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Calculating the Variation of the Universal Parameter (Variable) Using Kepler's Equation for Different Orbits

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Abstract

Stumpff functions are an infinite series that depends on the value of z. This value results from multiplying the reciprocal semi-major axis with a universal anomaly. The purpose from those functions is to calculate the variation of the universal parameter (variable) using Kepler's equation for different orbits. In this paper, each range for the reciprocal of the semi-major axis, universal anomaly, and z is calculated in order to study the behavior of Stumpff functions C(z) and S(z). The results showed that when z grew, Stumpff functions for hyperbola, parabola, and elliptical orbits were also growing. They intersected and had a tendency towards zero for both hyperbola and parabola orbits, but for elliptical orbits, Stumpff functions exhibited unstable behavior, with distinct, insignificant peaks disappearing as z increased. In comparison to other published studies in this section, the results showed good agreement.

Keywords: Universal Variables, Universal Anomaly, Stumpff Functions, Reciprocal of the semi-major axis, Universal Kepler’s Equation.

حساب تباين المعلم العالمي (المتغير) بأستخدام معادلة كبلر لمدارات متعددة

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الخلاصة

دوال ستامب هي متسلسة لا منتهيه تعتمد على قيمة z. تنتج هذه القيمة من حاصل ضرب نصف المحور الكبير المتبادل مع المتغيير الكوني. الغرض من هذه الدوال هو حساب التغيير في المتغيير الكوني. تم في هذا البحث حساب نصف المحور الكبير المتبادل والانحراف الكوني وقيمة z لغرض دراسة سلوك دوال ستامب في مدارات مختلفة. اشارت النتائج الى ان زيادة قيمة z فأن دالتي ستامب سوف تزداد في مدار القطع الناقص ومدار القطع المكافيء و المدار البيضوي. حيث انها سوف تتقاطع و تتجة نحو الصفر في المدر القطع المكافيء لكن بالنسبة للمدار البيضوي تكتسب هذه الداله سلوك غير مستقرمع قمم صغيرة منفصله و متلاشية بزيادة قيم z. كان هناك توافق جيد عندما تم مقارنة النتائج مع الدراسات المنشورة في هذا المجال.

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1. Introduction

Astronomy is the scientific study of all celestial bodies and phenomena. Astronomy's main objective was to observe the locations of the moon, the sun, and the planets in the solar system for navigational purposes and scientific interests until the telescope was designed in addition to the laws of motion and gravity were discovered in the 17^{th} century [1, 2]. Astrophysics has been part of astronomy since the late 19th century. Astrophysics is the study of using data analysis, physics, remote sensing, and image processing to understand the nature of celestial objects and the physical developments behind their creation, as well as radiation emission [3, 4]. Researchers are also very interested in the gases and dust particles that surround the Earth and are found in the space between stars. The diversity of atoms in nature can be linked to a cosmos that was formed in its early stages completely of hydrogen, helium, and a tiny amount of lithium, according to studies of the nuclear processes of power stars. Cosmology, the study of cosmic evolution, focuses on large-scale events. Cosmology was formerly only a field of speculation, but astrophysics has transformed it into a contemporary science with verifiable predictions [5, 6]. In classical mechanics, the two-body problem is a prediction of the movement of two bodies that are practically treated as point particles. This requires that the two bodies only interact with each other; all other bodies are ignored, and the only force acting on each body comes from the other [7, 8]. The universal variable formulation is a solution to Kepler's two-body problem. It is an extended form of Kepler's equation that takes into account parabolic and hyperbolic orbits in addition to elliptical orbits [9]. Thus, it is applicable to many circumstances in the solar system that involve orbits with a wide range of eccentricities [10]. Centripetal force and centrifugal force are two forces that work on a body (a satellite) in the solar system and maintain it in its own orbit. While the centrifugal force seeks to push the satellite away from the Earth's core, the centripetal force seeks to pull the satellite toward it. As a result, their influence is cancelled. An important aspect of understanding the dynamics of the elliptical orbit is how the gravitational force varies along the orbital route according to the elliptical orbit [11]. The body in an elliptical route follows the same path until it completes the period since the ellipse is a closed curve. A simple particular case of an ellipse is the circular orbit [12]. A body that moves along a parabolic path is on its way to infinity and will never return along the same route again [13]. Scientists use a variety of methods to solve Kepler's equation [14, 15]. Peter used the Bessel function to solve Kepler's equation [16]. Kepler's equation is solved iteratively and non-iteratively by Omar, Rasha, and Abdul-Rahman in order to calculate the eccentric anomaly [17, 18]. In this work, the optimal reference orbit is determined by studying theoretical astronomy.

2. Methodology

2.1. Representation of Kepler's Universal Equation

The variable anomaly at time (t_o+t) is calculated iteratively by means of solving Kepler's universal variable equation as follows [10]:

$$\sqrt{\mu}\,\Delta t = \frac{r_o v_{ro}}{\sqrt{\mu}}\chi^2 C(\alpha\chi^2) + (1 - \alpha r_o)\chi^3 S(\alpha\chi^2) + r_o\chi \quad (1)$$

Where r_o is the radius of the body at $t = t_o$, v_{ro} is the radial velocity of the body at $t = t_o$, χ is the universal anomaly measured in $km^{1/2}$ and expressed by [10]:

$$\chi = \begin{cases} \frac{h}{\sqrt{\mu}}\tan\frac{\theta}{2} & \text{Parabola} \\ \sqrt{a}E & \text{Ellipse} \\ \sqrt{-a}F & \text{Hyperbola} \end{cases} \quad (2)$$

Where h is the angular momentum, μ is the Earth's gravitational constant, that results from multiplying the mass of the Earth by the gravitational constant, a is the semi-major axis, θ is the true anomaly for elliptical orbit, E is the eccentric anomaly for elliptical orbit, F is the eccentric anomaly for hyperbola orbit, and α is the reciprocal of the semi-major axis, it is represented by [7, 10]:

$$\alpha = \frac{1}{a} \tag{3}$$

If its value, $\alpha < 0$ for hyperbola orbit, $\alpha = 0$ for parabola orbit, and $\alpha > 0$ for elliptical orbit, as [7, 10]:

2.2. Stumpff Functions

The infinite series defines the Stumpff function classes C(z) and S(z) as [10]:

$$S(z) = \sum_{k1=0}^{\infty} (-1)^{k} \frac{z^{k1}}{(2k1+3)!} = \frac{1}{6} - \frac{z}{120} + \frac{z^2}{5040} + \frac{z^3}{362880} + \frac{z^4}{39916800} + \frac{z^5}{6277020800} + \cdots \tag{4}$$

$$C(z) = \sum_{k1=0}^{\infty} (-1)^{k1} \frac{z^{k1}}{(2k1+2)!} = \frac{1}{2} - \frac{z}{24} + \frac{z^2}{720} + \frac{z^3}{40320} + \frac{z^4}{3628800} + \frac{z^5}{479001600} + \tag{5}$$

z is a dimensionless parameter represented by [10]:

$$z = \alpha\chi^2 \tag{6}$$

The final representation of Stumpff functions is [10]:

$$S(z) = \begin{cases} \frac{\sqrt{z} - \sin\sqrt{z}}{(\sqrt{z})^3} & z > 0 \\ \frac{\sinh\sqrt{-z} - \sqrt{-z}}{(\sqrt{z})^3} & z < 0 \\ \frac{1}{6} & z = 0 \end{cases} \tag{7}$$

$$C(z) = \begin{cases} \frac{1 - \cos\sqrt{z}}{z} & z > 0 \\ \frac{\cosh\sqrt{-z} - \sqrt{-1}}{-z} & z < 0 \\ \frac{1}{2} & z = 0 \end{cases} \tag{8}$$

2.3. Steps of Solution

The steps below are necessary to start the program and accomplish the study's requests:

- Input the semi-major axis value.
- Calculate the reciprocal of the semi-major axis.
- Calculate the universal anomaly.
- Calculate z value.
- Calculate Stumpff functions values.
- Calculate Kepler’s equation for universal variables.
- Plot the results.
- Re-calculate the steps above for another type of orbit (go to the first step).

This program is applicable to any type of orbit in celestial mechanics.

3. Results and Discussion

Three types of orbits (parabola, ellipse, and hyperbola) are used to apply the results for the study; Table 1 defines the z limitations that are employed in the program. The first row describes the type of the orbit. The second row illustrates the limits of z. Stumpff functions from Fig. 1 are both non-negative values. They propagate infinitely as the value of z approaches -40. Besides, they have a tendency toward zero. Fig. 2 shows that C(z) and S(z) are both positive values, and as the value of z reaches -120, it increases without limitation. Fig. 3 clarifies that C(z) and S(z) are both positive values. As the z value becomes closer to -200, it continues to increase noticeably. From Fig. 4, Stumpff functions increase without bounds as z approaches 40; they cross at z = 20, then deviate as the value of z increases. Fig. 5 simplifies that as the value of z gets closer to 120, C(z) function expands infinitely and S(z) function has positive values. C(z) and S(z) from Fig. 6 both increase as the value of z approaches 200; they are crossing at z = 60, 110, and 190. The C(z) function for the elliptical orbit contains a slight peak that decreases as z approaches infinity. The S(z) function has positive values, and when approaches infinity, Stumpff function grows, as shown in Fig. 7. Fig. 8 demonstrates that Stumpff functions expand without bounds as z approaches infinity. The C(z) function gradually fades as the value of z increases to 2000. S(z) function still has positive values and approaches infinity, as shown in Fig. 9. The results are compared with [6] and they show high convergence.

Table 1: Describes z limits utilized in the program.

Type of orbit	Hyperbola	Parabola	Ellipse
Limits of z	-40 to 0	0 t0 40	0 to 400
	-120 to 0	0 to 120	400 to 1200
	-200 to 0	0 to 200	1200 to 2000

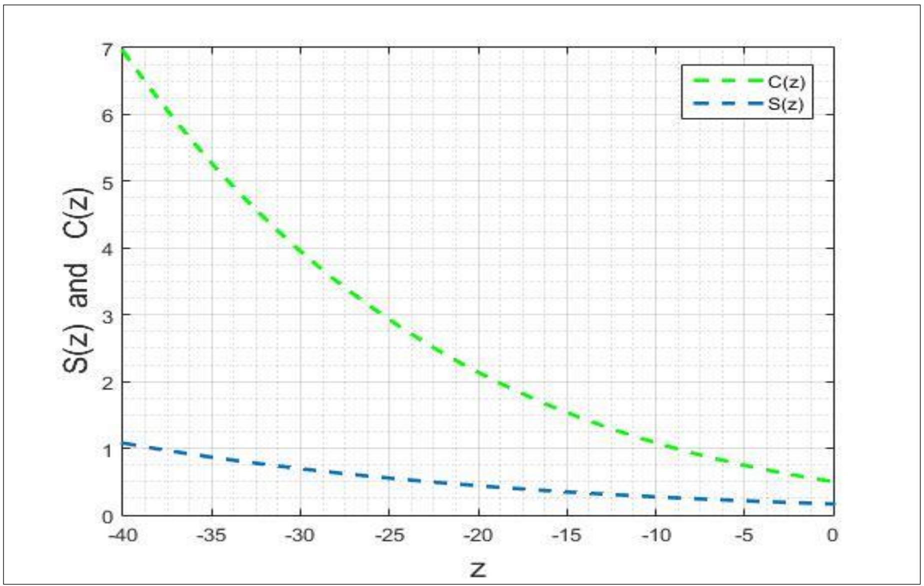

Figure 1: Stumpff functions for hyperbola orbit (z = - 40 t0 0).

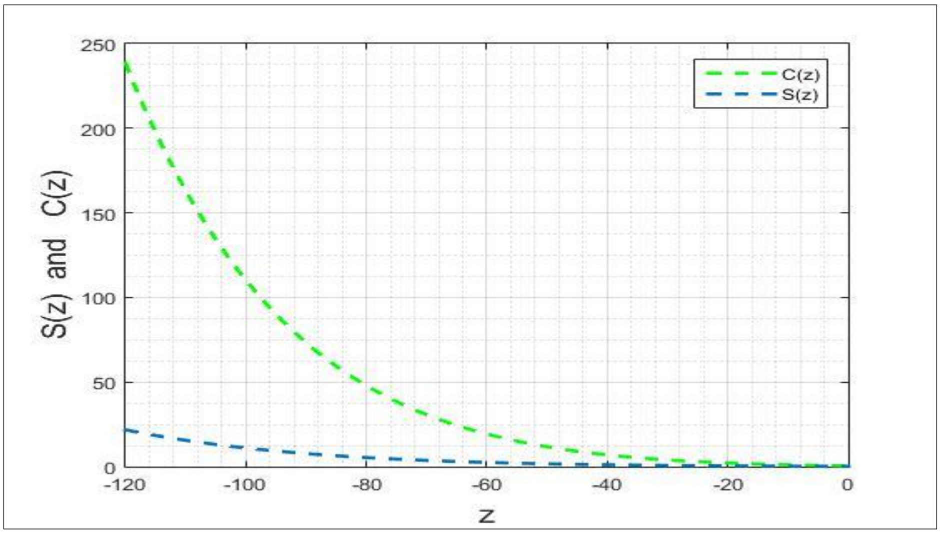

Figure 2: Stumpff functions for hyperbola orbit (z = - 120 to 0).

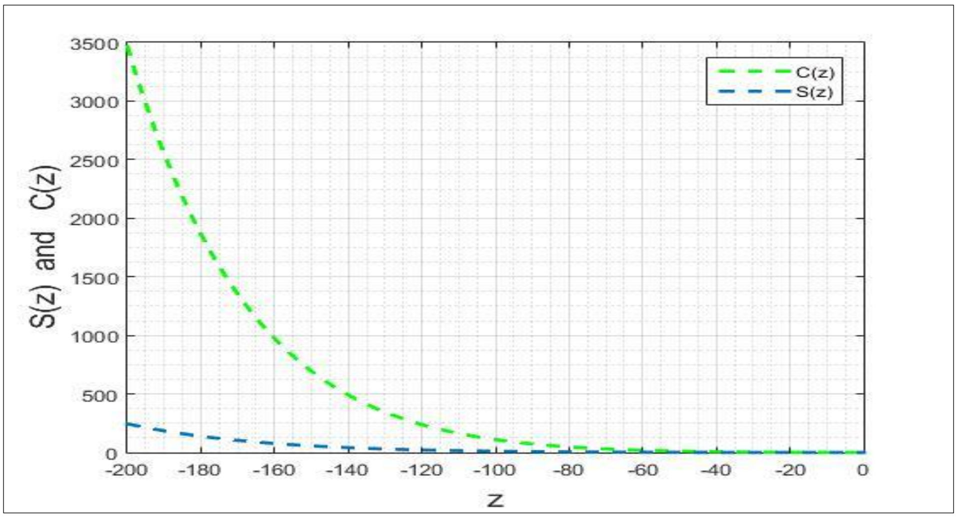

Figure 3: Stumpff functions for hyperbola orbit (z = - 200 t0 0).

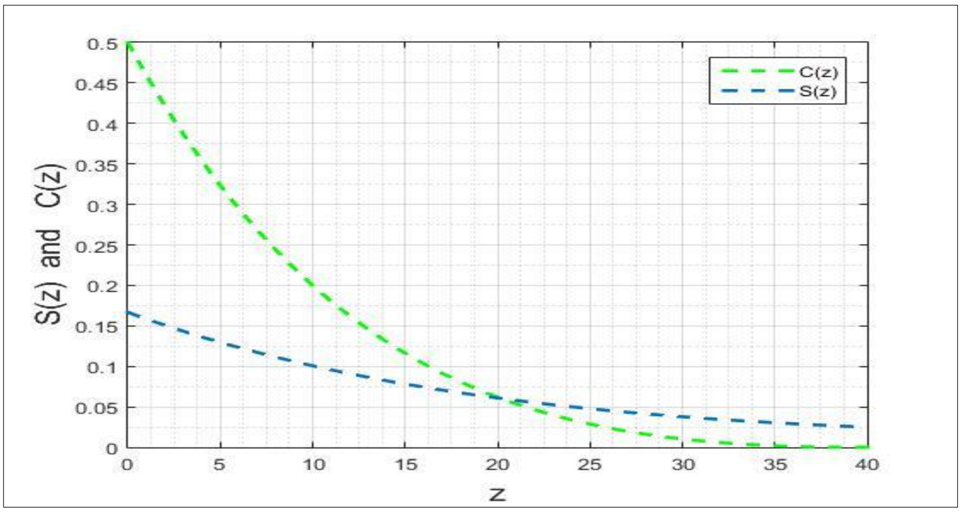

Figure 4: Stumpff functions for parabola orbit (z = 0 to 40).

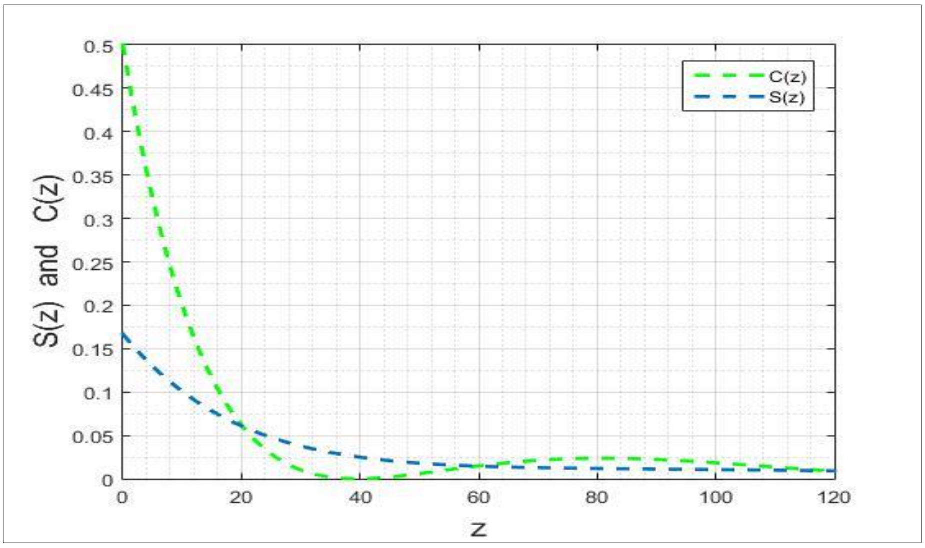

Figure 5: Stumpff functions for parabola orbit (z = 0 to 120).

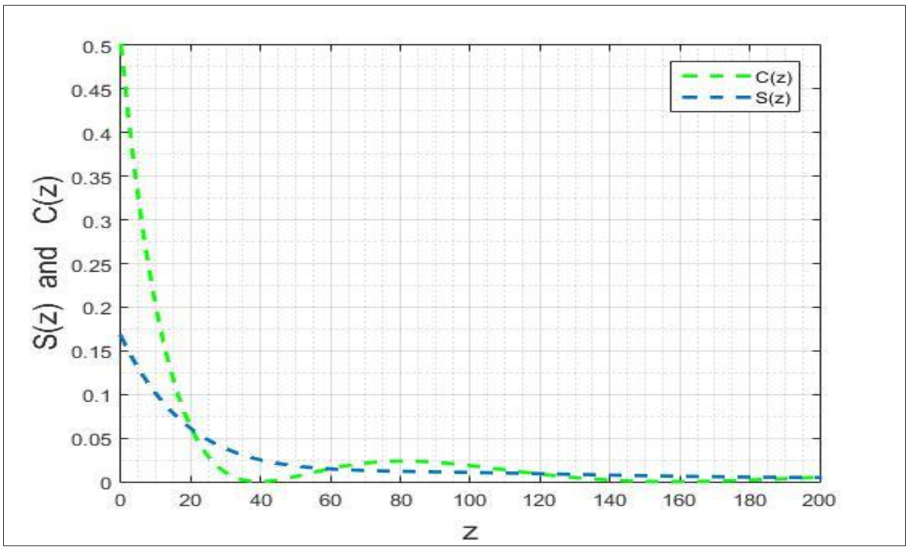

Figure 6: Stumpff functions for parabola orbit (z = 0 to 200).

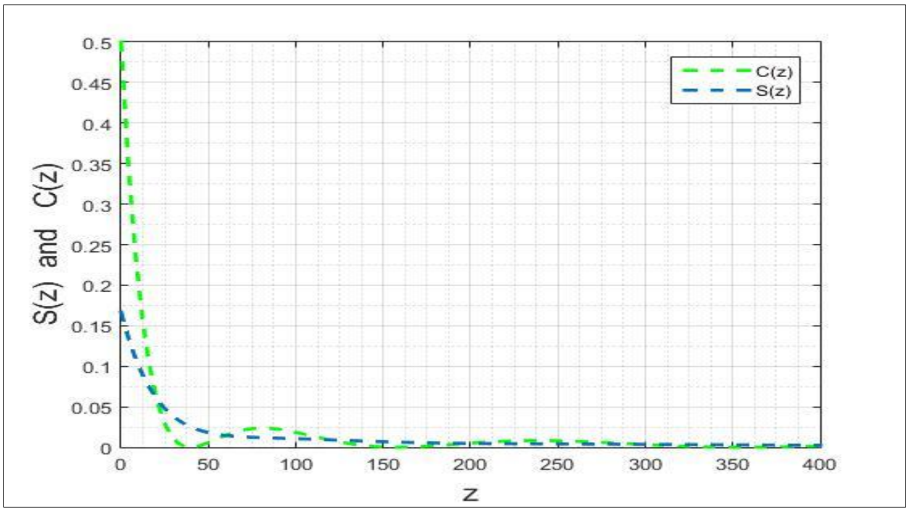

Figure 7: Stumpff functions for ellipse orbit (z = 0 to 400).

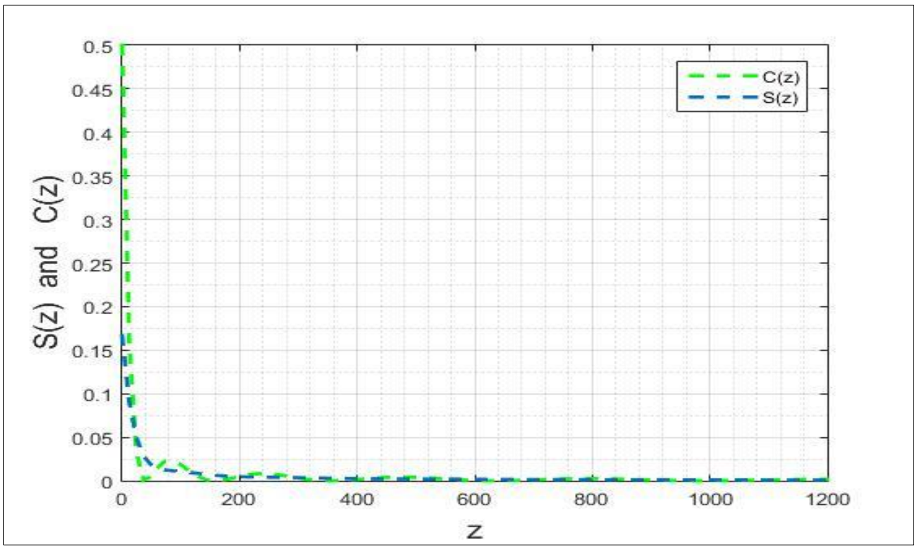

Figure 8: Stumpff functions for ellipse orbit (z = 400 to 1200).

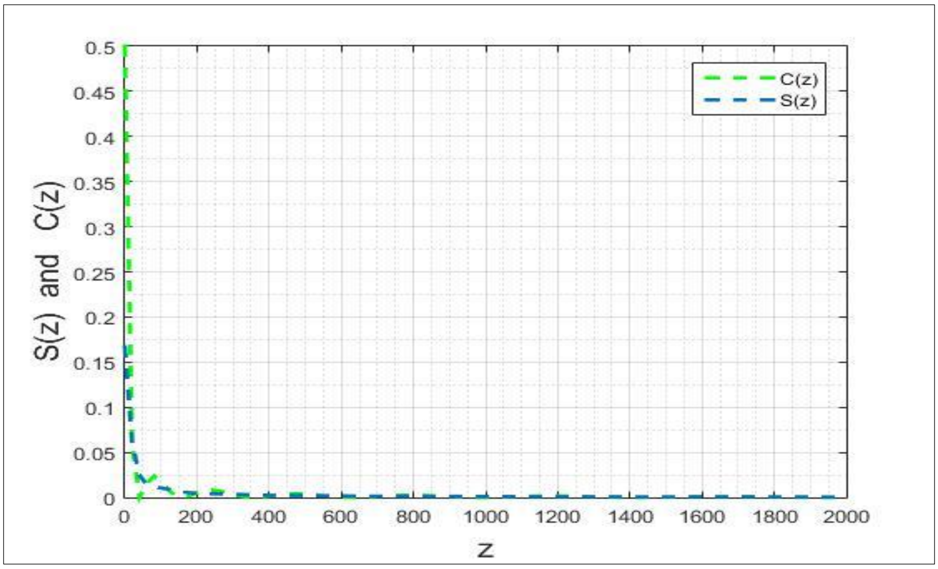

Figure 9: Stumpff functions for ellipse orbit (z = 0 to 2000).

4. Conclusions

The following are the conclusions based on the discussion:

1. Stumpff functions for hyperbola orbits have a tendency to infinity as the value of z directed towards minus infinity excludes S(z) function. It approaches 0.
2. For parabola orbits, the two functions intersect with each other at z = 20 and they decrease a little with z. For additional ranges for z, C(z) has a little change while S(z) decreases step by step.
3. For elliptical orbits, the two functions exhibit unsteady behavior at the beginning of x-axis with different little peaks vanishing as z increases. C(z) has a value from (0-0.5), while S(z) has a value from (0-0.17), then those two functions will be steady as z approaches to infinity.

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