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Pure Essential Submodules

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Abstract.
In this work, we generalized the notion of essential submodules by introduce the notion that namely pure essential submodules. We also introduce two notions pure relative complement submodules and pure uniform modules, as well as some other related concepts. The fundamental characteristics of these ideas are investigated.

Keywords: Essential submodule, pure essential submodules, pure relative complement submodules, pure uniform modules.

المقاسات الجزئية الأساسية النقية

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الخلاصة
في هذا البحث نقوم بتعميم فكرة المقاسات الجزئية الأساسية من خلال تقديم فكرة المقاسات الجزئية الأساسية النقية. كما نقدم أيضًا مفهومين للمقاسات الجزئية التكميلية النسبية النقية والمقاسات المنتظمة النقية، وبعض المفاهيم الأخرى ذات الصلة. تم استكشاف الخصائص الأساسية لهذه المفاهيم.

1. Introduction

Consider R be a commutative ring with identity and M be unitary R-module. Anderson and Fuller in [1] referred to as the submodule N of a left R- module M is called pure if $IM \cap N = IN$ for every ideal I of the ring R [2-4]. A submodule N of M is called an essential (or M is called an essential extension of N) ($N \leq_e M$), if $N \cap L \neq 0$, for each non-zero L submodule of M [5-7]. An R-module M namely an uniform if each non-zero submodule of M is essential [8]. In [9] presented a nation of semi-essential sub modules as an extension from a class of essential submodules. Researchers claimed so a non-zero submodule $H$of M is known as semi-essential, if $H \cap P \neq (0)$ for all non-zero prime submodule P in M. In 1997, AL-Thani investigated a new concept, which is known as P-essential submodules, where a submodule K of an R- Mod M is called P-essential in M and denoted ($K \trianglelefteq' M$), in case for every pure submodule L of M, $K \cap L = (0)$ implies $L = (0)$, [10].

Zhou and Zhang in [11] defined a generalization of essential submodules, which are called s-essential, as follow a submodule U is called s-essential in module M, if $U \cap T \neq 0$

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for each non-zero T is small submodule of M. Recall that an R-submodule T of an R-module M is called small (briefly $N \ll M$) if, for all $K \leq M$ with $T + K = M$ implies $K = M$ [6, P.106]

The primary goal of this study is to introduce concepts of pure essential submodules and pure uniform modules as popularization of the essential submodule and uniform module. In Section 2 we study some of characterizations for pure essential submodules and extend certain well-known features of essential submodules to pure essential submodules. And we provide restrictions which that a submodule of a finitely generated faithful multiplication R-module gets pure essential. While in the Section 3, we extend a notion of relative complement of a submodule by new generalization which is called pure relative complement. In Section 4, we offer the pure uniform module notion as an extension from a uniform notion. Additionally, we make generalizations as description additionally to a few from some uniform module characteristics for pure uniform modules.

2. Pure -essential submodules

This section introduces a new submodule class also referred to pure essential submodules, we study some of the characteristics for this kind of submodules.

Definition 2.1:

A submodule K of a left R-module M is called pure essential submodule of M, briefly (Pr-essential), and denoted for ($K \trianglelefteq_{pr} M$), if $K \cap L = (0)$, indicates that L is a pure submodule of M.

Remarks and examples 2.2:

1. It is clear that every essential is Pr- essential. Thus $2Z$ in Z as a Z- Mod is a Pr- essential.

2. The converse of (1) in general is not true. For example: in Z_6 as Z - module, $(\overline{3}) \cap (\overline{2}) = (\bar{0})$, implies $(\overline{2})$ is a pure sub module of Z_6, so $(\overline{3})$ is a Pr- essential in Z_6, but $(\overline{3})$ is not essential in Z_6 since $(\overline{2}) \neq (\bar{0})$, also $(\overline{2})$ is a Pr- essential in Z_6 as Z - Mod but not an essential.

3. Each direct summand of an R- Mod is pr-essential. Actually each submodule in a semi simple module is a Pr- essential.

4. A Pr- essential need not to be p- essential. For example: in Z_6 as Z - Mod, $(\overline{3})$ is a Pr- essential in Z_6 since $(\overline{3}) \cap (\overline{2}) = (\bar{0})$, implies $(\overline{2}) \leq_p Z_6$, but it is not a p-essential since $(\overline{2}) \neq (\bar{0})$.

5. In Z_{36} as Z -Mod $(\overline{4}) \oplus (\overline{9}) = Z_{36}$.$(\overline{4}), (\overline{9})$ are pure submodules of Z_{36} as Z -Mod. $(\overline{18})$ is not a Pr- essential in Z_{36} since $(\overline{18}) \cap (\overline{12}) = (\bar{0})$, but $(\overline{12})$ is not pure in Z_{36}.

6. (0) is not a Pr- essential submodule for any module M.

7. M is a Pr- essential submodule of any R- Mod M.

Proposition 2. 3:

Let N , K , L be submodules of an R- Mod M, such that $N \leq K \leq L \leq M$. If $N \trianglelefteq_{pr} M$, then $K \trianglelefteq_{pr} L$.

Proof: Let $H \leq L$ such that $K \cap H = (0)$. Since $H \leq L$, then $H \leq M$ also $N \cap H \leq K \cap H = (0)$, hence $N \cap H = (0)$, since $N \trianglelefteq_{pr} M$, then H is pure submodule in M. But $H \leq L$, then by [12] H is a pure submodule in L.

Corollary 2.4:

Consider A, B and C be submodules from an R- Mod M such that $A \leq B \leq C \leq M$, with C be a pure submodule of M. If $A \trianglelefteq_{pr} M$, then $A + B \trianglelefteq_{pr} C$.

Proof: Since $A \leq A + B$, then by Proposition 2.3, $A + B \trianglelefteq_{pr} C$.

Remark 2.5:
Let $f\colon M_1 \longrightarrow M_2$ be an R-homomorphism. If $N \trianglelefteq_{pr} M_1$. Then $f(N)$ need not be pr-essential in M_2. As the following example show: Let $f\colon Z \longrightarrow Z_2$ define as
$f(n) = \begin{cases} \bar{1} & n\ is\ odd; \\ \bar{0} & n\ is\ even. \end{cases}$
$(\bar{2}) = 2Z \leq Z$, $2Z \trianglelefteq_{pr} Z$ but $f(2Z) = \bar{0}$ is not a pr- essential in Z_2.
Proposition 2.6:
Let $f\colon M_1 \longrightarrow M_2$ be an R- epimorphism. If $N \trianglelefteq_{pr} M_2$, then $f^{-1}(N) \trianglelefteq_{pr} M_1$.

Proof: Let $L \leq M_1$ such that $f^{-1}(N) \cap L = (0)$. $N \cap f(L) = f(0)$, then $N \cap f(L) = (0)$, since $N \trianglelefteq_{pr} M_2$, then $f(L) \leq_p M_2$, by [13] $f^{-1}(f(L)) \leq_p M_1$. Therefore, $L \leq_p M_1$.
Proposition 2.7:
Let A, B be submodules from a left R- Mod M. If $A \cap B \trianglelefteq_{pr} M$, then $B \trianglelefteq_{pr} M$ and $A \trianglelefteq_{pr} M$.
Proof: Let L be a submodule of M such that $A \cap L = (0)$. $(A \cap B) \cap L \leq A \cap L = (0)$, then $(A \cap B) \cap L = (0)$. Since $A \cap B \trianglelefteq_{pr} M$, implies that $L \leq_p M$. By similar way we get that $B \trianglelefteq_{pr} M$.
Remark 2.8: Let A, B be submodules of M. If $A \trianglelefteq_{pr} M$ and $B \trianglelefteq_{pr} M$. Then $A \cap B$ need not be a Pr- essential in M.For example: in Z_{36} as Z – Mod. $(\bar{4}) \trianglelefteq_{pr} Z_{36}$, since $(\bar{4}) \cap (\bar{9}) = (\bar{0})$ implies $(\bar{9}) \leq_p Z_{36}$. Also, $(\bar{6}) \trianglelefteq_{pr} Z_{36}$, since $(\bar{6}) \leq_e Z_{36}$. But $(\bar{4}) \cap (\bar{6}) = (\overline{12})$, which is not Pr- essential in Z_{36}, since $(\overline{12}) \cap (\overline{18}) = (\bar{0})$ but $(\overline{18})$ is not a pure in Z_{36}.
Remark 2.9:
If $N \trianglelefteq_{pr} M$. Then $N \cap K \trianglelefteq_{pr} M \cap K$, for any submodule K of M.
Proof: Let $H \leq M \cap K$, then $H \leq M$ and $H \leq K$, such that $(N \cap K) \cap H = (0)$.
Hence, $N \cap (K \cap H) = (0)$, then $N \cap H = (0)$ (since $H \leq K$). Since $N \trianglelefteq_{pr} M$, then $H \leq_p M$. But $H \leq M \cap K$ then $H \leq_p M \cap K$.

An R- Mod M is said to be pure simple, if the only pure submodule of M are (0) and M, [14].
Proposition 2.10:
Let M be a pure simple R- Mod. If $K \trianglelefteq_{pr} M$ with $N \leq K$ and $N \leq_p M$, then $\frac{K}{N} \trianglelefteq_{pr} \frac{M}{N}$.
Proof: Let $\frac{L}{N} \leq \frac{M}{N}$ such that $\frac{K}{N} \cap \frac{L}{N} = (0)$ implies $K \cap L = N$. Since $N \leq_p M$, then $K \cap L \leq_p M$. Since M is pure simple, and $K \cap L \neq M$, then $K \cap L = (0)$. But $K \trianglelefteq_{pr} M$, hence $L \leq_p M$, by [12] $\frac{L}{N} \leq_p \frac{M}{N}$.
Proposition 2.11:
Let N and K be submodules of M, with $K \leq N$. If $\frac{N}{K} \trianglelefteq_{pr} \frac{M}{K}$ and $K \leq_p M$, then $N \trianglelefteq_{pr} M$.
Proof: Let $H \leq M$ such that $N \cap H = (0)$, then $\frac{N \cap H}{K} = \frac{(0)}{K}$ so $\frac{N}{K} \cap \frac{H}{K} = (0)$, since $\frac{N}{K} \trianglelefteq_{pr} \frac{M}{K}$, implies $\frac{H}{K} \leq_p \frac{M}{K}$ .Since $K \leq_p M$, then by [12] $H \leq_p M$.
Theorem 2.12:
Let M be a pure simple R- Mod and $A \leq B \leq M$. Then $A \trianglelefteq_{pr} M$ if and only if $A \trianglelefteq_{pr} B$ and $B \trianglelefteq_{pr} M$.
Proof: $\Longrightarrow$) Let $L \leq M$ such that $B \cap L = (0)$. To prove that $L \leq_p M$.
$A \cap L \leq B \cap L = (0)$, so that $A \cap L = (0)$. Since $A \trianglelefteq_{pr} M$, then $L \leq_p M$.
Let $C \leq B$ such that $A \cap C = (0)$. To prove that $C \leq_p B$.

As $C \leq M$ and $A \trianglelefteq_{pr} M$, then $C \leq_p M$. Hence, by [12, Remark 2.1, p. 14] we obtain $C \leq_p B$.

⇐) Let $H \leq M$ such that $A \cap H = (0)$. To prove that $H \leq_p M$.
$(A \cap H) \cap B = (0) \cap B$ then $A \cap (H \cap B) = (0)$, since $A \trianglelefteq_{pr} B$, then $H \cap B \leq_p M$, since M is a pure simple, then either $H \cap B = (0)$ or $H \cap B = M$ which is a contradiction. So, $B \cap H = (0)$, since $B \trianglelefteq_{pr} M$, implies $H \leq_p M$.

Recall that we say a multiplication of an R- Mod M if for every sub module N of M, there is an ideal I in R so that $N = IM$, also M is called faithful if ann$M$ = 0 [15], [16].

Theorem 2.13:

An ideal I of a ring R is a pr-essential if and only if IM is a Pr-essential submodule of M. Where M be a finitely generated faithful multiplication R- Mod.

Proof: ⇐) Assume that J is an ideal in R, such that $I \cap J = (0)$. To prove that $J \leq_p R$. $(I \cap J)M = (0)M$, implies $IM \cap JM = (0)$, since $IM \trianglelefteq_{pr} M$, then $JM \leq_p M$. Hence, by [15] we have $J \leq_p R$.

⟹) Let $H \leq M$ such that $IM \cap L = (0)$. To prove that $H \leq_p M$. Considering that M is multiplication, then $H = JM$, for some $J \leq R$. $IM \cap JM = (0)$, then $(I \cap J)\ M = (0)M$. Since M is a finitely generated faithful multiplication, hence by [17] we have $I \cap J = (0)$. But $I \trianglelefteq_{pr} R$, implies that $J \leq_p R$ [18]. Hence, JM is pure in M. Thus $H \leq_p M$ and $IM \trianglelefteq_{pr} M$.

A left R- Mod M is said to be a cancelation module, if whenever $IM = JM$, at I and J representing ideals of R, then $I = J$, [19].

Corollary 2.14:

Let M be a cancelation R- Mod, and I any ideal of R. If $IM \trianglelefteq_{pr} M$, then $I \trianglelefteq_{pr} R$.

Proof: Clear by Theorem 2.13.

Corollary 2.15:

Let M be a cancelation and multiplication R- Mod. If $A \trianglelefteq_{pr} M$,then $[A\colon M] \trianglelefteq_{pr} R$.

Proof: Since M is multiplication implies that $A = [A\colon M]\ M$, [17]. But $A \trianglelefteq_{pr} M$, hence by Corollary 2.14, $[A\colon M] \trianglelefteq_{pr} R$, where $[A\colon M]$ is the residual of A by M, which is the set of all r in R such that $rM \subseteq A$, [15].

An R- Mod M is said to be F-regular (regular), if every submodule of M is pure, [20].

Proposition 2.16:

Let M be a regular R- Mod and let $E \leq M$, then $E \trianglelefteq_{pr} M$ if and only if $E \leq_e M$.

Proof: Clear.

Recall that a submodule $Z(M)$ of a left R- Mod M, where $Z(M)$ ={$m \in M \mid ann(m) \leq_e R$} is said to be singular submodule of M, [5].

Proposition 2.17:

If $N \trianglelefteq_{pr} M$ and $Z(N) = (0)$, then $Z(M) \leq_p M$.

Proof: Since $Z(N) = N \cap Z(M)$ [5], then $N \cap Z(M) = (0)$. But $N \trianglelefteq_{pr} M$, implies $Z(M) \leq_p M$.

Let N be a submodule of an R- Mod M. A submodule K in M is called a relative complement for N in M, if K is maximal with respect to the property $N \cap K = 0$. (i.e., if there exists $K' \leq M$ such that $K \leq K'$ with $K' \cap N = 0$ implies $K = K'$, [5]. In this part, we study a pure relative complement notion as a broadening of the relative complement submodule concept.

Definition 2.18:
Let N and K be submodules of an R- Mod M, K is called pure- relative complement (Pr-relative complement) of N in M. If K is maximal pure submodule with respect to property $K \cap N = (0)$.

Remarks and examples 2.19:

1. It is clear that every relative complement is Pr-relative complement. Thus ($\overline{2}$) in Z_6 as Z-Mod is a Pr-relative complement of ($\overline{3}$) in Z_6 as Z- Mod.

2. If $N \leq M$, then a Pr-relative complement for N in M may not be unique. For example: Consider $M = Z_2 \oplus \{\overline{0}\} = \{ (\overline{0}, \overline{0}), (\overline{1}, \overline{0})\}$

$K_1 = \{\overline{0}\} \oplus Z_2 = \{ (\overline{0}, \overline{0}), (\overline{0}, \overline{1})\}$, $K_2 = Z(\overline{1}, \overline{1}) = \{ (\overline{0}, \overline{0}), (\overline{1}, \overline{1})\}$

K_1 and K_2 are pure submodules in M, $N \cap K_1 = (\overline{0}, \overline{0})$, implies that K_1 is maximal pure sub module of M, $N \cap K_2 = (\overline{0}, \overline{0})$, implies that K_2 is maximal pure submodule of M. Thus K_1 and K_2 are Pr-relative complement for N in M.

3. If N is a Pr-relative complement to K in M, then K may not be a Pr-relative complement to N in M. For example: Consider $N = (\overline{2})$ and $K = (\overline{0})$ are submodules of Z_4 as Z-Mod. $(\overline{0})$ is a Pr-relative complement to $(\overline{2})$in Z_4. But $(\overline{2})$ is not a Pr-relative complement to $(\overline{0})$in Z_4, since $(\overline{0}) \cap Z_4 = (0)$,where $(\overline{2})$ is not maximal pure submodule of Z_4.

4. In Z_6 as Z- Mod $(\overline{2}) \cap (\overline{3}) = (\overline{0})$. $(\overline{3})$ is a Pr-relative complement to $(\overline{2})$in Z_6, and $(\overline{2})$ is a Pr-relative complement to $(\overline{3})$in Z_6.

Proposition 2.20:
If $M = N \oplus K$, where N and K are submodules of an R- Mod M. Then K is a Pr-relative complement to N in M, and N is a Pr-relative complement to K in M.

Proof: $N \cap K = (0)$, since $M = N \oplus K$, then by [9] N and K are pure sub-modules of M. Let there exist $K' \leq M$ such that $K \leq K' \leq M$ and $N \cap K' = (0)$, since $M = N + K$ and $K \leq K' \leq M$, then $M = N + K' = N + K$, implies that $K = K'$, hence K is a Pr-relative complement to N in M. By similar way we prove that N is a Pr-relative complement to K in M.

Proposition 2.21:
Let A, B be submodules of an R- Mod M, with $A \trianglelefteq_{pr} M$. If B is a Pr-relative complement to A in M, then $A \oplus B \trianglelefteq_{pr} M$.

Proof: Let $C \leq M$ such that$(A \oplus B) \cap C = (0)$. $(A \oplus B) \cap C = (0)$, then $A \cap (B + C) = (0)$, since $A \trianglelefteq_{pr} M$, implies $B + C \leq_p M$. But B is a Pr-relative complement to A in M, then B is a maximal pure in M. $B \leq B + C$, hence $B = B + C$, implies $C \leq B$ so (0)= $(A \oplus B) \cap C$, (0) = $(A \cap C) + (B \cap C)$, then (0) = $(A \cap C) + C$, hence $C = (0)$. Since (0) is a pure in M. Thus $C \leq_p M$.

Remark 2.22:
If $N \oplus K \trianglelefteq_{pr} M$. Then K may not be a Pr-relative complement to N in M. For example: consider Z_{36}as Z- Mod. Let $N = (\overline{9})$ and $K = (\overline{12})$ are submodules of Z_{36}as Z- Mod.
$N \oplus K = (\overline{9}) \oplus (\overline{12}) = (\overline{3}) \trianglelefteq_{pr} Z_{36}$, but $(\overline{12})$ is not a Pr-relative complement to $(\overline{9})$. Since $(\overline{12})$ is not maximal pure with respect to $(\overline{9}) \cap (\overline{12}) = (\overline{0})$.

3. Pure – uniform modules

A non-zero left R- Mod M namely a uniform, if each non-zero submodule in M is an essential, [5]. We provide here, a pure uniform an R- Mod notion as a broadening of the uniform module notion. We also expand a several uniform module features to pure-uniform modules.

Definition 3.1:
A non-zero an R- Mod M is said to be pure-uniform, briefly (Pr-uniform), if for all nonzero sub module in M is Pr-essential.

Remarks and examples 3.2:
1. It is clear that each uniform R- Mod is a Pr-uniform. For example: Z_8 as Z- Mod is a Pr-uniform.
2. In general the convers of (1) is not true. For example: Z_{24} as Z-Mod is a Pr-uniform but not uniform. Since $(\bar{3})$ is not an essential submodule of Z_{24}.
3. Every semi simple module is a Pr-uniform. For example: Z_6 as Z- Mod.

Theorem 3.3:
Let M be a finitely generated faithful multiplication R-Mod. Then M is a Pr-uniform module if and only if R is a Pr-uniform ring.

Proof: $\Longrightarrow$)Let $I \leq R$, then $IM \leq M$. Since M is Pr-uniform module, implies $IM \trianglelefteq_{pr} M$. By Theorem 2.13, $I \trianglelefteq_{pr} R$. Thus R is a Pr-uniform ring.
$\Longleftarrow$) Suppose that $L \leq M$, since M is multiplication R-Mod, then $L = IM$, where $I \leq R$. But R is a Pr-uniform ring, hence $I \trianglelefteq_{pr} R$. By Theorem 2.13, $IM \trianglelefteq_{pr} M$. Thus M is Pr-uniform module.

Corollary 3.4:
Let M be a cancelation R-Mod, and let I be any ideal of the ring R. If M is Pr-uniform module, then R is a Pr-uniform ring.

Proof: By Corollary 2.14.

Proposition 3.5:
Let f: $M_1 \longrightarrow M_2$ be an R- epimorphism. If M_2 is a Pr-uniform module, then M_1 is a Pr-uniform module.

Proof: Let $L \leq M_1$, so $f(L) \leq M_2$, since M_2 is a Pr-uniform module, then $f(L) \trianglelefteq_{pr} M_2$. By Proposition 2.6, $f^{-1}(f(L)) \trianglelefteq_{pr} M_1$. Hence, $L \trianglelefteq_{pr} M_1$.

Proposition 3.6:
Let $M = M_1 \oplus M_2$ be R-Mod. If M_1or M_2 is a Pr-uniform module, then M is Pr-uniform module.

Proof: Suppose that ρ: $M \longrightarrow M_1$ and j: $M_1 \longrightarrow M$. Let $L \leq M$, then $\rho(L) \leq M_1$. Since M_1 is a Pr-uniform module, hence $\rho(L) \trianglelefteq_{pr} M_1$. $L = j(\rho(L)) \trianglelefteq_{pr} M$. Thus, M is Pr-uniform module.

Remember to a non-zero left R-Mod M is said to be fully essential if for each non-zero semi-essential submodule in M is an essential submodule of M, [8].

Definition 3.7:
A non-zero R-Mod M is said to be fully Pr-essential if for each non-zero Pr-essential submodule of M is an essential submodule of M.

Examples 3.8:
1. Z_8 as Z-Mod is a fully Pr-essential.
2. Z_6 as Z-Mod is not fully Pr –essential. Since $(\bar{2})$ is Pr- essential submodule but not essential submodule of Z_6.

Proposition 3.9:
Let M be a left R-Mod. Then M is Pr-uniform and fully Pr-essential if and only if M is uniform.

Proof: $\Longrightarrow$) Assume that $0 \neq N \leq M$, since M is a Pr-uniform, then $N \trianglelefteq_{pr} M$. But M is fully Pr-essential, so $N \leq_e M$, hence M is uniform.
$\Longleftarrow$) Clear.

Corollary 3.10:
Let M be a regular R-Mod. Then M is uniform if and only if M is a Pr-uniform.

4. Conclusions:

In this paper, a generalization of essential submodule, relative complement sub module and uniform module has been introduced which are called pure essential submodules, pure relative complement submodules and pure uniform modules respectively. We also show the following:

- If $N \trianglelefteq_{pr} M$, then $K \trianglelefteq_{pr} L$, where $N \leq K \leq L \leq M$.
- Let $f: M_1 \longrightarrow M_2$ be an R- epimorphism. If $N \trianglelefteq_{pr} M_2$, then $f^{-1}(N) \trianglelefteq_{pr} M_1$.
- If $A \cap B \trianglelefteq_{pr} M$, then $B \trianglelefteq_{pr} M$ and $A \trianglelefteq_{pr} M$, where A , B are submodules of M.
- If $\frac{N}{K} \trianglelefteq_{pr} \frac{M}{K}$ and $K \leq_p M$, then $N \trianglelefteq_{pr} M$, with $K \leq N \leq M$.
- If B is a Pr-relative complement to A in M, then $A \oplus B \trianglelefteq_{pr} M$, with $A \trianglelefteq_{pr} M$.
- If M is Pr-uniform module, then R is a Pr-uniform ring.

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