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Fekete Szegö Problem for a Subclass of Analytic Function Involving Quasi Subordination

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Abstract

In this paper, we investigate a particular subclass of the set of analytic functions in a unit disk A. This subclass is identified by quasi-subordination, and for functions in this class, we give precise bounds for the Fekete-Szegö functional (that is, $|c_3 - \delta c_2^2|$).Well-defined borders are produced by the study, and we also discuss various outcomes for new classes while establishing connections with the ones that already exist.

Keywords: Analytic functions, Majorization, Quasi- subordination, Fekete-Szegö inequality.

مشكلة Fekete−Szegö لفئة فرعية من الدالة التحليلية التي تنطوي على شبه التبعية

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الخلاصة

في هذه المقالة، نستكشف فئة فرعية معينة من مجموعة الدوال التحليلية في قرص الوحدة A . يتم تحديد هذه الفئة الفرعية عن طريق شبه التبعية، ونحن نقدم حدودًا دقيقة للدالة (اي Fekete−Szegö $|c_3 - \delta c_2^2|$) للدوال في هذه الفئة. أنتجت الدراسة نطاقًا محددًا بشكل جيد، وتحدثت أيضًا عن نتائج مختلفة لفئات جديدة مع إقامة روابط مع تلك الموجودة بالفعل.

1. Introduction

Let $\mathcal{A}$ represent a collection of normalized functions of the form

$$f(w) = w + \sum_{k=2}^{\infty} c_k w^k, \tag{1.1}$$

which is analytic in $\mathfrak{A} = \{w \in \mathbb{C}: |w| < 1\}$. Let $\mathcal{H}$ be the set of all elements in $\mathcal{A}$ that are univalent in $\mathfrak{A}$. Let $h(w)$ be an analytic function in $\mathfrak{A}$ with $|h(w)| \leq 1. w \in \mathfrak{A}$, so

$$h(w) = \mathfrak{N}_0 + \mathfrak{N}_1 w + \mathfrak{N}_2 w^2 + \cdots, \tag{1.2}$$

where $\mathfrak{N}_0, \mathfrak{N}_1, \mathfrak{N}_2$, ...are umbers. Let $\kappa(w)$ be an analytic function in$\mathfrak{A}$, with $\kappa(0) = 1$, $\kappa'(0) > 0$, and a positive real component, such

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$$\kappa(w) = 1 + \mathcal{F}_1 w + \mathcal{F}_2 w^2 + \cdots, \quad (1.3)$$

where $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots$ are real numbers and $\mathcal{F}_1 > 0$. Unless otherwise specified, we will assume that the functions $\hbar$ satisfy the preceding constraints throughout this study.

As has been determined that for f ∈ Hgiven by (1.1), there exists sharp upper limits for |c_3-δc_2^2 | when δ is real [1]. Consequently, the Fekete- Szegö issue for F has been well-known for estimating the sharp upper limits for |c_3-δc_2^2 | with δ being an arbitrary real or complex integer for any compact collection F of elements in A.Several scholars have computed sharp Fekete-Szegö limits for various $\mathcal{H}$ subclasses, including [2-11] . More information on the Fekete-Szegö issue and the q-derived operator may be found in the publications of Alsoboh and Darus [12] and Elhaddad and Darus [13]. The study by Srivastava et al. [14] has a very good resource on the Fekete-Szegö inequality and the Horadam polynomials.

We review the majorization and subordination principles between the two analytic functions in $\mathfrak{A}$, $f(w)$ and $r(w)$. If an analytic function $t(w)$ exists in $\mathfrak{A}$, with $t(0) = 0$ and $|t(w)| < 1$, $w \in \mathfrak{A}$, such that $f(w) = r(t(w))$, then $f(w)$ is subordinate to $r(w)$, written $f(w) \prec r(w)$, $w \in \mathfrak{A}$. Furthermore, if r is univalent in $\mathfrak{A}$, then $f(w) \prec r(w)$ is equivalent to $f(0) = r(0)$ and $f(\mathfrak{A}) \subset r(\mathfrak{A})$, if r is univalent in $\mathfrak{A}$. We know that $f(w)$ is majorized by $r(w)$, written $f(w) \ll r(w), w \in \mathfrak{A}$, if there exists an analytic function $\hbar(w)$, $w \in \mathfrak{A}$, with $|\hbar(w)| \leq 1$, such that $f(w) = \hbar(w) r(w)$, $w \in \mathfrak{A}$.
Robertson [15] introduced a novel concept that unifies majorization and subordination: quasi-subordination. For any two $f(w)$ and $r(w)$ analytic functions is namely quasi subordinate to $r(w)$, written as $f(w) \prec_q r(w)$, $w \in \mathfrak{A}$, if there is analytic functions$\hbar$ and t with $t(0) = 0$, $|\hbar(w)| \leq 1$ and $|t(w)| < 1$ such that $f(w) = \hbar(w) r(t(w)), w \in \mathfrak{A}$. Note that if $\hbar(w) = 1$, then $f(w) = r(t(w)), w \in \mathfrak{A}$, so $f(w) \prec r(w)$ in $\mathfrak{A}$. Also, note that if $t(w) = w$, then $f(w) = \hbar(w) r(w), w \in \mathfrak{A}$ and hence $f(w) \prec\prec$ r(w) in $\mathfrak{A}$. More research on quasi-subordination may be found in [16-19].

Let Υ be the collection of analytic functions in $\mathfrak{A}$ of the form

$$t(w) = t_1 w + t_2 w^2 + t_3 w^3 + \ldots, \quad (1.4)$$

fulfilling the criterion $|t(w)| < 11$, $w \in \mathfrak{A}$. To establish our primary result, we need the following lemma [20].

Lemma 1.1. If $t \in \Upsilon$, then for any complex number δ, we have $|t_1| \leq 1. |t_2 - \delta t_1^2| \leq 1 + (|\delta| - 1)|t_1|^2 \leq max\{1, |\delta|\}. t(w) = w \; or \; t(w) = |w|^2$ exhibit the sharpness of the result.

In light of current trends in quasi-subordination, we define the next new collection sub classes $\mathcal{A}$.

Definition 1.2. A function $f(w)$ in $\mathcal{A}$ is said to be in the collection $\mathcal{G}(\alpha, \beta, \gamma, \kappa)$. If

$$\left(\frac{w f'(w)}{f(w)}\right)^{\alpha} \left(\frac{f(w)}{w}\right)^{\beta} + \gamma(f'(w) - 1) \prec_q (\kappa(w) - 1), w \in \mathfrak{A}.$$

where $\alpha > 0$ and $0 \leq \gamma \leq 1$.

Remark1.3. A function $f(w)$ in $\mathcal{A}$ is said to be in the collection $\mathcal{G}(\alpha,\gamma,\kappa)$. If

$$\left(\frac{wf'(w)}{f(w)}\right)^{\alpha}+\gamma(f'(w)-1)\prec_q(\kappa(w)-1), w\in\mathfrak{A}.$$

where $\alpha>0, 0\le\beta<1$ and $0\le\gamma\le1$.

Remark 1.4. A function $f(w)$ in $\mathcal{A}$ is said to be in the collection $\mathcal{G}(\alpha,\beta,\kappa)$. If

$$\left(\frac{wf'(w)}{f(w)}\right)^{\alpha}\left(\frac{f(w)}{w}\right)^{\beta}\prec_q(\kappa(w)-1), w\in\mathfrak{A}.$$

where $\alpha>0, 0\le\beta<1$ and $0\le\gamma\le1$.

Remark 1.5. A function $f(w)$ in $\mathcal{A}$ is said to be in the collection $\mathcal{G}(\alpha,\beta,1,\kappa)$. If

$$\left(\frac{wf'(w)}{f(w)}\right)^{\alpha}\left(\frac{f(w)}{w}\right)^{\beta}+(f'(w)-1)\prec_q(\kappa(w)-1), w\in\mathfrak{A}.$$

where $\alpha>0, 0\le\beta<1$ and $0\le\gamma\le1$.

Numerous interesting implications of this outcome are highlighted.

2 Main results

Theorem 2.1. Let $\alpha>0, 0\le\beta<1$ *and* $0\le\gamma\le1$. If $f(w)\in\mathcal{G}(\alpha,\beta,\gamma,\kappa)$, then

$$|c_2|\le\frac{\mathcal{F}_1}{(\beta+\alpha+2\gamma)}, \tag{2.1}$$

and for any complex number $\delta\in\mathbb{C}$,

$$|c_3-\delta c_2^2|\le\frac{\mathcal{F}_1}{(\beta+2\alpha+3\gamma)}\max\left(1.\left|T\mathcal{F}_1-\frac{\mathcal{F}_2}{\mathcal{F}_1}\right|\right), \tag{2.2}$$

where

$$T=\left(\frac{\delta(\beta+2\alpha+3\gamma)}{\left(\alpha(\beta+\frac{1}{2}(\alpha^2-3))\right)^2}-\frac{1}{\alpha(\beta+\frac{1}{2}(\alpha^2-3))}\right). \tag{2.3}$$

The result is sharp

Proof.

Let $f\in\mathcal{G}(\alpha,\beta,\gamma,\kappa)$.Then, an analytic function $\hbar(w)$ and a Schwarz function $t(w)$ exists such that

$$\left(\frac{wf'(w)}{f(w)}\right)^{\alpha}\left(\frac{f(w)}{w}\right)^{\beta}+\gamma(f'(w)-1)=\kappa(w)(\kappa((t(w))-1), w\in\mathfrak{A}. \tag{2.4}$$

Sequential derivatives of f and their series expansions from (1.1) yield

$$\left(\frac{wf'(w)}{f(w)}\right)^{\alpha}\left(\frac{f(w)}{w}\right)^{\beta}+\gamma(f'(w)-1)$$
$$=(\beta+\alpha+2\gamma)c_2w+(\beta+2\alpha+3\gamma)c_3w^2+\alpha(\beta+\frac{1}{2}(\alpha^2-3))c_2^2w^2+\cdots. \tag{2.5}$$

We get the same result from (1.2), (1.3), and (1.4).

$$\kappa(w)(\kappa((t(w))-1)=\mathfrak{N}_0\mathcal{F}_1t_1w+[\mathfrak{N}_1\mathcal{F}_1t_1+\mathfrak{N}_0(\mathcal{F}_1t_2+(\mathcal{F}_2t_1^2)w^2+\cdots. \tag{2.6}$$

Using (2.5) and (2.6) in (2.4), we obtain

$$c_2 = \frac{\mathfrak{R}_0 \mathcal{F}_1 t_1}{(\beta + \alpha + 2\gamma)}, \tag{2.7}$$

and

$$c_3 = \frac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)} \left[\mathfrak{R}_1 t_1 + \mathfrak{R}_0 \left\{t_2 + \left(\frac{\mathcal{F}_1}{(\beta + \alpha + 2\gamma)} + \frac{\mathcal{F}_2}{\mathcal{F}_1}\right) t_1^2\right\}\right]. \tag{2.8}$$

Thus, for $\in \mathbb{C}$, we get

$$c_3 - \delta c_2^2 = \frac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)} \left[\mathfrak{R}_1 t_1 + \left(t_2 + \frac{\mathcal{F}_2}{\mathcal{F}_1} t_1^2\right) + \mathfrak{R}_0 - T\mathfrak{R}_1 \mathfrak{R}_0^2 t_1^2\right]. \tag{2.9}$$

where T is defined in (2.3).

Since $\hbar(w)$ is analytic and bounded by one in $\mathfrak{A}$, we now have by [13]

$$|\mathfrak{R}_0| \leq 1 \text{ and } \mathfrak{R}_1 = (1 - \mathfrak{R}_0^2)x \quad x \leq 1. \tag{2.10}$$

The statement (2.1) comes from (2.7) using (2.10) and Lemma 1.1, and we gain (2.9) and (2.10) from (2.9) and (2.10).

$$c_3 - \delta c_2^2 = \frac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)} \left[x t_1 + \left(t_2 + \frac{\mathcal{F}_2}{\mathcal{F}_1} t_1^2\right) + \mathfrak{R}_0 - (T\mathcal{F}_1 t_1^2 + x t_1 \mathfrak{R}_0^2\right]. \tag{2.11}$$

If $R_0 = 0$, then (2.11) yields

$$|c_3 - \delta c_2^2| = \frac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)}. \tag{2.12}$$

In contrast, if $\mathfrak{R}_0 \neq 0$, we establish a function.

$$L\mathfrak{R}_0 = x t_1 + \left(t_2 + \frac{\mathcal{F}_2}{\mathcal{F}_1} t_1^2\right) + \mathfrak{R}_0 - (T\mathcal{F}_1 t_1^2 + x t_1 \mathfrak{R}_0^2. \tag{2.13}$$

Notice, (2.13) is a quadratic in $\mathfrak{R}_0$ and so analytic in $|\mathfrak{R}_0| \leq 1$. And $|L(\mathfrak{R}_0)|$ clearly reaches its greatest value at $\mathfrak{R}_0 = e^{i\theta}$, $0 \leq |\theta| \leq 2\pi$. Thus

$$max|L(\mathfrak{R}_0)| = \max_{0 \leq 0 \leq 2x} |L(e^{i\theta})| = |L(1)| = \left|t_2 - \left(\mathcal{F}_1 - \frac{\mathcal{F}_2}{\mathcal{F}_1}\right) t_1^2\right|.$$

Therefore, it follows from (2.11) that

$$|c_3 - \delta c_2^2| = \frac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)} \left|t_2 - \left(T\mathcal{F}_1 - \frac{\mathcal{F}_2}{\mathcal{F}_1}\right) t_1^2\right|. \tag{2.14}$$

We get this from Lemma 1.1.

$$|c_3 - \delta c_2^2| = \frac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)}\, max\left(1. \left|T\mathcal{F}_1 - \frac{\mathcal{F}_2}{\mathcal{F}_1}\right|\right). \tag{2.15}$$

Now, based on (2.12) and (2.15), the statement (2.2) is implied. To demonstrate the sharpness, we define $f(w)$ as

$$\left(\left(\frac{wf'(w)}{f(w)}\right)^{\alpha} \left(\frac{f(w)}{w}\right)^{\beta} + \gamma(f'(w) - 1)\right) = \kappa(w),$$

or

$$\left(\left(\frac{wf'(w)}{f(w)}\right)^{\alpha} \left(\frac{f(w)}{w}\right)^{\beta} + \gamma(f'(w) - 1)\right) = \kappa(w^2),$$

or

$$\left(\left(\frac{wf'(w)}{f(w)}\right)^{\alpha} \left(\frac{f(w)}{w}\right)^{\beta} + \gamma(f'(w) - 1)\right) = w(\kappa(w) - 1).$$

This complete the proof.

Corollary 2.2. Let $\alpha > 0$, $0 \leq \beta < 1$ *and* $0 \leq \gamma \leq 1$. If $f(w) \in \mathcal{G}(\alpha, \gamma, \kappa)$, then

$$|c_2| \leq \frac{\mathcal{F}_1}{(\alpha+2\gamma)}$$

and for any complex number $\delta \in \mathbb{C}$,

$$|c_3 - \delta c_2^2| \leq \frac{\mathcal{F}_1}{(2\alpha+3\gamma)} \max\left(1. \left|T\mathcal{F}_1 - \frac{\mathcal{F}_2}{\mathcal{F}_1}\right|\right).$$

where

$$T = \left(\frac{\delta(2\alpha+3\gamma)}{\left(\alpha(\frac{1}{2}(\alpha^2-3))\right)^2} - \frac{1}{\alpha(\frac{1}{2}(\alpha^2-3))}\right).$$

Corollary 2.3. Let $\alpha > 0$, $0 \leq \beta < 1$ and $0 \leq \gamma \leq 1$. If $f(w) \in \mathcal{G}(\alpha, \beta, \kappa)$, then

$$|c_2| \leq \frac{\mathcal{F}_1}{(\beta+\alpha)}$$

and for any complex number$\delta \in \mathbb{C}$,

$|c_3 - \delta c_2^2| \leq \frac{\mathcal{F}_1}{(\beta+2\alpha)} \max\left(1. \left|T\mathcal{F}_1 - \frac{\mathcal{F}_2}{\mathcal{F}_1}\right|\right)$. where

$$T = \left(\frac{\delta(\beta+2\alpha)}{\left(\alpha(\beta+\frac{1}{2}(\alpha^2-3))\right)^2} - \frac{1}{\alpha(\beta+\frac{1}{2}(\alpha^2-3))}\right).$$

Corollary 2.4. Let $\alpha > 0$, $0 \leq \beta < 1$ *and* $0 \leq \gamma \leq 1$. *If* $f(w) \in \mathcal{G}(\alpha, \beta, 1, \kappa)$, then

$$|c_2| \leq \frac{\mathcal{F}_1}{(\beta+\alpha+2)}$$

and for any complex number$\delta \in \mathbb{C}$,

$$|c_3 - \delta c_2^2| \leq \frac{\mathcal{F}_1}{(\beta+2\alpha+3)} \max\left(1. \left|T\mathcal{F}_1 - \frac{\mathcal{F}_2}{\mathcal{F}_1}\right|\right),$$

where

$$T = \left(\frac{\delta(\beta+2\alpha+3)}{\left(\alpha(\beta+\frac{1}{2}(\alpha^2-3))\right)^2} - \frac{1}{\alpha(\beta+\frac{1}{2}(\alpha^2-3))}\right).$$

We base our next insightful finding on majorization.

Theorem 2.5. Let > 0 , $0 \leq \beta < 1$ and $0 \leq \gamma \leq 1$. If $f(w) \in \mathcal{A}$ satisfies

$$\left(\frac{wf'(w)}{f(w)}\right)^\alpha \left(\frac{f(w)}{w}\right)^\beta + \gamma(f'(w)-1) \prec\prec (\kappa(w)-1), w \in \mathfrak{A}. \quad (2.16)$$

Then

$$|c_2| = \frac{\mathcal{F}_1}{(\beta+\alpha+2\gamma)} \quad (2.17)$$

and for any complex numberδ,

$$|c_3 - \delta c_2^2| \leq \frac{\mathcal{F}_1}{(\beta+2\alpha+3\gamma)}\ max\left(1. \left|T\mathcal{F}_1 - \frac{\mathcal{F}_2}{\mathcal{F}_1}\right|\right), \quad (2.18)$$

where T is as defined by (2.3).

Proof. Suppose that (2.16) holds. There is an analytic function $\hbar(w)$, from the principle of majorization, such that

$$\left(\frac{wf'(w)}{f(w)}\right)^\alpha \left(\frac{f(w)}{w}\right)^\beta + \gamma(f'(w)-1) = \hbar(w)(\kappa(w)-1), w \in \mathfrak{A}. \quad (2.19)$$

Putting $\omega(w) \equiv w$ (so that $t_1 = 1, t_N = 0, n \geq 2$), after completing the proof of Theorem 2.1, we arrive at the intended outcomes (2.17) and (2.18). To demonstrate the sharpness, we define $f(w)$ as

$$1 + \left(\frac{wf'(w)}{f(w)}\right)^{\alpha}\left(\frac{f(w)}{w}\right)^{\beta} + \gamma(f'(w) - 1) = (\kappa(w)), w \in \mathfrak{A}.$$

Which complete of the proof.

Our subsequent noteworthy outcome is connected to $\mathcal{G}(\alpha,\beta,\gamma,\kappa)$.

Theorem 2.6. Let $\alpha > 0\,, 0 \leq \beta < 1$ and $0 \leq \gamma \leq 1$. If $f(w) \in \mathcal{G}(\alpha,\beta,\gamma,\kappa)$, then

$$|c_2| \leq \frac{\mathcal{F}_1}{(\beta + \alpha + 2\gamma)}$$

and for any $\delta \in \mathbb{C}$,

$$|c_3 - \delta c_2^2| \leq \frac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)} \max\left(1. \left|T\mathcal{F}_1 - \frac{\mathcal{F}_2}{\mathcal{F}_1}\right|\right),$$

where $T$is as stated in (2.3).

Proof. Let ($f \in \mathcal{G}(\alpha,\beta,\gamma,\kappa)$,). Taking $h(w) \equiv 1, w \in \mathfrak{A}$, we get $\mathfrak{N}_0 = 1, \mathfrak{N}_n = 0, n \in N$ and we get the intended outcomes by adhering to Theorem 2.1. To demonstrate the sharpness, we define $f(w)$ as

$$\left(\frac{wf'(w)}{f(w)}\right)^{\alpha}\left(\frac{f(w)}{w}\right)^{\beta} + \gamma(f'(w) - 1) = \kappa(w),$$

or

$$\left(\frac{wf'(w)}{f(w)}\right)^{\alpha}\left(\frac{f(w)}{w}\right)^{\beta} + \gamma(f'(w) - 1) = \kappa(w^2),$$

which complete the proof.

For real δ, we now establish sharp constraints for $|c_3 - \delta c_2^2|$ for $f(w) \in \mathcal{G}(\alpha,\beta,\gamma,\kappa)$.

Theorem2.7. Let $\alpha > 0\,, 0 \leq \beta < 1$ and $0 \leq \gamma \leq 1$. If $f(w) \in \mathcal{G}(\alpha,\beta,\gamma,\kappa)$, then for real δ, we have

$$|c_3 - \delta c_2^2| \leq \begin{cases} \dfrac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)}\left[\mathcal{F}_1\left(\dfrac{1}{(\beta + \alpha + 2\gamma)} - \dfrac{\delta(\beta + 2\alpha + 3\gamma)}{(\beta + \alpha + 2\gamma)^2}\right) + \dfrac{\mathcal{F}_2}{\mathcal{F}_1}\right] & (\delta \leq \tau_1) \\ \dfrac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)} & (\tau 1 \leq \delta \leq \tau 1 + 2u) \\ -\dfrac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)}\left[\mathcal{F}_1 = \left(\dfrac{1}{(\beta + \alpha + 2\gamma)} - \dfrac{\delta(\beta + 2\alpha + 3\gamma)}{(\beta + \alpha + 2\gamma)^2}\right) + \dfrac{\mathcal{F}_2}{\mathcal{F}_1}\right] & (\delta \geq \tau_1 + 2u) \end{cases} \quad (2.20)$$

where

$$\tau_1 = \frac{(\beta + \alpha + 2\gamma)}{(\beta + 2\alpha + 3\gamma)} - \frac{(\beta + \alpha + 2\gamma)^2}{(\beta + 2\alpha + 3\gamma)}\left(\frac{1}{\mathcal{F}_1} - \frac{\mathcal{F}_2}{\mathcal{F}_1^2}\right), \quad (2.21)$$

and

$$u = \frac{(\beta + \alpha + 2\gamma)^2}{(\beta + 2\alpha + 3\gamma)\mathcal{F}_1}. \quad (2.22)$$

Proof. Let our real values be δ. Then, in the following scenarios, (2.20) may be obtained from (2.2) and (2.3), respectively:

$$T\mathcal{F}_1 - \frac{\mathcal{F}_2}{\mathcal{F}_1} \leq -1, -1 \leq \delta\mathcal{F}_1 - \frac{\mathcal{F}_2}{\mathcal{F}_1} \leq 1 \text{ and } T\mathcal{F}_1 - \frac{\mathcal{F}_2}{\mathcal{F}_1} \geq 1.$$

Where T is defined in (2.3). We also like to mention the following:
(i) Equality holds for $\delta < \tau_1$ or $\delta > \tau_1 + 2u$ ⟺if $(w) \equiv 1$ and $t(w) = w$ or one of its rotations.
(ii) Equality holds for $\tau_1 < \delta < \tau_1 + 2u$ ⟺ $h(w) \equiv 1$ and $w(w) = w^2$ or one of its rotations.
(iii) Equality holds for $\delta = \tau_1$ ⟺if $h(w) \equiv 1$ and $t(w) = \frac{w(w+\theta)}{1+\theta w}, 0 \le \theta \le 1$, or one of its rotation, while for $\delta = \tau_1 + 2u$, the equality holds if and only if $h(w) \equiv 1$ and $t(w = -\frac{w(w+\theta)}{1+\theta w}, 0 \le \theta \le 1$, or one of its rotations.

The following actions can improve the second part of the assertion in (2.20) for the real value of δ even more:

Theorem 2.8. Let $\alpha > 0, 0 \le \beta < 1$ *and* $0 \le \gamma \le 1$. If $f(w) \in \mathcal{G}(\alpha, \beta, \gamma, \kappa)$, then for real δ, we have

$$|c_3 - \delta c_2^2| + (\delta - \tau_1)|c_2^2|^2 \le \frac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)} \quad (\tau_1 \le \delta \le \tau_1 + u), \tag{2.23}$$

and

$$|c_3 - \delta c_2^2| + (\tau_1 + 2u - \delta)|c_2^2| \le \frac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)} \ (\tau_1 + u \le \delta \le \tau_1 + 2u), \tag{2.24}$$

where τ_1 and u are given by (2.21) and (2.22), respectively.
Proof. Let $f(w) \in \mathcal{G}(\alpha, \beta, \gamma, \kappa)$. For a real δ satisfying $\tau_1 \le \delta \le \tau_1 + u$ and using (2.7) and (2.14), we get

$$|c_3 - \delta c_2^2| + (\delta - \tau_1)|c_2^2|^2 \le \frac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)} \left[|t_2| - \frac{\mathcal{F}_1(\beta + 2\alpha + 3\gamma)}{(\beta + \alpha + 2\gamma)^2} (\delta - \tau_1 - u)|t_1|^2 + \frac{\mathcal{F}_1(\beta + 2\alpha + 3\gamma)}{(\beta + \alpha + 2\gamma)^2} (\delta - \tau_1)|t_1|^2 \right].$$

Therefore, by using Lemma 1.1, we obtain

$$|c_3 - \delta c_2^2| + (\delta - \tau_1)|c_2^2|^2 \le \frac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)} [1 - |t_1|^2 + |t_1|^2]$$

which yields the assertion (2.23).
If$\tau_1 + u \le \delta \le \tau_1 + 2u$, then again from (2.7), (2.14) and Lemma 1.1, we have

$$|c_3 - \delta c_2^2| + (\tau_1 + 2u - \delta)|c_2^2|^2 \le \frac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)} \left[|t_2| - \frac{\mathcal{F}_1(\beta + 2\alpha + 3\gamma)}{(\beta + \alpha + 2\gamma)^2} (\tau_1 + 2u - \delta)|t_1|^2 + \frac{\mathcal{F}_1(\beta + 2\alpha + 3\gamma)}{(\beta + \alpha + 2\gamma)^2} (\delta - \tau_1)|t_1|^2 \right] \le \frac{\mathcal{F}_1}{(\beta + 2\alpha + 3\gamma)} [1 - |t_1|^2 + |t_1|^2].$$

which estimates (2.24).

3 Conclusions

In conclusions, our study of the Fekete-Szegö coefficient functional of the quasi-subordination class has illuminated the characteristics and actions of analytic functions inside the open unit disk. Through the use of quasi-subordination to construct a particular subclass, we have found sharp constraints on the Fekete-Szegö functional, concentrating on $|c_3 - \delta c_2^2|$ for functions in this subclass.
The results of the research not only advance knowledge of the particular quasi-subordination class but also have ramifications for more general classes and link to well-known

mathematical frameworks. The derived sharp limits improve our understanding of the characteristics and analysis of holomorphic functions, offering a more complex view of their behavior under quasi-subordination.

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