2 z + \frac{3\Gamma(\delta)(\tau+2)(\tau+1)\tau!}{\Gamma(2\vartheta+\delta)\,\tau!\,(2)!} a_3 z^2 + \frac{4\Gamma(\delta)(\tau+3)(\tau+2)(\tau+1)\tau!}{\Gamma(3\vartheta+\delta)\,\tau!\,(3)!} a_4 z^3 + \cdots = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

$$1+\frac{2(\tau+1)\Gamma(\delta)}{\Gamma(\vartheta+\delta)}a_2z+\frac{3(\tau+2)(\tau+1)\Gamma(\delta)}{2\Gamma(2\vartheta+\delta)}a_3z^2+\frac{2(\tau+3)(\tau+2)(\tau+1)\Gamma(\delta)}{3\Gamma(3\vartheta+\delta)}a_4z^3+\cdots$$
$$=1+c_1z+c_2z^2+c_3z^3+\cdots. \tag{7}$$

In Equation (7), the coefficients for the respective like powers $z^0, z,$ and z^3 are equated, results in

$$\left.\begin{aligned} a_2 &= \frac{1}{2}\frac{c_1\,\Gamma(\vartheta+\delta)}{(\tau+1)\,\Gamma(\delta)}\\ a_3 &= \frac{2}{3}\frac{c_2\,\Gamma(2\vartheta+\delta)}{(\tau+2)(\tau+1)\,\Gamma(\delta)}\\ a_4 &= \frac{3}{2}\frac{c_3\,\Gamma(3\vartheta+\delta)}{(\tau+3)(\tau+2)(\tau+1)\,\Gamma(\delta)} \end{aligned}\right\}. \tag{8}$$

It is simple to demonstrate that by the second Hankel functional $|a_2a_4-a_3^2|$ should be changed to reflect the values of a_2, a_3, and a_4 from Equation (8)

$$|a_2a_4-a_3^2| = \left|\frac{c_1\,\Gamma(\vartheta+\delta)}{2\,(\tau+1)\,\Gamma(\delta)}.\frac{3c_3\,\Gamma(3\vartheta+\delta)}{2\,(\tau+3)(\tau+2)(\tau+1)\,\Gamma(\delta)} - \frac{4c_2^2[\Gamma(2\vartheta+\delta)]^2}{9\,(\tau+2)^2(\tau+1)^2\,[\Gamma(\delta)]^2}\right|$$

$$= \left|\frac{3c_1c_3\,\Gamma(\vartheta+\delta)\,\Gamma(3\vartheta+\delta)}{4(\tau+3)(\tau+2)(\tau+1)^2[\Gamma(\delta)]^2} - \frac{4c_2^2[\Gamma(2\vartheta+\delta)]^2}{9(\tau+2)^2(\tau+1)^2[\Gamma(\delta)]^2}\right|.$$

Lemma 2.2 is the method we use to get the correct bound on

$$=\frac{1}{(\tau+2)(\tau+1)^2[\Gamma(\delta)]^2}\left|\frac{3c_1c_3\Gamma(\vartheta+\delta)\Gamma(3\vartheta+\delta)}{4(\tau+3)} - \frac{4c_2^2[\Gamma(2\vartheta+\delta)]^2}{9(\tau+2)}\right|. \tag{9}$$

Now, to simplify our calculation, we let $u = 3\Gamma(\vartheta+\delta)\Gamma(3\vartheta+\delta)$, and $v = [\Gamma(2\vartheta+\delta)]^2$. Equation (9) can therefore be written as

$$\frac{1}{(\tau+2)(\tau+1)^2[\Gamma(\delta)]^2}\left|\frac{uc_1c_3}{4(\tau+3)} - \frac{4vc_2^2}{9(\tau+2)}\right|.$$

Lemma 2.2 Equation (5) and the c_2 and c_3 values from Equation (4) are substituted into Equation (9), yielding

$$=\frac{1}{(\tau+2)(\tau+1)^2[\Gamma(\delta)]^2}\left|\frac{u\left[c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)z\right]}{16(\tau+3)} - \frac{v\left[c_1^4+2c_1^2(4-c_1^2)x+(4-c_1^2)^2x^2\right]}{9(\tau+2)}\right|$$

$$=\frac{1}{(\tau+2)(\tau+1)^2[\Gamma(\delta)]^2}\left|\frac{uc\left[c^3+2c(4-c^2)x-c(4-c^2)x^2-2(4-c^2)(1-|x|^2)z\right]}{16(\tau+3)} - \frac{v\left[c^4+2c^2(4-c^2)x+(4-c^2)^2x^2\right]}{9(\tau+2)}\right|$$

$$= \frac{1}{(\tau+2)(\tau+1)^2[\Gamma(\delta)]^2}\left|\frac{u\mathfrak{c}^4-u\mathfrak{c}^2(4-\mathfrak{c}^2)x^2}{16(\tau+3)}+\frac{2u\mathfrak{c}^2(4-\mathfrak{c}^2)x+2u\mathfrak{c}(4-\mathfrak{c}^2)(1-|x|^2)z)}{16(\tau+3)}-\frac{v\mathfrak{c}^4+2v\mathfrak{c}^2(4-\mathfrak{c}^2)x}{9(\tau+2)}-\frac{v(4-\mathfrak{c}^2)^2x^2}{9(\tau+2)}\right|$$

$$= \frac{1}{(\tau+2)(\tau+1)^2[\Gamma(\delta)]^2}\left|\frac{u\mathfrak{c}^2[\mathfrak{c}^2-(4-\mathfrak{c}^2)x^2]}{16(\tau+3)}+\frac{u\mathfrak{c}^2(4-\mathfrak{c}^2)x}{8(\tau+3)}-\frac{v\mathfrak{c}^2[\mathfrak{c}^2+2(4-\mathfrak{c}^2)x]}{9(\tau+2)}-\frac{v(4-\mathfrak{c}^2)^2x^2}{9(\tau+2)}\right|.$$

Utilizing the facts that $|z| < 1$ and the tringle inequity based on a foundation of $\mathfrak{c}_1 = \mathfrak{c}$ and $\mathfrak{c} \in [0,2]$ demonstrate that

$$\frac{1}{(\tau+2)(\tau+1)^2[\Gamma(\delta)]^2}\left|\frac{u\mathfrak{c}_1\mathfrak{c}_3}{4(\tau+3)}-\frac{4v\mathfrak{c}_2^2}{9(\tau+2)}\right|$$

$$\leq \frac{1}{(\tau+2)(\tau+1)^2[\Gamma(\delta)]^2}\left|\frac{u\mathfrak{c}^2[\mathfrak{c}^2-(4-\mathfrak{c}^2)\wp^2]}{16(\tau+3)}+\frac{u\mathfrak{c}^2(4-\mathfrak{c}^2)\wp}{8(\tau+3)}-\frac{v\mathfrak{c}^2[\mathfrak{c}^2+2(4-\mathfrak{c}^2)\wp]}{9(\tau+2)}-\frac{v\wp^2}{9(\tau+2)}\right| = F(\mathfrak{c},\wp) \qquad (10)$$

for $0 \leq \wp = |x| \leq 1$.

The upper bound for Equation (10) is thought to reach the internal point of $\wp \in [0,1]$ and $\mathfrak{c} \in [0,2]$, according to our assumed. The closed square $[0,2] \times [0,1]$ is then optimized for the function $F(\mathfrak{c},\wp)$. In order recognize Equation (10) from $\wp$, we get

$$\frac{\partial F}{\partial \wp} = \frac{-2u\mathfrak{c}^2(4-\mathfrak{c}^2)\wp}{16(\tau+3)}+\frac{u\mathfrak{c}^2(4-\mathfrak{c}^2)}{8(\tau+3)}-\frac{2v\mathfrak{c}^2(4-\mathfrak{c}^2)}{9(\tau+2)}-\frac{2v(4-\mathfrak{c}^2)^2\wp}{9(\tau+2)}$$

$$\frac{\partial F}{\partial \wp} = -\left[\frac{2v\mathfrak{c}^2(4-\mathfrak{c}^2)}{9(\tau+2)}+\frac{2v(4-\mathfrak{c}^2)^2\wp}{9(\tau+2)}\right] = -\left[\frac{2v\mathfrak{c}^2(4-\mathfrak{c}^2)+2v(4-\mathfrak{c}^2)^2\wp}{9(\tau+2)}\right]$$

$$\frac{\partial F}{\partial \wp} = -\left[\frac{2v(4-\mathfrak{c}^2)\{\mathfrak{c}^2+(4-\mathfrak{c}^2)\wp\}}{9(\tau+2)}\right]. \qquad (11)$$

We can see from Equation (11) that $\frac{\partial F}{\partial \wp} > 0$ for $\wp > 0$. Due to the fact that Equation (11) is a growing function of $\wp$, it is impossible for it to reach its maximum inside of the enclosed space $[0,2] \times [0,1]$. Additionally, we have to be fixed $\mathfrak{c} \in [0,2]$

$$\max_{0\leq\wp\leq1} F(\mathfrak{c},\wp) = F(\mathfrak{c},1),$$

and by adding $\wp = 1$ to Equation (11), we have

$$\frac{\partial F}{\partial \wp} = -\left[\frac{2v(4-\mathfrak{c}^2)\{\mathfrak{c}^2+(4-\mathfrak{c}^2)\}}{9(\tau+2)}\right]. \qquad (12)$$

We can see from Equation (12) that $F'(\mathfrak{c},1) \leq 0\ \forall\ \mathfrak{c} \in [0,2]$. In the range $\mathfrak{c} \in [0,2]$, $F(\mathfrak{c},1)$ is hence a decreasing operation of $\mathfrak{c}$. Its maximum values at various locations in F has to be near the border of $\mathfrak{c} \in [0,2]$. But for $\mathfrak{c} = 1$, F reaches its greatest value since $f(\mathfrak{c},1) \geq F(2,1)$.

When $\wp = 1$ and $\mathfrak{c} = 1$, and the upper constraint for Equation (10) is reached, the following occurs:

$$|a_2a_4 - a_3^2| \le \frac{1}{(\tau+2)(\tau+1)^2[\Gamma(\delta)]^2}\left|\frac{u(1)^2[(1)^2-(4-(1)^2)(1)^2]}{16(\tau+3)} + \frac{u(4-(1)^2)(1)}{8(\tau+3)} - \frac{v(1)^2[(1)^2+2(4-(1)^2)(1)]}{9(\tau+2)} - \frac{v(1)^2}{9(\tau+2)}\right|$$
$$\le \frac{1}{(\tau+2)(\tau+1)^2[\Gamma(\delta)]^2}\left|\frac{u}{4(\tau+3)} - \frac{16v}{9(\tau+2)}\right|. \quad (13)$$

By substituting $u = 3\Gamma(\vartheta+\delta)\Gamma(3\vartheta+\delta)$ and $v = [\Gamma(2\vartheta+\delta)]^2$, in Equation (13). We have the upper bound

$$|a_2a_4 - a_3^2| \le \frac{1}{(\tau+2)(\tau+1)^2[\Gamma(\delta)]^2}\left|\frac{3\Gamma(\vartheta+\delta)\Gamma(3\vartheta+\delta)}{4(\tau+3)} - \frac{16[\Gamma(2\vartheta+\delta)]^2}{9(\tau+2)}\right| \quad (14)$$

where $\{Re(\vartheta), Re(\delta)\} > 0; z \in \mathbb{U}$.

In Equation (4) and Equation (5), we discover that $\mathfrak{c}_2 = 2$ and $\mathfrak{c} = 0$ by setting $\mathfrak{c}_1 = 1$ and selecting $x = 1$. When these values are substituted in Equation (14), equality is achieved, demonstrating the accuracy of our solution is sharp.

The proof of our theorem comes to an end here. □

The functions in $\mathbb{H}_{\vartheta,\delta}^{\tau} f(z)$ stated by hold equality

$$f'(z) = \frac{1+z^2}{1-z^2}.$$

The following corollary demonstrates how we would achieve the result that agrees with Janteng et al. [10] for the selection $\vartheta = 1/2$, $\delta = 1$, and $\tau = 0$ into Theorem 3.1.

Corollary 3.3. Let $f \in \mathbb{R}$ in the class $\mathbb{H}_{1/2,1}^{0} f(z)$ given by Equation (1) be. Then

$$|a_2a_4 - a_3^2| \le \frac{2.68}{9}.$$

The end result is sharp.

4. Conclusions

In this research paper, a new function $\mathbb{H}_{\vartheta,\delta}^{\tau} f(z)$ was found resulting from the multiplication of the Mittag-Leffler function $\mathfrak{J}_{\vartheta,\delta} f(z)$ and Analogue of Ruscheweyh operator $\mathfrak{R}_q^{\tau} f(z)$ using convolution or Hadamard product, where the new function is also analytic. After that, we used the Hankel determinant to prove new properties, including that the function resulting from the multiplication of the two functions is sharp. Finally, we found a new result: If the variables of the resulting function $\vartheta = \frac{1}{2}$, $\delta = 1$, and $\tau = 0$, then the function is also sharp.

References

[1] H. M. Srivastava, A. Kilicman, Z. E. Abdulnaby, R. W. Ibrahim, "Generalized convolution properties based on the modified Mittag-Leffler function," *Journal of Nonlinear Sciences and Applications,* vol.10, no.8, pp. 4284–4294. 2017.

[2] S. Ruscheweyh, "New criteria for univalent functions," *Proceedings of the American Mathematical Society,* vol. 49, no. 1, pp. 109–115, 1975.

[3] J. W. Noonan, D. K. Thomas, "On the second Hankel determinant of areally mean and p-valent function," *Transactions of the American Mathematical Society,* vol. 223, pp. 337–346, 1976.

[4] D. G. Cantor, "Power series with integral coefficients," *Bulletin of the American Mathematical Society,* vol. 69, no.3, pp. 362–366, 1963.

[5] R. Wilson, " Determinantal criteria for meromorphic functions," *Proceedings of the London Mathematical Society,* vol. 3, no.1, pp. 357–374, 1954.

[6] K. I. Noor, "Hankel determinant problem for the class of functions with bounded boundary rotation," *Revue Roumaine de Mathématiques Pures et Appliquées,* vol. 28, no. 8, pp. 731–739, 1983.

[7] K. I. Noor, S. A. Al-Bany, "On Bazilevic functions," *International Journal of Mathematics and Mathematical Sciences,* vol.10, no.1, pp. 79–88, 1987.

[8] R. Ehrenborg, "The Hankel determinant of exponential polynomials," *The American Mathematical Monthly,* vol. 107, no. 1, pp. 557–560, 2000.

[9] J. W. Layman, "The Hankel transform and some of its properties," *Journal of Integer Sequences.,* vol. 4, no. 1, pp. 1-11, 2001.

[10] M. Fekete, G. Szego, "Eine Bemerkung uber ungerade schlichte funktionen," *Journal of the London Mathematical Society,* vol. 1, no. 2, pp. 85–89, 1933.

[11] F. R. Keogh, E. P. Merkes, "A coefficient inequality for certain classes of analytic functions," *Proceedings of the American Mathematical Society,* vol. 20, no. 1, pp. 8–12, 1969.

[12] A. Janteng, S. A. Halim, M. Darus, " Coefficient Inequality for a Function whose Derivative has a Positive Real Part," *JIPAM. Journal of Inequalities in Pure & Applied Mathematics,* vol. 7, no. 2, pp. 1-5, 2006.

[13] M. N. M. Pauzi, M. Darus, S. Siregar, "Second Hankel determinant for a class defined by modified Mittag-Leffler with generalized polylogarithm functions," *Journal of Mathematics and Computer Science,* 2018.

[14] A. Janteng, S. A. Halim, M. Darus, "Hankel Determinant for Starlike and Convex Functions," *International Journal of Mathematical Analysis.* (*Ruse*)*,* vol. 1, no. 13, pp. 619–625, 2007.

[15] O. Al-Refai, M. Darus, " Second Hankel determinant for a class of analytic functions Defined by a fractional operator," *European Journal of Scientific Research,* vol. 28, no. 2, pp. 234–241, 2009.

[16] A. Abubaker, M. Darus, " Hankel determinant for a class of analytic functions involving a generalized linear differential operator," *International Journal of Pure & Applied Mathematics,* vol. 69, no. 4, pp. 429–435, 2011.

[17] M. H. Al-Abbadi, M. Darus, "Hankel Determinant for certain class of analytic function defined by generalized derivative operator," *Tamkang Journal of Mathematics,* vol. 43, no. 3, pp. 445–453, 2012.

[18] D. Bansal, "Upper bound of second Hankel determinant for a new class of analytic functions," *Applied Mathematics Letters,* vol. 26, no. 1, pp. 103–107, 2013.

[19] U. Grenander, G. Szego, "Toeplitz Forms and their Application," *University of California Press, Berkeley–Los Angeles,* 1958.