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Common Best Proximity Points in CRÇ$_b$ −Metric Spaces

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Abstract

Throughout this paper, a new generalization of complex valued CRÇ$_b$ -metric space is used. This definition was stated in published research as a previous work. In this paper, some new results on common best proximity points for non-self-mappings are obtained in these spaces. It is an extension of previous results proven in classical complex metric spaces. The requirements of the work are as follows: first, we present some equivalent statements for convergence and Cauchy sequences. Second, we define complex b- rectangular metric space by using the function distance of a recent space. Finally, we present the concepts of commute proximity, weak Q- property and weakly dominate mappings. These concepts are illustrated by some examples. Furthermore, some new results on common best proximity points for non-self-mappings are obtained in these spaces. As well as the extension of previous results have been proven in classical complex metric spaces. The work required the employment of conditions on mappings, which are commute proximity and continuity and one of the mappings weakly dominates other. In addition to other appropriate conditions. As a direct result, the existence of a common unique fixed point has been proved.

Keywords: Complex metric spaces, Best proximity point, Commute proximally, Weakly dominates, Fixed point.

أفضل نقاط القرب المشتركة في CRÇ$_b$− الفضاء المتري

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الخلاصة

خلال هذا البحث تم استخدام تعميم جديد للفضاء المتري ذو القيمة المعقدة يسمى الفضاء CRG$_b$− المتري المستطيل المعقد. ورد هذا التعريف في بحث منشور كعمل سابق . في هذه الورقة، تم الحصول على بعض النتائج الجديدة حول أفضل نقاط القرب المشتركة لتطبيقين غير الذاتيين في هذه الفضاءات. وهو امتداد للنتائج السابقة المثبتة في الفضاءات المترية المعقدة الكلاسيكية. متطلبات العمل هي، أولا، تقديم بعض العبارات المكافئة للتقارب ومتتابعات كوشي، ثانيا، تعريف "الفضاء المتري المستطيل ب المركب" باستخدام المسافة الدالة لمسافة حديثة، وأخيرا، تقديم مفاهيم القرب المتنقل، خاصية Q− الضعيفة والتطبيقات المهيمنة

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بضعف. ويتم توضيح هذه المفاهيم من خلال بعض الأمثلة وقد تطلب العمل توظيف شروط على التطبيقات وهي القرب المتبادل و الاستمرارية و أن أحد التطبيقين يهيمن على الآخر بشكل ضعيف. بالإضافة إلى غيرها من الشروط المناسبة. وكنتيجة مباشرة تم إثبات وجود نقطة صامدة وحيدة مشتركة.

1. Introduction

In complete metric spaces, the first appearance of the existence of a unique fixed point for a simple contraction mapping was made by the mathematician Banach 1922. After that, the results and applications of this theorem were repeated, making the metric fixed point field one of the most active areas of research during the last century, see [1]. The work in this field has expanded in various axes:

- Proving Banach Theorem in different spaces, such as generalizations of the metric space itself or in other spaces like: b-metric space, G-metric space, cone metric space, Cat(k) space, normed space, modular space, Hilbert space, etc.

-Proving theorems for more than one contraction mappings, so the study turns to the existence of common fixed points, coincidence or coupled points; replacing contraction by anther more general one such as non-expansive mapping, pseudo contraction, etc.

As well as employing these axes in different branches in mathematics and other sciences.

As a brief mention of a simple aspect of that achievement, see [2-11], as well as their references.

The traditional Ky Fan's theorem in best approximations included: there is a best approximation x (in normed space) $\mathbb{X}$ from the point $v \in H \subset \mathbb{X}$ such that $\| v - T(v) \| = dist(T(v), H)$, where H is convex compact and $T$is continuous mapping on C, [12]. Many extensions for this theorem and its generalization for a non-self-mapping T in topological vector space are done for various aim, see Vetrivel and et.al. [13], O'Regan and et.al. [14], Basha and Shahzad [15], Mohammed and Abed [16] and Lateef [17]. Azam Fisher and Khan ([18] and [19]) presented the complex-valued distance function as generalization of the usual distance with presenting some results on fixed point. Recently, many authors proved theorems on the existence of best proximity points and its related fixed points in generalizations of complex –valued spaces, like by Aghayanet et al. [19], Meena [20], Choudhury and et al. [21], Aghayan and et al. [22] and Ege and et al. [23]. Here, we will adopt the CR$Ç_b$-metric spaces to study the existence of best proximity elements for non-self-mappings. In many branches of mathematics and other related fields, the complex valued distance is the useful such as in number theory, algebraic geometry, applied Mathematics as well as in physics including thermodynamics, hydrodynamics, electrical engineering, mechanical and engineering. This gives an incentive to introduce a generalization of the complex-valued metric space. The aim of this work is showing some new results in type of complex metric spaces.

2. Preliminaries

Let $\mathbb{R}$:= the set of real numbers with order relations $\leq, <$.

Let $\mathbb{C}$:= the set of complex numbers with order relations $\precsim$ and $\prec$ when define as:

For $\delta_1, \delta_2 \in \mathbb{C}$, $\delta_1 \precsim \delta_2$ if and only if
Re $(\delta_1) \leq$ Re (δ_2) and Im $(\delta_1) \leq$ Im (δ_2).

Also, $\delta_1 \prec \delta_2$ means that $\delta_1 \neq \delta_2$, if and only if
Re $(\delta_1) <$ Re (δ_2) and Im $(\delta_1) <$ Im (δ_2) [9].

In addition, $\delta_1 \precsim \delta_2$ holds if one of the following conditions is satisfied:

(i) $Re(\delta_1) = Re(\delta_2)$ and $Im(\delta) = Im(\delta_2)$,

(ii) $Re(\delta_1) < Re(\delta_2)$ and $Im(\delta_1) = Im(\delta_2)$,

(iii) $Re(\delta_1) = Re(\delta_2)$ and $Im(\delta_1) < Im(\delta_2)$,
(iv) $Re(\delta_1) < Re(\delta_2)$ and $Im(\delta_1) < Im(\delta_2)$.

In particular, [20] we write $\delta_1 \precsim \delta_2$ if $\delta_1 \neq \delta_2$ and one of previous cases (i), (ii), and (iii) is satisfied where can be write $\delta_1 \prec \delta_2$ if only (iv) is satisfied.

Also, note that, for δ_1, δ_2 and $\delta_3 \in \mathbb{C}$

(1) If $0 \precsim \delta_1 \prec \delta_2$ then $|\delta_1| < |\delta_2|$, where $|.|$ is the modulus of a complex number.

(2) If $\delta_1 \prec \delta_2$ and $\delta_2 \prec \delta_3$ then $\delta_1 \prec \delta_3$.

(3) If $a, b \in \mathbb{R}$ and $a \leq b$ then $a\delta \precsim b\delta$, $\forall\ \delta \in \mathbb{C}$.

(4) If $\mathrm{a}, \mathrm{b} \in \mathbb{R}$ and $0 \leq a \leq b$ and $\delta_1 \precsim \delta_1$ then $a\delta_1 \precsim b\delta_2$

Next, recall some basic tools of ordering on $\mathbb{C}$.

Definition 2.1 [2]

Let D subset of $\mathbb{C}$

(i) If there exists $g \in \mathbb{C}$ such that $z \precsim g\ \forall z \in D$ then D is bounded above and g is an upper bound.

(ii) If there exists $y \in \mathbb{C}$ such that $y \precsim z\ \forall z \in D$ then D is bounded below and y is a lower bound.

Definition 2.2 [2]

(i) For $D \subset \mathbb{C}$ with D is bounded above by g_1 such that for every upper bound g of D, $g_1 \precsim g$ then g_1 is called $\sup D$.

(ii) For $D \subset \mathbb{C}$ with D is bounded below by y_1 such that for every upper bound y of D, $y_1 \precsim y$ then y_1 is called $inf D$.

Suppose that $D \subset \mathbb{C}$ is bounded above then $\exists\ p = a + ib$ such that $z = u + iv \precsim p = a + ib, \forall z \in D$ this means that

(1) $u \precsim a$ and $v \precsim b, , \forall z = u + iv \in D$.

(2) If $E = \{u : z = u + iv \in D\} \subseteq \mathbb{R}$ and $F = \{v : z = u + iv \in D\} \subseteq \mathbb{R}$ are two bounded above. So, $u^* = \sup E$ and $v^* = inf\ F$ exist. Then $z^* = u^* + iv^* = \sup D$.

(3) Similarly, if D is bounded below. Then $z^* = u^* + iv^* = inf\ D$ where $u^* = \sup E$ and $v^* = inf\ F$.

(4) Any $D \subset \mathbb{C}$ which is bounded above has supremum. Also, any $D \subset \mathbb{C}$ which is bounded below has infimum.

Definition 2.3 [24]

Let $\mathbb{X}$ be a non-empty set and $s > 1$. mapping $Ç_b : \mathbb{X}^3 \to \mathbb{C}$ is called complex- valued rectangular $Ç_b$–metric for every $x, w, h \in \mathbb{X}$ and every two points $\vartheta \neq \alpha \in \mathbb{X} \backslash \{x, w, h\}$ such that

(i) $Ç_b(x, w, h) = 0$, if $x = w = h$,

(ii) $0 \prec Ç_b(x, x, w)$, for all $x, w \in \mathbb{X}$ with $x \neq w$,

(iii) $Ç_b(x, x, w) \precsim Ç_b(x, w, h)$, for all $x, w, h \in \mathbb{X}$ with $w \neq h$,

(iv) $Ç_b(x, w, h) = Ç_b(x, w, h) = Ç_b(w, x, h) = \cdots$(All three variables are symmetrical),

(v) $Ç_b(x, w, h) \precsim s\big(Ç_b(x, \vartheta, \vartheta) + Ç_b(\vartheta, \alpha, \alpha) + Ç_b(\alpha, w, h)\big)$, (Pentagonal inequality), for every $x, w, h \in \mathbb{X}$ and every two points $\vartheta \neq \alpha \in \mathbb{X} \backslash \{x, w, h\}$.

Then the pair $(\mathbb{X}, Ç_b)$ is called a complex- valued rectangular $Ç_b$–metric space. (Shortly CR$Ç_b$MS).

Lemma 2.4 [24]

Let a, b, c be a non-negative real number the following statement holds, for any $r \in \mathbb{N}$

$$(a + b + c)^r \preccurlyeq 3^{r-1}(a^r + b^r + c^r), \text{ for all } r \in \mathbb{N}.$$

To illustrate Definition 2.3, we solve the following example

Example 2.5

Let $\mathbb{X} = \mathbb{R}$ and we define $Ç_b : \mathbb{X}^3 \to \mathbb{C}$ by formula

$$Ç_b(x, w, h) = (|x - w| + |w - h| + |x - h|)^r$$
$$+i(|x - w| + |w - h| + |x - h|)^r.$$

Note that $\forall\, x, w, h \in \mathbb{X}$ and non-negative points $c \neq b \in \mathbb{X}\backslash\{x, w, h\}$, and all $(r \geq 1)$. The condition from (i) to (iv) of Definition 2.3 is hold. To show that (v) is hold. By Lemma 2.4, we get

$$\begin{aligned} Ç_b(x, w, h) &= (|x - w| + |w - h| + |x - h|)^r \\ &\quad +i(|x - w| + |w - h| + |x - h|)^r \\ &\preccurlyeq (|x - c| + |c - w| + |w - h| + |x - c| + |c - h|)^r \\ &\quad +i(|x - c| + |c - w| + |w - h| + |x - c| + |c - h|)^r \\ &\preccurlyeq 3(|x - c| + |x - c| + |c - b| + |c - b|)^r \\ &\quad +p\,(|b - w| + |w - h| + |h - b| + |b - h|)^r \\ &\quad +3i(|x - c| + |x - c| + |c - b| + |c - b|)^r \\ &\quad +(|b - w| + |w - h| + |h - b| + |b - h|)^r \\ &\quad +3^{p-1}i(|b - w| + |w - h| + |h - b| + |b - h|)^r \\ &= 3\big(Ç_b(x, c, c) + Ç_b(c, b, b) + Ç_b(b, w, h)\big). \end{aligned}$$

Hence, $(\mathbb{X}, Ç_b)$ is a CR$Ç_b$MS with $s = 3^{r-1}$, for all $b, c \in \mathbb{X} \backslash\{x, w, h\}$,

Lemma 2.6 [24]

Let $(\mathbb{X}, Ç_b)$ be a R$Ç_b$MS and $\{x_r\}$ be a sequence in $\mathbb{X}$, then

(i) $\{x_r\}$ to be $Ç_b$ −converges to $x \iff |Ç_b(x, x_r, x_t)| \to 0$ as $r, t \to \infty$.

(ii) $\{x_r\}$ to be $Ç_b$ −Cauchy to $x \iff |Ç_b(x_r, x_t, x_l)| \to 0$ as $r, t, l \to \infty$.

Another needed definition is the continuity of a mapping defined on $\mathbb{X}$.

Definition 2.7

Let $(\mathbb{X}, Ç_b)$ be a CR$Ç_b$MS, mapping $T : \mathbb{X} \to \mathbb{X}$ is said to be continuous if for any three $Ç_b$ −convergent sequences $\{x_r\}$, $\{w_r\}$ and $\{h_r\}$ to x, w and h, respectively then $\{T(x_r, w_r, h_r)\}$ is converges to $T(x, w, h)$.

It is easy to get the following properties

Proposition 2.8

Let $(\mathbb{X}, Ç_b)$ be a CR$Ç_b$MS. Then the following are equivalent:

(i) $\{x_r\}$ to be $Ç_b$ −converges to point x;

(ii) $|Ç_b(x, x_r, x_r)| \to 0$ as $r \to \infty$;

(iii) $|Ç_b(x_r, x, x)| \to 0$ as $r, t \to \infty$;

(iv) $|Ç_b(x_r, x_t, x)| \to 0$ as $r, t \to \infty$.

It is necessary to address some properties of a CR$Ç_b$MS $(\mathbb{X}, Ç_b)$ that may not be realized so that we can address theories in light of these failures, [22] such as

(i) $\mathbb{X}$ is not necessary Hausdorff.

(ii) $Ç_b$ is not necessarily continuous.

(iii) Every $Ç_b$ −convergent sequence is $Ç_b$-Cauchy sequence.

(iv) Every $Ç_b$ −convergent sequence has unique limit.

Therefore, $(\mathbb{X}, Ç_b)$ is a Hausdorff CRÇ$_b$MS with $Ç_b$is continuous in three variables. The following proposition is a good tool to give some new definitions in next section.

Proposition 2.9

Let $(\mathbb{X}, Ç_b)$ be a CRÇ$_b$MS then the mapping $dÇ_b$ is a complex valued rectangular b-metric on $\mathbb{X}$, for any $x, w \in \mathbb{X}$

$$dÇ_b(x, w) = Ç_b(x, w, w) + Ç_b(w, x, x).$$

Proof: We must prove the conditions in [18, Definition 3.1] are fulfilled for $dÇ_b(x, w)$.

(1) If $dÇ_b(x, w) = 0$, then $Ç_b(x, w, w) + Ç_b(w, x, x) = 0$, implies that $x = w$. Part (i) $(Ç_b(x, w, h) = 0$, if $x = w = h)$.

(2) If $x = w$, then $Ç_b(x, w, w) + Ç_b(w, x, x) = 0$, and $dÇ_b(x, w) =0$.

(3) $dÇ_b(x, w) = Ç_b(x, w, w) = Ç_b(w, x, x)$
$$= Ç_b(w, w, x) + Ç_b(x, x, w) = dÇ_b(w, x).$$

(4) For points $\varphi \neq e \in \mathbb{X} \setminus \{x, w\}$, it follows that

$$\begin{aligned} dÇ_b(x, w) &= Ç_b(x, w, w) + Ç_b(w, x, x) \\ &\preccurlyeq s[Ç_b(x, \varphi, \varphi) + Ç_b(\varphi, e, e) + Ç_b(e, w, w)] \\ &\quad + s[Ç_b(w, e, e) + Ç_b(e, \varphi, \varphi) + Ç_b(\varphi, x, x)] \\ &= s[Ç_b(x, \varphi, \varphi) + Ç_b(\varphi, x, x) + Ç_b(\varphi, e, e) \\ &\quad + Ç_b(e, \varphi, \varphi) + Ç_b(e, w, w) + Ç_b(w, e, e)] \\ &= s[dÇ_b(x, \varphi) + dÇ_b(\varphi, e) + dÇ_b(e, w)]. \end{aligned}$$

3. Main results

Through this work, we suppose that $\mathcal{H}$ and K are non-empty subsets of CRÇ$_b$MS, $(\mathbb{X}, Ç_b)$. Therefore, $\{ dÇ_b(h, x): h \in \mathcal{H}, \ x \in K\} \subseteq \mathbb{C}$ is always bounded below by $y_0 = 0 + 0i$. Hence, $inf\{ dÇ_b(h, x): h \in \mathcal{H}, \ x \in K\}$ exists. Define

$$dÇ_b(\mathcal{H}, K) = inf\{ dÇ_b(h, x): h \in \mathcal{H}, \ x \in K\},$$

$\mathcal{H}_0 = \{h \in \mathcal{H} : dÇ_b(h, x) = dÇ_b(\mathcal{H}, K)$ for some $x \in K\}$,

$K_0 = \{x \in K : dÇ_b(h, x) = dÇ_b(\mathcal{H}, K)$ for some $h \in \mathcal{H}\}$,

where

$$dÇ_b(\mathcal{H}, K) = inf\ \{ dÇ_b(h, x): h \in \mathcal{H}, x \in K\}.$$

From above it clear that $\forall\, h \in \mathcal{H}_0$ there exist $x \in K_0$ such that $dÇ_b(h, x) = dÇ_b(\mathcal{H}, K)$ and the converse, is true.

Below, the definition of approximation is reformulated in CRÇ$_b$MS, $(\mathbb{X}, Ç_b)$.

Definition 3.1

An element $h \in \mathcal{H}$ is called common best proximity point of the mappings $T, S: \mathcal{H} \to K$ if

$$dÇ_b(h, Sh) = dÇ_b(\mathcal{H}, K) = dÇ_b(h, Th). \quad (1)$$

Example 3.2

Consider the CRÇ$_b$MS $(\mathbb{X}, Ç_b)$ for $s \geq 1$. Let $\mathbb{X} = \mathbb{C}$ and $dÇ_b: \mathbb{X}^2 \to \mathbb{C}$ define as

$$dÇ_b(x, w) = Ç_b(x, w, w) + Ç_b(w, x, x). \quad (2)$$

Let $dÇ_b(\delta_1, \delta_2) = |x_2 - x_1|^2 + i|w_2 - w_1|^2$ where $\delta_1 = x_1 + iw_1$, $\delta_2 = x_2 + iw_2$ and $r = 2$. Consider the subsets $\mathcal{H}$ and K of $\mathbb{X}$ given by

$$\begin{aligned} \mathcal{H} &= \{\delta \in \mathbb{C} : Re(\delta) = -1, \ \ 0 \leq Im(\delta) \leq 3\} \\ &\quad \cup \{\delta \in \mathbb{C} : Re(\delta) = 1, \ \ 0 \leq Im(\delta) \leq 3\} \\ K &= \{\delta \in \mathbb{C} : Re(\delta) = -2, \ \ 0 \leq Im(\delta) \leq 3\} \\ &\quad \cup \{\delta \in \mathbb{C} : Re(\delta) = 2, \ \ 0 \leq Im(\delta) \leq 3\}. \end{aligned}$$

Then we conclude $\mathcal{H}$ and K are closed and bounded, $\mathcal{H}_0 = \mathcal{H}, K_0 = K$

and $d\c{C}_b(\mathcal{H},K)=1+0i$.

Let $T,S:\mathcal{H}\to K$ define as

$$T\delta = 2|x| + iw, \text{ for each } \delta = x + iw \in \mathcal{H}$$

and $$S\delta = 2|x| + i\frac{w}{2}, \text{ for each } \delta = x + iw \in \mathcal{H}.$$

Then T,S satisfy Equation (1) of Definition 3.1, let $h=1+0i$ then

$$d\c{C}_b(h,Sh) = |1-2| + |0+0|i = 1+0i$$

$d\c{C}_b(h,Th) = 1+0i = d\c{C}_b(\mathcal{H},K)$.

Definition 3.3

The pair $(\mathcal{H},K)$ is said to have weak $Q-$ property if and only if for any $h_1,h_2\in\mathcal{H}$ and $x_1,x_2\in K$

$$\begin{cases} d\c{C}_b(h_1,x_1) = d\c{C}_b(\mathcal{H},K) \\ d\c{C}_b(h_2,x_2) = d\c{C}_b(\mathcal{H},K) \end{cases} \text{ implies } d\c{C}_b(h_1,h_2) \precsim d\c{C}_b(x_1,x_2) \text{ where } \mathcal{H}\neq\emptyset.$$

Example 3.4

Consider $\mathcal{H} = \{-3+iw:\ 0\le w\le 1\}$,

$$K = \{3+iw:\ 0\le w\le 1\}.$$

Define $d\c{C}_b(x,w) = |R(x)-R(w)|^2 + i|Im(x)-Im(w)|^2$.

Such that $d\c{C}_b(\mathcal{H},K) = |3+3|^2 + i|0-0|^2 = |3+3|^2 + i|1-1|^2 = 81+0i$.

To show that weak $Q-$ property.

Let $\delta_1,\delta_2\in\mathcal{H},\ x_1,x_2\in K$ such that

δ_1=$-3+0.5i,\ \delta_2=-3+0.7i$ and $x_1=3+0.5i, x_2=3+0.7i$.

To calculate $d\c{C}_b(\delta_1,x_1) = |R(\delta_1)-R(x_1)|^2 + i|Im(\delta_1)-Im(x_1)|^2$

$|-3-3|^2 + i|0.5-0.5|^2 = 81+0i$.

By the same manner we have $d\c{C}_b(\delta_2,x_2) = |-3-3|^2 + i|0.7-0.7|^2 = 81+0i$.

Hence, we get $d\c{C}_b(\delta_1,x_1) = d\c{C}_b(\delta_2,x_2) = 81+0i$.

And $d\c{C}_b(x_1,x_2) \preccurlyeq d\c{C}_b(\delta_1,\delta_2)$ since

$$d\c{C}_b(x_1,x_2) = |3-3|^2 + i|0.7-0.5|^2$$
$$= d\c{C}_b(\delta_2,x_2) = |-3+3|^2 + i|0.7-0.5|^2 = 0+0.04i.$$

Which proves the weak $Q-$ property for $\mathcal{H}$ and K.

Definition 3.5

The mappings $T,S:\mathcal{H}\to K$ is said to commute proximally if the following condition satisfied $d\c{C}_b(u,Sh) = d\c{C}_b(v,Tx) = d\c{C}_b(\mathcal{H},K)$ implies $Sv=Th$.

Example 3.6

Let $\mathcal{H},K\subset\mathbb{X}=\mathbb{C}$. Such that

$$\mathcal{H} = \{\delta\in\mathbb{X}:\ 0\le Re(\delta)\le 3 \text{ and } 0\le Im(\delta)\le 3\}$$
$$K = \{z\in\mathbb{X}:\ 2\le Re(z)\le 5 \text{ and } 2\le Im(\delta)\le 5\}.$$

Where $\delta=a+bi,\ z=c+di$. Define

$$\c{C}_b(x,w,w) = |x-w| + i|x-w|$$
$$d\c{C}_b(x,w) = |R(x)-R(w)| + i|Im(x)-Im(w)|$$

the mappings $T,S:\mathcal{H}\to K$ define as $T(\delta)=\delta+1+i$ and $S(\delta)=\delta+1+2i$.

Where $x=-1+i, h=2+i,\ u=1+i, v=2+0i$, for all $x,h,u,v\in\mathbb{X}$.

Compute $d\c{C}_b(\mathcal{H},K) = |5-3| + i|5-3| = |2-0| + i|2-0| = 2+2i$.

$$Tx = -1+i+1+i = 0+2i.$$

Now check distance $d\c{C}_b(u,Sh) = |1-3| + i|1-3| = 2+2i$.

$$= |2-0| + i|0-2| = d\c{C}_b(v,Tx) = d\c{C}_b(\mathcal{H},K) = 2+2i.$$

and $Sv = 2+0i+1+2i = 3+2i$,

$Th = 2+i+1+i = 3+2i$.

Therefore, $Sv = Th$.
So, we conclude T and S satisfy the condition of proximally commuting mappings in $\mathbb{X}$.

Definition 3.7

A mapping $T: \mathcal{H} \to K$ is said to dominates a mapping $S: \mathcal{H} \to K$ proximally if for each $0 \leq \mu < \frac{1}{s}$, for all $s > 1$, such that $\forall\, m_1, m_2, v_1, v_2, h_1, h_2 \in \mathcal{H}$ they satisfy the condition that

$$[d\c{C}_b(m_1, Sh_1) = d\c{C}_b(m_2, Sh_2) = d\c{C}_b(\mathcal{H}, K)] = d\c{C}_b(v_1, Sh_1) = d\c{C}_b(v_2, Sh_2) \Longrightarrow d\c{C}_b(m_1, m_2) \prec \mu d\c{C}_b(v_2, v_2).$$

Definition 3.8

Assume $(\mathbb{X}, \c{C}_b)$ be a CR $\c{C}_b$MS, a mapping $S : \mathcal{H} \to K$ is called weakly dominate mappings $T : \mathcal{H} \to K$ proximally if $\exists\, \mu < \frac{1}{s}, s > 1$ such that, for all $m_1, m_2, v_1, v_2, h_1, h_2 \in \mathcal{H}$.

$$[d\c{C}_b(m_1, Sh_1) = d\c{C}_b(m_2, Sh_2) = d\c{C}_b(\mathcal{H}, K)] = d\c{C}_b(v_1, Sh_1) = d\c{C}_b(v_2, Sh_2) \Longrightarrow d\c{C}_b(m_1, m_2) \preccurlyeq \mu\, \mathcal{D}_{m_1, m_2, v_1, v_2}$$

where $\mathcal{D}_{m_1,m_2,v_1,v_2} = \text{Re}\, \mathcal{D}_{m_1,m_2,v_1,v_2} + i\, Im\, \mathcal{D}_{m_1,m_2,v_1,v_2}$

and

$$Re\mathcal{D}_{m_1,m_2,v_1,v_2} = max \begin{Bmatrix} Red\c{C}_b(m_1, m_2), Red\c{C}_b(m_1, v_1), Re\ d\c{C}_b(m_1, v_2) \\ + \frac{Re\ d\c{C}_b(v_1,m_2) + Re\ d\c{C}_b(v_2,m_1)}{2} \end{Bmatrix}$$

and

$$Im\mathcal{D}_{m_1,m_2,v_1,v_2} = max \begin{Bmatrix} Imd\c{C}_b(m_1, m_2), Imd\c{C}_b(m_1, v_1), Imd\c{C}_b(m_1, v_2) \\ + \frac{Imd\c{C}_b(v_1,m_2) + Imd\c{C}_b(v_2,m_1)}{2} \end{Bmatrix}.$$

Example 3.9

Consider the CR$\c{C}_b$MS $(\mathbb{X}, \c{C}_b)$ for $s \geq 1$. Let $\mathbb{X} = \mathbb{C}$ and $d\c{C}_b: \mathbb{X}^2 \to \mathbb{C}$ define as

$$\c{C}_b(x, w, h) = max\,\{|x - w|, |w - h|, |x - h|\}^2 + i\, max\,\{|x - w|, |w - h|, |x - h|\}^2$$

and $\quad d\c{C}_b(x, w) = \c{C}_b(x, w, w) + \c{C}_b(w, x, x)$.

Consider the subsets $\mathcal{H}$ and K of $\mathbb{X}$ given by

$$\mathcal{H} = \{\delta \in \mathbb{C} : Re(\delta) = -2, \quad 0 \leq Im(\delta) \leq 3\},$$
$$K = \{\delta \in \mathbb{C} : Re(\delta) = 2, \quad 0 \leq Im(\delta) \leq 3\}.$$

Then we conclude, $\mathcal{H}_0 = \mathcal{H}$, and $K_0 = K$ and $d\c{C}_b(\mathcal{H}, K) = 16 + 0i$.

Let $T, S : \mathcal{H} \to K$ define as

$$T\delta = 2 + iw, \text{ for all } 0 \leq w \leq 1,$$
$$S\delta = \begin{cases} 2 + 0.2i, & 0 \leq w < 1, \\ 2 + 0.25i, & w = 1, \end{cases}$$

for each $\delta = -2 + iw \in \mathcal{H}$.

Now, suppose that $n_1 = -2 + 0.2i$, $n_2 = 2 + 0.25i$, $m_1 = m_2 = 2 + i$.

On the other hand, let $x_1 = 2 + 0.333i$, $x_2 = -2 + i \in \mathcal{H}$

$$d\c{C}_b(n_1, Sx_1) = |-2 - 2|^2 + i|0.2 - 0.2|^2 = 16 + i0 = d\c{C}_b(\mathcal{H}, K)$$
$$= d\c{C}_b(n_2, Sx_2) = |2 + 2|^2 + i|0.25 - 0.25|^2 = 16 + i0$$
$$= d\c{C}_b(m_1, Tx_1) = |2 + 2|^2 + i|1 - 1|^2 = 16 + 0\text{i}$$
$$d\c{C}_b(m_2, Tx_2) = d\c{C}_b(\mathcal{H}, K).$$

But note that for any non-negative real number $\mu < 1$

$d\c{C}_b(n_1, n_2) \precnsim d\c{C}_b(m_1, m_2)$ since $4 + i0.05 \precnsim \mu(0 + i0)$, for any $\mu < 1$

S and T proximity.

On the other hand, for $s = 3$ and $\mu = \frac{1}{3}$, we get T is weakly dominates S proximally.

Now, we get a result about existence of common best proximity point.

Theorem 3.10

Let $(\mathbb{X}, Ç_b)$ be a $Ç_b$ −complete CR$Ç_b$MS, with $s > 1$. $\emptyset \neq \mathcal{H}, K \subset \mathbb{X}$. Assume that $\mathcal{H}_0, K_0$ non-empty and $\mathcal{H}_0$ is closed. Let $T, S : \mathcal{H}_0 \to K_0$ be two non-self-mappings such that

(i) T weakly dominates S proximally,

(ii) S, T commute proximally,

(iii) S and T are continuous,

(iv) $S(\mathcal{H}_0) \subseteq K_0$ and $S(\mathcal{H}_0) \subseteq T(\mathcal{H}_0)$.

Then there exists a unique element $h \in \mathcal{H}$ such that

$$dÇ_b(h, Th) = dÇ_b(h, Sh) = dÇ_b(\mathcal{H}, K).$$

Proof: Suppose that $h_0 \in \mathcal{H}_0$. Since $S(\mathcal{H}_0) \subseteq T(\mathcal{H}_0)$ then $\exists\, h_1 \in \mathcal{H}_0$ such that $Sh_0 = Th_1$. Continuous in this manner to get $h_1 \in \mathcal{H}_0$ such that $\exists\, h_{r+1} \in \mathcal{H}_0$ satisfying $Sh_r = Th_{r+1}$, for each $r \in \mathbb{N}$.

Since $S(\mathcal{H}_0) \subseteq K_0$, there exists an element $n_r \in \mathcal{H}$ such that

$$dÇ_b(Sh_r, n_r) = dÇ_b(\mathcal{H}, K), \text{for each } r \in \mathbb{N}. \quad (2)$$

By chosen h_r and $n_r \in \mathcal{H}$ it follows that

$$dÇ_b(Sh_r, n_r) = dÇ_b(Sh_{r+1}, n_{r+1}) = dÇ_b(\mathcal{H}, K) = dÇ_b(Th_r, n_{r-1}) = dÇ_b(Th_{r+1}, n_r).$$

Since T weakly dominates S proximity then we conclude

$$dÇ_b(n_r, n_{r+1}) \preccurlyeq \mu\, \mathcal{D}_{n_r, n_{r+1}, n_{r-1}, n_r}, \text{ where } \mu < 1,$$

$$Re\, \mathcal{D}_{n_r, n_{r+1}, n_{r-1}, n_r} = \mu \max \left\{ \begin{matrix} Re\, dÇ_b(n_{r-1}, n_r), Re\, dÇ_b(n_{r-1}, n_r), Re\, dÇ_b(n_r, n_{r+1}), \\ \frac{Re\, dÇ_b(n_{r-1}, n_{r+1}) + Re\, dÇ_b(n_r, n_r)}{2s} \end{matrix} \right\}$$

and

$$Im\, \mathcal{D}_{n_r, n_{r+1}, n_{r-1}, n_r} = \mu \max \left\{ \begin{matrix} Im\, dÇ_b(n_{r-1}, n_r), Im\, dÇ_b(n_{r-1}, n_r), Im\, dÇ_b(n_r, n_{r+1}), \\ \frac{Im\, dÇ_b(n_{r-1}, n_{r+1}) + Im\, dÇ_b(n_r, n_r)}{2s} \end{matrix} \right\}.$$

From above by focus $Re\, dÇ_b(n_r, n_{r+2})$ and conclude that $Im dÇ_b(n_r, n_{r+2})$, therefore finally

$$\begin{aligned} Re\, dÇ_b(n_r, n_{r+1}) &\leq \mu\, max \left\{ Re\, dÇ_b(n_{r-1}, n_r), \frac{Re\, dÇ_b(n_{r-1}, n_{r+1})}{2s} \right\} \\ &\leq \mu\, max \left\{ Re\, dÇ_b(n_{r-1}, n_r), \frac{Re\, dÇ_b(n_{r-1}, n_r), Re\, dÇ_b(n_r, n_{r+1})}{2} \right\}. \end{aligned}$$

Now, we must prove that $\{h_r\}$ be a $Ç_b$ −Cauchy sequence, at the first if

$$Re\, dÇ_b(n_r, n_{r+1}) \leq \mu\, Re\, dÇ_b(n_{r-1}, n_r).$$

Then we get

$$Re\, dÇ_b(n_r, n_{r+1}) \leq \mu^r Re\, dÇ_b(n_0, n_1). \quad (3)$$

Also, by similarly way we have

$$Im\, dÇ_b(n_r, n_{r+1}) \leq \mu^r Im\, dÇ_b(n_0, n_1).$$

Now, for $r, t \in \mathbb{N}$ such that $t > r$ we get

$$\begin{aligned} Re\, dÇ_b(n_r, n_t) &\leq s[\, Re\, dÇ_b(n_r, n_{r+1}) + Re\, dÇ_b(n_{r+1}, n_t)] \\ &\leq s\, Re\, dÇ_b(n_r, n_{r+1}) + s^2[Re\, dÇ_b(n_{r+1}, n_{r+2}) + Re\, dÇ_b(n_{r+2}, n_t)] \\ &\leq s\, Re\, dÇ_b(n_r, n_{r+1}) + s^2 Re\, dÇ_b(n_{r+1}, n_{r+2}) + \cdots \\ &\quad + s^{r-t-1}[Re\, dÇ_b(n_{t+1}, n_{t+1}) + Re\, dÇ_b(n_{t-1}, n_t)] \end{aligned}$$

$$\leq s\, Re\, dÇ_b(n_r, n_{r+1}) + s^2 Re\, dÇ_b(n_{r+1}, n_{r+2}) + \cdots$$
$$+ s^{r-t-1} Re\, dÇ_b(n_{t+1}, n_{t+1}) + s^{r-t} Re\, dÇ_b(n_{t-1}, n_t).$$

From (2) and $s\mu < 1$, we have

$$Re\, dÇ_b(n_r, n_t) \leq (s\mu^r + s^2\mu^{r+1} + \cdots + s^{t-r}\mu^{t-1}) Re\, dÇ_b(n_0, n_1)$$
$$= s\mu^r(1 + s\mu + \cdots + (s\mu)^{t-r-1}) Re\, dÇ_b(n_0, n_1)$$
$$\leq \frac{s\mu^r}{1-s\mu} Re\, dÇ_b(n_0, n_1) \to 0 \text{ as } r, t \to \infty.$$

The second case

$$Re\, dÇ_b(n_r, n_{r+1}) \leq \mu \frac{Re\, dÇ_b(n_{r-1}, n_r) + Re\, dÇ_b(n_r, n_{r+1})}{2}$$
$$\leq \frac{\mu/2}{1-\mu/2} Re\, dÇ_b(n_{r-1}, n_r).$$

Let $\beta = \frac{\mu/2}{1-\mu/2} < 1$, and $s\beta < 1$.

So, we conclude

$Re\, dÇ_b(n_r, n_{r+1}) \leq \beta^r Re\, dÇ_b(n_0, n_1)$. Continuous with same argument for any $t > r$ where $r, t \in \mathbb{N}$ we get

$$Re\, dÇ_b(n_r, n_t) \to 0 \text{ as } r, t \to \infty.$$

Similarly, for any $t > r$ where $r, t \in \mathbb{N}$ we get

$$Im\, dÇ_b(n_r, n_t) \to 0 \text{ as } r, t \to \infty.$$

This implies that

$$dÇ_b(n_r, n_t) \to 0 \text{ as } r, t \to \infty.$$

Then $\{n_r\}$ is a $Ç_b$-Cauchy sequence also by completeness of $\mathbb{X}$, and since $\mathcal{H}_0$ is closed. Then $\exists\ n \in \mathcal{H}_0$ such that $n_r \to n$. By hypothesis, mappings S and T are commuting proximally and by (1) hence $Tn_r = Sn_{r-1}$, for every $n \in \mathbb{N}$.

Since T and S are continuous it implies that

$$Tn = \lim_{r\to\infty} Tn_r = \lim_{r\to\infty} Sn_{r-1} = Sn.$$

As $Sn \in S(\mathcal{H}_0) \subseteq K_0\ \exists\ h \in \mathcal{H}_0$ such that

$$dÇ_b(h, Sn) = dÇ_b(\mathcal{H}, K) = dÇ_b(h, Tn). \tag{4}$$

Since T and S are commute proximally then $Sh = Th$.

Also, $Sh \in S(\mathcal{H}_0) \subseteq K_0$, then there exists $\delta \in \mathcal{H}_0$ such that

$$dÇ_b(\delta, Sh) = dÇ_b(\mathcal{H}, K) = dÇ_b(\delta, Th). \tag{5}$$

Since T is weakly dominates S then from (3) and (4) we have

$$dÇ_b(h, \delta) \preccurlyeq \alpha\, \mathcal{D}_{h,\delta,h,\delta}$$
$$= \mu\left(\text{Re}\, dÇ_b(h, \delta) + i\, Im\, dÇ_b(h, \delta)\right)$$
$= \mu\, dÇ_b(h, \delta)$, therefore $\delta = h$. So, we get

$$dÇ_b(x, Sx, Sx) = dÇ_b(\mathcal{H}, K) = dÇ_b(h, Th). \tag{6}$$

To show that S and T have the same unique best proximity point.

Assume that y^* in $\mathcal{H}$ is another common best proximity point of S and T, then

$$dÇ_b(y^*, Sy^*) = dÇ_b(\mathcal{H}, K) = dÇ_b(y^*, Ty^*). \tag{7}$$

Since T weakly dominates S proximally then from (5) and (6), we have

$dÇ_b(h, y^*) \preccurlyeq \mu\, dÇ_b(h, y^*)$, which implies, $x = y^*$.

Then S and T have a unique common best proximity point.

Corollary 3.11

Let $(\mathbb{X}, Ç_b)$ be a $Ç_b$ −complete CR$Ç_b$MS with s > 1. Let $S, T: \mathbb{X} \to \mathbb{X}$ be continuous self-mapping on $\mathbb{X}$ and T commutes with S. For $\alpha < \frac{1}{s}$ suppose that

$S(\mathbb{X}) \subset T(\mathbb{X})$ such that for every $x \in \mathbb{X}$

$dÇ_b(Sx, Sw) \preccurlyeq \alpha\ \mathcal{D}_{S_x, S_w, Tx, Tw}$, since

$\mathcal{D}_{S_x, S_w, Tx, Tw} = Re\, \mathcal{D}_{S_x, S_w, Tx, Tw} + i\, Im\, \mathcal{D}_{S_x, S_w, Tx, Tw}$, such that

$$Re\,\mathcal{D}_{S_x,S_{w},Tx,Tw} = max \begin{Bmatrix} Re\; dÇ_b(Tx,Tw), Re\; dÇ_b(Tx,Sx), \\ Re\; dÇ_b(Tw,Sw) \\ ,\frac{Re\; dÇ_b(Sw,Tx)+Re dÇ_b(Sx,Tw)}{2s} \end{Bmatrix},$$

$$Im\,\mathcal{D}_{S_x,S_{w},Tx,Tw} = max \begin{Bmatrix} Im\; dÇ_b(Tx,Tw), Im\; dÇ_b(Tx,Sx) \\ ,Im\; dÇ_b(Tw,Sw) \\ ,\frac{Im\; dÇ_b(Sw,Tx)+ Im\; dÇ_b(Sx,Tw)}{2s} \end{Bmatrix}.$$

Then T and S have common fixed point which be unique.

4. Conclusions

In this work, the authors have adopted the idea of a Complex Valued Rectangular $Ç_b$-Metric space which gives a topological structure not always Hausdorff. We present some new results on common best proximity points in CR$Ç_b$- metric spaces, which are introduce a new generalization of complex-valued metric spaces called CR$Ç_b$- metric spaces, and we define the concept of common best proximity point in CR$Ç_b$- metric spaces and prove some new results on the existence of such points. Finally, we provide some examples to illustrate the difference between domination and weak domination

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