THE IMPACT OF DISEASE AND HARVESTING ON THE DYNAMICAL BEHAVIOR OF PREY PREDATOR MODEL

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Abstract.
In this paper, a harvested prey-predator model involving infectious disease in prey is considered. The existence, uniqueness and boundedness of the solution are discussed. The stability analysis of all possible equilibrium points are carried out. The persistence conditions of the system are established. The behavior of the system is simulated and bifurcation diagrams are obtained for different parameters. The results show that the existence of disease and harvesting can give rise to multiple attractors, including chaos, with variations in critical parameters.

تأثير المرض والحصاد على السلوك الديناميكي لنظام الفريسة والمفترس

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المستخلص
تناولنا في هذا البحث نظام حصاد الفريسة والمفترس والذي يتضمن مرض في الفريسة. وجود، ووحدانية وقيد الحل للنظام نوقشة. تحليل الاستقرارية لجميع النقاط الثابتة الممكنة للنظام درس. شروط الاصرار للنظام وجدت. بمحاكاة سلوك النظام عدد من مخططات التفرع رسمت ولقيم معالم مختلفة. بينت النتائج بان وجود المرض والحصاد في النظام يؤدي الى وجود جواذب مختلفة بضمنها الفوضى.

1. Introduction:

It is well know that, in nature species does not exist alone. In fact, any given habitat may contain dozens or hundreds of species, some times thousands. Consequently, the possibility of spread of the disease in a community becomes larger as the number of infected species in the habitat increases. Accordingly, the study of the effect of disease on the dynamical behavior of interacting species has a vital biological significance in ecology. In the last two decades, numbers of prey-predator models with infectious diseases have been investigated [1-5]. All these models, reached at the conclusion that disease may cause vital changes in the dynamics of an ecosystem.

On the other side, the study of population dynamics with harvesting is a subject of mathematical bio-economics, and it is related to the optimal management of renewable resources [6]. Therefore the impact of harvesting on the dynamical behavior of interacting species has been considered by many researchers [7-9]. Most of these studies reached to the following conclusions: harvesting may be used as a biological control for the coexistence of the species, but unregulated harvesting might lead to extinction in one or more species. Recently, Chattopadhyay et al [9], proposed and analyzed a harvested prey-predator system with infection on prey population. They assumed that, the predator feeds on the susceptible prey population according to Holling type-II functional response (nonlinear type), while it feeds on infected prey population according to Lotka-Volterra type of functional response (linear type). They observed that harvesting of infected prey may be used as a biological control for the persistence in an infected prey-predator system.

In this paper, the harvested prey-predator model of Chattopadyay et al, is modified by assuming that the predator species feeds on both the susceptible prey species and the infected prey species, according to the Holling type-II functional response. The possibility of occurrence of chaotic behavior is considered. The impact of disease and harvesting on the dynamical behavior (especially chaotic dynamic) are studied analytically as well as numerically. The persistence conditions of the model are established.

2. The Mathematical Model:

Let $S(t)$ and $I(t)$ be the numbers of the susceptible and infected prey population at time t respectively. Let $Z(t)$ be the number of the predator population at time t. The dynamics of a harvested prey-predator model with infection on prey population can be represented by the following set of differential equations:

$$\frac{dS}{dt} = S\left[r\left(1-\frac{S+I}{K}\right)-cI-\frac{\mu Z}{\gamma+S+bI}-E_1\right] \quad (1a)$$

$$\frac{dI}{dt} = cSI-\frac{\alpha IZ}{\gamma+S+bI}-\lambda I-E_2 I \quad (1b)$$

$$\frac{dZ}{dt} = -\theta Z+\frac{g\,SZ}{\gamma+S+bI}+\frac{hIZ}{\gamma+S+bI}-E_3 Z \quad (1c)$$

Where the positive parameters ($r, K, c, \alpha, \gamma, \lambda, \theta, h, \mu, b, g$) are defined as following: The constants r and K are, respectively, the intrinsic growth rate and carrying capacity of the prey species in the absence of predation and harvesting; The constants $c, \alpha,$ and γ represent the infection rate, maximum attack rate, and the half saturation coefficient, respectively; The constants λ and θ denote to the death rates of the infected prey and the predator, respectively; The constant h represents the growth rate of predator due to predation of infected prey and hence it can be written as $h = e\alpha$ with $0 < e < 1$; The constant μ represents the amount of handled susceptible prey in unit time; The constant b denotes to attack rate of infected prey relative to susceptible prey; Finally, the constant g represents the growth rate of the predator due to predation of susceptible prey. Moreover the non-negative constants $E_1, E_2,$ and E_3 are the harvesting efforts for the susceptible prey, infected prey and predator, respectively.

Obviously, the right hand sides of Eqs. (1a-1c) are continuously differentiable functions on $R_+^3 = \{(S, I, Z) \in R^3, S \geq 0, I \geq 0, Z \geq 0\}$ and hence they are Lipschitzian functions. Therefore the solution of system (1) exists and is unique. Furthermore, the solution of system (1) with non-negative initial conditions is bounded as shown in the following theorem. It is easy to verify that, the necessary condition of coexistence of all species in system (1) is given by

$$r - E_1 > 0 \Leftrightarrow r > E_1 \quad (2)$$

Therefore, from now onward, we assume that condition (2) is always holds.

Theorem (1): All the solutions of system (1), which initiate in R_+^3 are uniformly bounded if the following condition holds

$$\frac{g}{\mu}\left(\frac{r}{K}+c\right)-\frac{h}{\alpha}c > 0 \quad (3)$$

Proof: Let $(S(t), I(t), Z(t))$ be any solution of the system (1) with positive initial conditions. Since we have

$$\frac{dS}{dt} \leq rS\left(1-\frac{S}{K}\right)-E_1 S.$$

Then according to the comparison theorem [10, pp. 31], we obtain that $\lim\limits_{t\to\infty} Sup\, S(t) \leq \frac{K(r-E_1)}{r}$, which gives $S(t) \leq \frac{K(r-E_1)}{r}; \forall t \geq 0$.

Let $W(t) = \frac{g}{\mu}S(t)+\frac{h}{\alpha}I(t)+Z(t) \quad (4)$

Now according to condition (3), the time derivation of Eq. (4) along the solution of system (1) can be written as

$$\frac{dW}{dt} \leq \frac{rg}{\mu}S-\frac{E_1 g}{\mu}S-\frac{h}{\alpha}I[\lambda+E_2]-Z[\theta+E_3].$$

So, if we assume $L = \min\{E_1, \lambda+E_2, \theta+E_3\}$, we get

$$\frac{dW}{dt} \leq \frac{rg}{\mu}S-L\left[\frac{g}{\mu}S+\frac{h}{\alpha}I+Z\right], \text{ or}$$

$$\frac{dW}{dt} \leq \frac{rg}{\mu}S-LW \Rightarrow \frac{dW}{dt}+LW \leq \beta$$

where $\beta = \frac{Kg(r-E_1)}{\mu}$. Again, by applying the comparison theorem on the above differential inequality we obtain that: $\lim_{t\to\infty} Sup W(t) \le \frac{\beta}{L} \Rightarrow W(t) \le \frac{\beta}{L}; \forall t \ge 0$

Hence all the solutions of system (1) that initiate in R_+^3 are confined in the region $B = \left\{(S, I, Z) \in R_+^3 : W < \frac{\beta}{L} + \varepsilon \text{ for any } \varepsilon > 0\right\}$.

Thus these solutions are uniformly bounded, and then the proof is complete. ■

It is well known that, the ecological system is said to be dissipative if the solution of the system, which initiate in R_+^3 is uniformly bounded as $t \to \infty$. Therefore, system (1) is dissipative.

3. Two-dimensional subsystems analysis:

There are two of two-species subsystems to be considered. The first subsystem is obtained by assuming the absence of predator species (i.e. $Z = 0$).

$$\frac{dS}{dt} = S\left[r\left(1 - \tfrac{S+I}{K}\right) - cI - E_1\right] = g_1(S, I)$$
$$\frac{dI}{dt} = I[cS - \lambda - E_2] = g_2(S, I) \qquad (5)$$

However, the second subsystem is obtained by assuming the absence of infected species (i.e. $I = 0$)

$$\frac{dS}{dt} = S\left[r\left(1 - \tfrac{S}{K}\right) - \tfrac{\mu Z}{\gamma + S} - E_1\right]$$
$$\frac{dZ}{dt} = Z\left[-\theta + \tfrac{gS}{\gamma + S} - E_3\right] \qquad (6)$$

Now, the existence and stability analyses of all possible equilibrium points of these subsystems are carried out and the following results are obtained:

Subsystem (5) has at most three non-negative equilibrium points, namely $q_0 = (0,0)$, $q_1 = \left(\frac{K(r-E_1)}{r}, 0\right)$ and $q_2 = (\bar{S}, \bar{I})$, where $\bar{S} = \frac{\lambda + E_2}{c}$; $\bar{I} = \frac{cK(r-E_1) - r(\lambda + E_2)}{c(r + cK)}$ .Clearly, q_0 and q_1 are always exist, while q_2 exists in the interior of R_+^2 of SI-plane under the following condition

$$cK(r - E_1) > r(\lambda + E_2) \qquad (7)$$

In addition to above, q_0 is unstable saddle point; while q_1 is locally asymptotically stable provided that

$$K < \frac{r(\lambda + E_2)}{c(r - E_1)} \qquad (8)$$

Note that, the satisfying of condition (8) means that there is no positive equilibrium point in the $Int.R_+^2$ of SI-plane or equivalent the infected population will face extinction while the susceptible population grows to its maximum value. Finally, q_2 is locally asymptotically stable in $Int.R_+^2$ of SI-plane whenever it exists.

Keeping the above in view, the global stability of the positive equilibrium point q_2 is investigated in the following theorem.

Theorem (2): Suppose that the positive equilibrium point q_2 of the subsystem (5) exists, then it is a globally asymptotically stable in the $Int.R_+^2$ of SI-plane.

Proof: Assume that $H(S, I) = \frac{1}{SI}$. Clearly, $H(S, I) > 0$ be C^1 function in the $Int.R_+^2$ of SI-plane. Now, since

$$\Delta(S, I) = \frac{\partial(Hg_1)}{\partial S} + \frac{\partial(Hg_2)}{\partial I} = \frac{-r}{IK} < 0$$

Hence $\Delta(S, I)$ does not change sign and is not identically zero in the $Int.R_+^2$ of the SI-plane. Then according to Bendixson-Dulic criterion [11, pp. 26], there is no periodic solution in the $Int.R_+^2$ of SI-plane.

Now, since all the solutions of the subsystem (5) are uniformly bounded and q_2 is a unique positive equilibrium point in the $Int.R_+^2$ of SI-plane.. Hence, by using the Poincare-Bendixson theorem [11], q_2 is a globally asymptotically stable. ■

Similarly, subsystem (6) has at most three non-negative equilibrium points, namely $Q_0 = (0,0)$, $Q_1 = q_1$ and $Q_2 = (S', Z')$ where $S' = \frac{\gamma(\theta + E_3)}{\psi}$; $Z' = \frac{\gamma g[K(r-E_1)\psi - r\gamma(\theta + E_3)]}{K\mu\psi^2}$ here $\psi = (g - \theta - E_3)$. Note that, Q_0 and Q_1 are always exist, while Q_2 exists in $Int.R_+^2$ of SZ-plane under the following conditions;

$$g > \theta + E_3 \qquad (9a)$$
$$K(r - E_1)\psi > r\gamma(\theta + E_3) \qquad (9b)$$

Now, it is easy to verify that Q_0 is unstable saddle point; while Q_1 is locally asymptotically stable provided that

$$K < \frac{r\gamma(\theta + E_3)}{(r - E_1)\psi} \qquad (10)$$

Obviously, satisfying condition (10) means that there is no positive equilibrium point in in $Int.R_+^2$ of SZ-plane or equivalently the predator population will face extinction while the susceptible population grows to its maximum

value. Finally, Q_2 is locally asymptotically stable in $Int.R_+^2$ of SZ-plane
if and only if

$$K < \frac{r(\gamma + 2S')}{(r - E_1)} \quad (11)$$

While, Q_2 is unstable and there is a stable limit cycle surrounding it provided that;

$$K > \frac{r(\gamma + 2S')}{(r - E_1)} \quad (12)$$

Moreover, the global stability of the positive equilibrium point Q_2 is investigated in the following theorem.

Theorem (3): Suppose that the positive equilibrium point Q_2 of the subsystem (6) exists and is locally asymptotically stable in $Int.R_+^2$ of SZ-plane. Then it is a globally asymptotically stable in the $Int.R_+^2$ of SZ-plane

Proof: Similar to proof of theorem (2) with the Dulac function $B(S,Z) = \frac{1}{SZ}$. ■

4. Stability analysis of system (1) with Persistence:

In this section, the existence of the equilibrium points, stability analysis and persistence of system (1) are discussed. It is observed that, the following equilibrium points exists for the system (1)

1. The equilibrium point $P_0 = (0,0,0)$ always exists.
2. The axial equilibrium point $P_1 = \left(\frac{K(r-E_1)}{r}, 0, 0\right)$ always exists.
3. The predator free equilibrium point $P_2 = (\bar{S}, \bar{I}, 0)$, where $\bar{S}$ and $\bar{I}$ are given in q_2, exists in the $Int.R_+^2$ of the SI-plane under condition (7).
4. The disease free equilibrium point $P_3 = (S', 0, Z')$, where S' and I' are given in Q_2, exists in $Int.R_+^2$ of SZ-plane under conditions (9a) and (9b).
5. The positive equilibrium point $P_4 = (S^*, I^*, Z^*)$; where $I^* = \frac{S^*\psi - \gamma(\theta + E_3)}{\theta b + E_3 b - h}$, $Z^* = \frac{(cS^* - \lambda - E_2)(\gamma + I^* b + S^*)}{\alpha}$, while S^* is a posi-tive root of the second order equation:

$$r\left(1 - \frac{S + I^*}{K}\right) - cI - \frac{\mu Z^*}{(\gamma + S + bI^*)} - E_1 = 0$$

exists uniquely in $Int.R_+^3$ if and only if one of the following sets of conditions holds:

$$\text{Max}\{\bar{S}, S'\} < S^* \text{ and } \frac{h}{b} < \theta + E_3 < g \quad (13a)$$

$$\bar{S} < S^* < S' \text{ and } \theta + E_3 < \frac{h}{b} \quad (13b)$$

$$\bar{S} < S^* \text{ and } g < \theta + E_3 < \frac{h}{b}. \quad (13c)$$

Now, the local dynamical behaviors of the system (1), around each of these equilibrium points, are carried out by computing the Jacobian matrix $J(P_i); i = 0,1,2,3,4$ and then compute the egenvalues for the resulting matrix. The following results are obtained:

The eigenvalues of $J(P_0)$ are $\lambda_{01} = r - E_1 > 0$, $\lambda_{02} = -(\lambda + E_2) < 0$, and $\lambda_{03} = -(\theta + E_3) < 0$. Hence, P_0 is unstable saddle point.

The eigenvalues of $J(P_1)$ are given by $\lambda_{11} = -(r + E_1) < 0$, $\lambda_{12} = \frac{cK(r-E_1)}{r} - \lambda - E_2$, and $\lambda_{13} = -\theta + \frac{gK(r-E_1)}{r\gamma + K(r-E_1)} - E_3$. Therefore, P_1 is locally asymptotically stable provided that the following condition holds:

$$K < \min.\left\{\frac{r(\lambda + E_2)}{c(r - E_1)}, \frac{r\gamma(\theta + E_3)}{(r - E_1)(g - \theta - E_3)}\right\} \quad (14)$$

Clearly condition (14) means that, the planer equilibrium points P_2 and P_3 do not exist. However, P_1 is unstable saddle point if and only if at least one of the planer points P_2 or P_3 exists.

The eigenvalues of $J(P_2)$ satisfy the following relations;

$$\lambda_{21} + \lambda_{22} = -\frac{r}{cK}(\lambda + E_2) < 0 \quad (15a)$$

$$\lambda_{21}.\lambda_{22} = \frac{[cK(r - E_1) - r(\lambda + E_2)]}{cK} > 0 \quad (15b)$$

$$\lambda_{23} = \frac{\bar{S}\psi + \bar{I}(h - \theta b - E_3 b) - \gamma(\theta + E_3)}{(\gamma + \bar{S} + b\bar{I})} \quad (15c)$$

where λ_{2j}; for $j = 1,2,3$ represent the egienvalues in the S, I, Z-direction respect-tively. Clearly, the real part of λ_{21} and λ_{22} have the same sign as that of $\lambda_{21} + \lambda_{22}$, moreover, P_2 is locally asymptotically in $Int.R_+^3$ or unstable saddle point, depending on whether the eigenvalue λ_{23} is negative or positive respectively.

Similarly, it is easy to verify that, the eigenvalues of $J(P_3)$ satisfy the following relations

$$\lambda_{31} + \lambda_{33} = -\frac{rS'}{K} + \frac{\mu Z'S'}{(\gamma + S')^2} < 0 \quad (16a)$$

$$\lambda_{31}.\lambda_{33} = \frac{\mu\, g\gamma\, S'Z'}{(\gamma + S')^3} > 0 \quad (16b)$$

$$\lambda_{32} = \frac{(cS' - \lambda - E_2)(\gamma + S') - \alpha Z'}{(\gamma + S')} \quad (16c)$$

Clearly, the real part of λ_{31} and λ_{33} have the same sign as that of $\lambda_{31} + \lambda_{33}$, moreover, P_3 is locally asymptotically stable in $Int.R_+^3$ or unstable saddle point, depending on whether the eigenvalue λ_{32} that describes the dynamics in the I-direction is negative or positive respectively.

Finally, the Jacobian matrix of system (1) at the positive equilibrium point P_4 is given by $J(P_4) = (a_{ij})_{3\times 3}$, where

$a_{11} = S^*\left(-\frac{r}{K} + \frac{\mu Z^*}{B^2}\right)$, $a_{12} = S^*\left(-\frac{r}{K} - c + \frac{\mu b Z^*}{B^2}\right)$

$a_{13} = -\frac{\mu S^*}{B} < 0$, $a_{21} = cI^* + \frac{\alpha I^* Z^*}{B^2} > 0$,

$a_{22} = \frac{\alpha b I^* Z^*}{B^2} > 0$, $a_{23} = -\frac{\alpha I^*}{B} < 0$,

$a_{31} = Z^*\left(\frac{\gamma g + I^*(bg-h)}{B^2}\right)$, $a_{32} = Z^*\left(\frac{\gamma h + S^*(h-bg)}{B^2}\right)$,

$a_{33} = 0$; with $B = \gamma + S^* + bI^*$.

Then the characteristic equation of $J(P_4)$ is given by

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \quad (17)$$

Where

$$A_1 = -(a_{11} + a_{22})$$
$$A_2 = -(-a_{11}a_{22} + a_{23}a_{32} + a_{12}a_{21} + a_{13}a_{31})$$
$$A_3 = a_{23}(a_{11}a_{32} - a_{12}a_{31}) + a_{13}(a_{22}a_{31} - a_{21}a_{32})$$

Therefore, by substituting the values of a_{ij}, and then simplify the terms we obtain:

$$A_1 = \frac{rB^2S^* + Z^*K(\mu S^* + \alpha bI^*)}{KB^2} \quad (18)$$

Thus $A_1 > 0$ if and only if the following condition holds

$$rB^2S^* > Z^*K(\mu S^* + \alpha bI^*) \quad (19)$$

Now, according to the signs of the elements of the Jacobian matrix $J(P_4)$ together with the following conditions:

$$0 < I^* < \frac{\gamma g}{h - bg} \text{ or } 0 < S^* < \frac{\gamma h}{bg - h} \quad (20)$$

$$1 < \frac{\alpha}{\mu} < \frac{\gamma h + S^*(h - bg)}{\gamma g + I^*(bg - h)} \quad (21)$$

It is easy to verify that

$$A_3 = \frac{S^*I^*Z^*}{KB^5}\Big[r\alpha B^2\big([\gamma h + S^*(h-bg)] - [\gamma g + I^*(bg-h)]\big).$$
$$+ cKB^2\big(\mu[\gamma h + S^*(h-bg)] - \alpha[\gamma g + I^*(bg-h)]\big)\Big] > 0$$

Moreover,

$$\Delta = A_1A_2 - A_3 = -(a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21})$$
$$+ a_{31}(a_{11}a_{13} + a_{12}a_{23})$$
$$+ a_{32}(a_{22}a23 + a_{13}a_{21})$$

Thus, by substituting the values of a_{ij}, and then simplifying the resulting terms, we obtain that:

$$\Delta = A_1A_2 - A_3 = \frac{1}{KB^5}[M_1 - hM_2]$$

Where

$$M_1 = S^*I^*B\big(rB^2S^* - Z^*K(\mu S^* + \alpha bI^*)\big)$$
$$\big(\alpha Z^*(c + \tfrac{r}{K}) - bZ^*(c\mu + \tfrac{r\alpha}{K}) + cB^2(c + \tfrac{r}{K})\big)$$
$$+ \mu\gamma g S^{*2}Z^*(rB^2 - \mu KZ^*)$$
$$+ \gamma g\alpha KS^*I^*Z^*\big(B^2(c + \tfrac{r}{K}) - \mu bZ^*\big)$$
$$+ bgKS^*I^*Z^*(\alpha I^* + \mu S^*)(c + \tfrac{r}{K})B^2$$
$$+ bgKS^*I^*Z^{*2}(\alpha - \mu)(\alpha I^*b + \mu S^*)$$

And

$$M_2 = KS^*I^*Z^*(\alpha I^* + \mu S^*)(\tfrac{r}{K} + c)B^2$$
$$+ KS^*I^*Z^{*2}(\alpha - \mu)(\alpha I^*b + \mu S^*) + \alpha^2\gamma K bI^{*2}Z^{*2}$$
$$+ \alpha\gamma K\mu S^*I^*Z^{*2} + c\mu\gamma KB^2S^*I^*Z^*$$

Obviously, in addition, to conditions (19) and (21), $M_1 > 0$ and $M_2 > 0$ if and only if the following condition holds:

$$b \le \frac{(\alpha Z^* + cB^2)(c + \frac{r}{K})}{Z^*(c\mu + \frac{r\alpha}{K})} \quad (22)$$

Further, it is easy to verify that $\Delta > 0$, if in addition to conditions (19), (21) and (22) the following condition satisfies:

$$h < \frac{M_1}{M_2} \quad (23)$$

Therefore, according to the above analysis, the following theorem can be easily proved.

Theorem (4): Assume that the positive equilibrium point P_4 exists in $Int.R_+^3$. Then P_4 is locally asymptotically stable if and only if the conditions (19), (21), (22) and (23) are hold.

Proof. Follow directly by applying Routh-Hurwitz criteria. ■

It is well known that, biologically the persistence of the system means that the survival of all population of the system in future time, however mathematically it means that strictly positive solution do not have omega limit set in the boundary planes of non-negative cone. Therefore, the persistence condition of system (1) is established in the next theorem. This theorem is applicable when there are no nontrivial periodic solutions in the boundary planes.

Theorem (4): Assume that there are no nontrivial periodic solutions in the boundary planes of system (1). Then the necessary conditions for the persistence of system (1) are

$\lambda_{23} \geq 0$ (24a)

$\lambda_{32} \geq 0$ (24b)

Where λ_{23} and λ_{32} are given in Eq. (15c) and (16c) respectively. However, the sufficient conditions for the persistence of the system (1) are

$\lambda_{23} > 0$ (25a)

$\lambda_{32} > 0$ (25b)

Proof: The boundedness of the solution is proved in theorem (1). Also λ_{23} and λ_{32} are the eigenvalues, which gives the stability in the positive direction orthogonal to the SI and SZ-planes respectively. So, if there are nontrivial periodic solutions in the SI-plane and if $\lambda_{23} < 0$ then there are orbits in the positive cone, which approach P_2. Therefore condition (24a) is one of the necessary conditions for the persistence. Similarly, with respect to P_3, the other necessary condition (24b) holds.

Now in order to prove that, conditions (25a) and (25b) are the sufficient conditions for persistence we have to consider system (1).

Let $\frac{ds}{st} = g_1(S,I,Z)$, $\frac{dI}{dt} = g_2(S,I,Z)$ and $\frac{dZ}{dy} = g_3(S,I,Z)$, then it is clear that the following conditions are satisfied:

1. $\frac{\partial g_1}{\partial I} = \frac{-r}{K} - c + \frac{\mu Zb}{B^2} < 0$; for the regions sufficiently close to the SI-plane. $\frac{\partial g_1}{\partial Z} = \frac{-\mu}{B} < 0$; $\frac{\partial g_2}{\partial S} = c + \frac{\alpha Z}{B^2} > 0$; $\frac{\partial g_2}{\partial Z} = \frac{-\alpha}{B} < 0$ $\frac{\partial g_3}{\partial S} = \frac{g(\gamma+bI)-hI}{B^2}$; $\frac{\partial g_3}{\partial I} = \frac{-g\,S\,b+h(\gamma+S)}{B^2}$. Further $\frac{\partial g_3}{\partial S}$ and $\frac{\partial g_3}{\partial I}$ remains positive for the regions sufficiently close to planes SZ and IZ.
2. The prey grows in the absence of predation, infection and harvesting that is $g(0,0,0) = r > 0$, and $\frac{\partial g_1}{\partial S} = \frac{-r}{K} + \frac{\mu Z}{B^2}$ then we obtain $\frac{\partial g_1}{\partial S}(S,0,0) = \frac{-r}{K} < 0$. However the predator population dies in the absence of the preys (i.e. $g_3(0,0,0) = -(\theta + E_3) < 0$).
3. There are no equilibrium points in the IZ-plane.
4. In the absence of susceptible prey the predator can't survive on the infected prey. Therefore, there exists two planer equilibrium points $P_2 = (\bar{S},\bar{I},0)$ and $P_3 = (S',0,Z')$ in the SI and SZ-planes respectively.

Hence, according to Freedman and Waltman theorem [12], the system (1) persists if the conditions (25a) and (25b) are satisfied. ■

Theorem (5): Suppose that the conditions (25a) and (25b) hold and there are a finite number of limit cycles in the SZ-plane. Then, the persistence conditions for system (1) take the form

$$\int_0^T g_2\left(\bar{\phi}(t),0,\bar{\psi}(t)\right)dt > 0 \qquad (26)$$

for each limit cycle $\left(\bar{\phi}(t),\bar{\psi}(t)\right)$ in the SZ-plane, here T is the time period of the limit cycle.

Proof: Let there exists a limit cycle in the SZ-plane, then the Jacobian matrix about the limit cycle $S(t) = \bar{\phi}(t)$, $I(t) = 0$, $Z(t) = \bar{\psi}(t)$ can be obtained as follows:

$$J(S,I,Z) = \begin{bmatrix} g_1 + S\frac{\partial g_1}{\partial S} & S\frac{\partial g_1}{\partial I} & S\frac{\partial g_1}{\partial Z} \\ 0 & g_2 & 0 \\ Z\frac{\partial g_3}{\partial S} & Z\frac{\partial g_2}{\partial I} & g_3 + Z\frac{\partial g_2}{\partial Z} \end{bmatrix}$$

Consider a solution of system (1) with positive conditions $(\alpha_1,\alpha_2,\alpha_3)$ sufficiently close to the limit cycle. Then from the jacobian matrix, $\frac{\partial I}{\partial \alpha_2}$ is a solution of the system

$$\frac{dI}{dt} = \left[g_2\left(\bar{\phi}(t),0,\bar{\psi}(t)\right)\right]I;\ I(0) = 1.$$

That is

$$\frac{\partial I}{\partial \alpha_2}(t,\alpha_1,\alpha_2,\alpha_3) = \exp\left[\int_0^t g_2\left(\bar{\phi}(t),0,\bar{\psi}(t)\right)dt\right]$$

Hence, by Taylor's expansion theorem, we have $I(t,\alpha_1,\alpha_2,\alpha_3) - I(t,\alpha_1,0,\alpha_3)$

$$\cong \exp\left[\int_0^t g_2\left(\bar{\phi}(t),0,\bar{\psi}(t)dt\right)\right]\alpha_2$$

Then I increases or decreases according as $\int_0^T g_2\left(\bar{\phi}(t),0,\bar{\psi}(t)\right)dt$, is positive or negative respectively.

Since P_3 and these limit cycles (in the SZ-plane) are the only possible limits of trajectories with positive initial conditions, then these trajectories go away from the SI and SZ-planes if the conditions (25a), (25b) and (26) are hold. Hence the proof is complete. ■

5. Numerical Analysis:

In this section, the global dynamics of the solution of system (1) in the positive octant is investigated numerically. One of the most important systematic tools for investigation of the dynamics is constructing a bifurcation diagram, as a function of changes in a control parameter keeping the rest of parameters fixed. System (1) is solved numerically using the Predictor-Corrector method with sixth order Runge-Kutta method, and then successive maxima of the predator species Z is plotted as a function of control parameter. Fig. 1(a). shows the bifurcation diagram of system (1) as a function of the intrinsic growth rate (r) in the range $0 \le r \le 9$. Keeping other parameters fixed as the following:

$$\begin{aligned} &K = 400, c = 0.06,\ \mu = 15, \gamma = 50, \\ &b = 0.5, E_1 = 0.0, \alpha = 4, \lambda = 3.4, \qquad (27) \\ &E_2 = 0.0, \theta = 5, g = 10, h = 2, E_3 = 0.0 \end{aligned}$$

The figure shows the sensitivity of the solution for the changing in the parameter r. It is observed that, there is alternate between the chaotic and periodic regions. Further, as r increases the chaotic regions become narrower and the system approaches to periodic behavior. Typical attracting set of the system (1) is drawn in Fig. 1(b) for $r = 1$ with the rest of parameters given by Eq. (27). Fig. 1(b) a long with its time series, as shown in Fig. 1(c), shows clearly the chaotic dynamic of system (1).

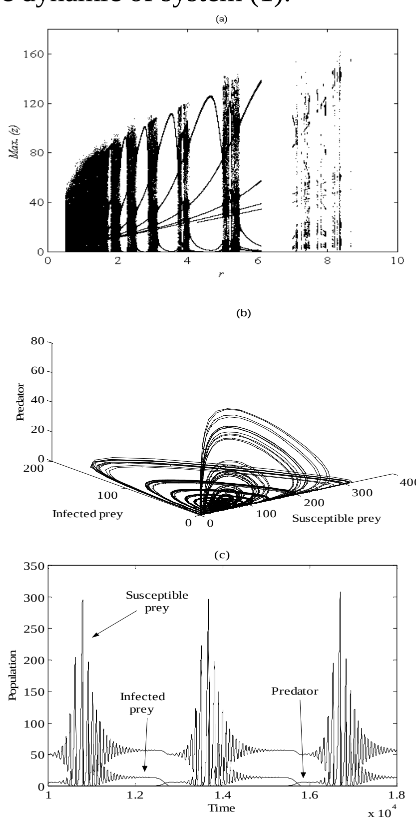

Figure 1.(a): Bifurcation diagram as a function of r in the range $0 \le r \le 9$, keeping other parameters as in Eq. (27).(b)Chaotic attractor for $r = 1$. (c) Time series for attractor in (a).

Another bifurcation diagram as a function of the infection rate (c), varying in the range $0.02 \le c \le 0.07$ keeping the rest of parameters as in Eq. (27) with $r = 1$, is drawn in Fig. 2(a) and it is blown up Fig. 2(b) in the range $0.02 \le c \le 0.03$, these figures demonstrate the rout to chaos through sequence of periodic doubling. Moreover, it is observed that, as c increases further $c > 0.065$, the predator species will face extinction and the solution will approach to predator free equilibrium point in the SI -plane.

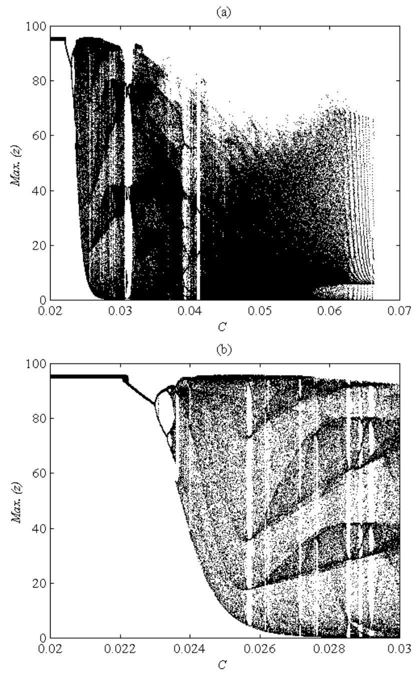

Figure 2. (a): Bifurcation diagram as a function of c in the range $0.02 \le c \le 0.07$, keeping other parameters as in Eq. (27) with $r = 1$.(b) Blown up of (a) in the range $0.02 \le c \le 0.03$.

The bifurcation diagram in Fig. 3(a). is drawn between the successive maxima of the predator species and the natural death rate of infected species (λ) in the range $2 \le \lambda \le 7$, keeping other parameters fixed as in Eq. (27) with $r = 1$. The figure confirms our expectation regarding to the sensitivity of the solution to the λ. The chaotic behavior of the solution is clearly visible in the range $2 < \lambda \le 6$, in between there is number of periodic regions, see for example $3.25 < \lambda < 3.3$, $3.45 < \lambda < 3.5$, $4.47 < \lambda < 4.6$.

In addition to the above it is observed that, as the infection rate increases further $\lambda > 9.5$ the infected species I faces extinction and then the $3D$-system (1) will be reduced to $2D$-subsystem (6). Typical attracting set a long with its time series are drawn in Figs. 3 (b-c) for

$\lambda = 9.6$ keeping other parameters as in Eq. (27) with $r = 1$.

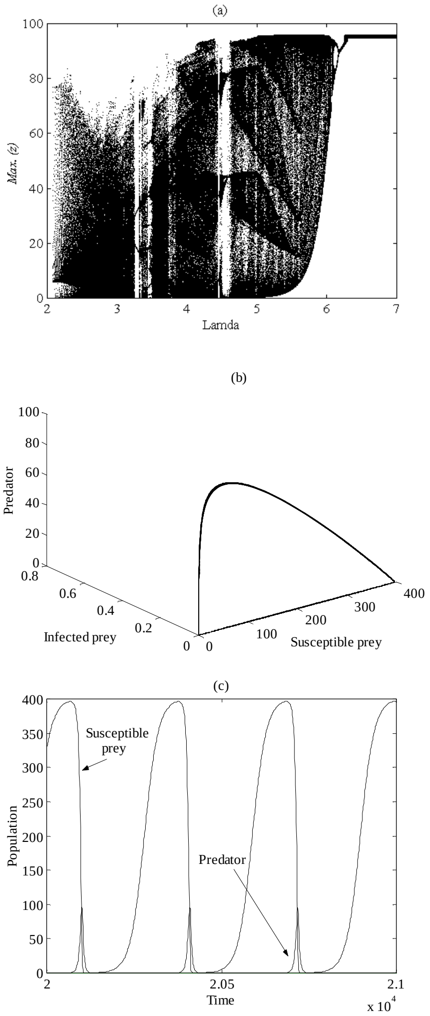

Figure 3.(a): Bifurcation diagram as a function of λ in the range $2 \le \lambda \le 7$, keeping other parameters as in Eq.(27) with $r = 1$.(b)Periodic attractor in $Int.R_+^2$ of SZ -plane. (c) Time series of Figure 3(b).

Now, In Figs. 4(a-c), the global dynamical behavior of the system (1) is discussed, through plotting the bifurcation diagrams as a function of the harvest rate $E_i(i = 1,2,3)$ in the ranges $0 \le E_1 \le 0.6$, $0 \le E_2 \le 6.5$, and $0 \le E_3 \le 0.25$ respectively, keeping other parameters fixed as in Eq. (27) with $r = 1$. In all of these figures, it is observed that, the system has different types of the dynamical behavior including chaos. Clearly as the parameters $E_i(i = 1,2,3)$ increase, the dynamic of the system (1) will return to periodic state from the chaotic state and hence the harvest rates have a stabilizing effect on the system (1).

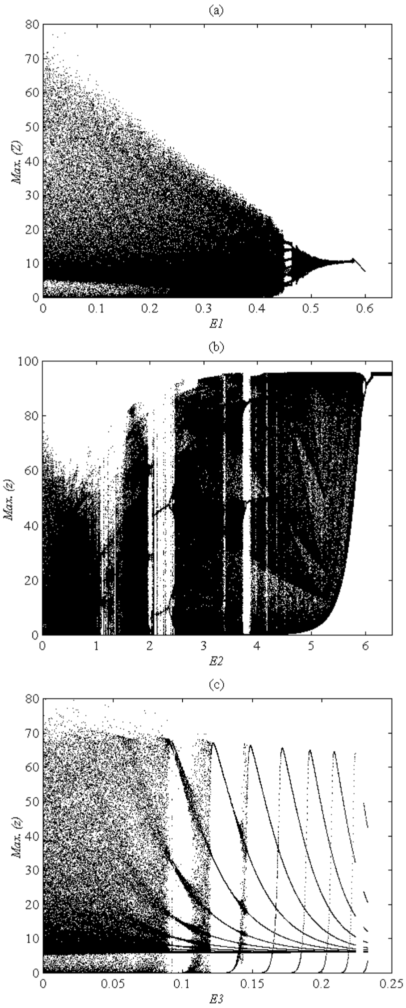

Figure 4: Bifurcation diagram as a function of harvest rates for the parameters given in Eq. (27) with $r = 1$. (a) $0 \le E_1 \le 0.6$. (b) $0 \le E_2 \le 6.5$. (c) $0 \le E_3 \le 0.25$

In the following an investigation for the effect of harvested rate E_2 of infected prey species on the dynamical behavior of system (1), the attracting sets of system (1) are drawn in Figs. (5a-5e) for $E_2 = 0, 0.2, 0.4, \; 1.0, 1.25$ respectively keeping other parameters fixed as in

$$r = 1, K = 400, c = 0.06, \mu = 15, \gamma = 50,$$
$$b = 0.5, E_1 = 0.4, \alpha = 4, \lambda = 3.4, \theta = 5, \qquad (28)$$
$$g = 10, h = 2, E_3 = 0.02$$

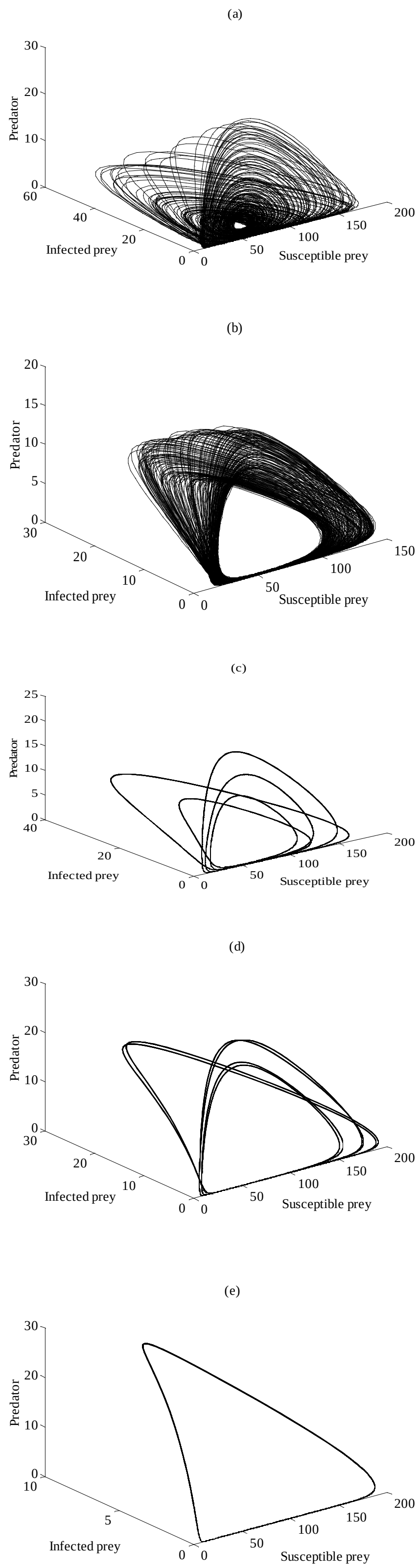

Figure 5:.Attracting sets of system (1) for parameters set as given in Eq. (28). (a) Chaotic attractor for $E_2 = 0.0$ (b) Chaotic attractor for $E_2 = 0.2$ (c) Long periodic attractor for $E_2 = 0.4$ (d) Long periodic attractor for (e) Period-1 attractor for $E_2 = 1.25$ in $Int.R^3_+$.

It is clear from Figures 5 (a-e) that, the system (1) still chaotic in case of $E_1 = 0.4, E_2 = 0.0,$ and $E_3 = 0.02$, while as E_2 increases the solution of system (1) returns to periodic attractor from the chaotic attractor as shown in case of $E_1 = 0.4$, $E_2 = 1.25$, and $E_3 = 0.02$.

6. Discussions and conclusions:

The mathematical model of chapter two is modified so that the predator species feeds on both the susceptible population and infected population according to the Holling type-II functional response. In order to investigate the effects of the disease and the harvesting on the dynamical behavior of the prey-predator model, the stability analysis of the modified model is carried out analytically as well as numerically. The conditions of the survival of all the species are established. It is observed that, on contrast to the prey-predator model given in chapter two, the predator species in the modified model can survive on the susceptible population even in case of the absence of the infected population. The complexity of the system increases due to adding the new non-linear term in the first and third equations of the system (1). Further, we can not found a Lyapunov function for the system (1), instead of that the global stability of the system (1) is studied numerically and the following conclusions can be drawn:

1. For small value of the intrinsic growth rate $(r < 8.5)$, the system (1) has rich dynamics including chaos and periodic as shown in Figure 1 (a-c). However, as r increases further the system return to stable dynamic.
2. For small value of infection rate (c), the system (1) undergo periodic dynamic due to the rarity of the infected prey and the existence of the alternate food (susceptible) for predator, while as c increase slightly the system (1) enter to the chaotic regions through sequence of periodic doubling as shown in Figure 2(a-b). Moreover, increases c further causes extinction in predator species due to the effect of disease on the susceptible prey species as well as predator species, and then the solution approach to a stable point in SI -plane.
3. The system (1) is very sensitive to the natural death rate of the infected prey (λ) as shown in Fig. 3(a). It is observed that, for $2 < \lambda \leq 6$ the system (1) has a rich dynamics including chaos, however as λ increase

further $(\lambda > 9.5)$, the system (1) return to periodic dynamic in $Int.R_+^2$ of SZ-plane due to the extinction of infected prey species as shown in Figure 3(b-c).

4. According to the Figures. 4(a-c) it is observed that, in general the chaotic behavior of the system (1) can be avoided, and then the system return to periodic state by increasing the harvesting rates E_i ($i = 1, 2, 3$) slightly. Moreover, it is observed that increasing E_3 further the system will be facing extinction and the trajectory will approaches to stable point in $Int.R_+^2$ of SI-plane, however increasing E_2 further (i.e. $E_2 > 6$) the $3D$-system (1) will reduces to $2D$-system (6) and then the trajectory approaches to periodic dynamic in SZ-plane.

5. From Figure 5(a-e), it is clearly that the harvesting rates E_i ($i = 1, 2, 3$) work as a control parameters on the system. Especially, when we choose E_i ($i = 1,3$) small and varying E_2 regularly.

Finally, according to the above observations to control the chaotic behavior of the system (1) and hence control the disease, the value of the intrinsic growth rate of the susceptible prey should be taken as $r \geq 8.5$.

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