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Pure Closed Submodules and Pure Extending Modules

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Abstract
In this paper closed submodules and extending modules are generalizing to pure closed submodules and pure extending modules, respectively by using pure submodules. In this work, some properties of these concepts are consideration.

Key words: *Closed submodule, Pure closed submodule, Extending module, Pure extending module.*

المقاسات الجزئية المغلقة النقية وتوسيع المقاسات النقية

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الخلاصة
في هذه الورقة البحثية من المقاسات الجزئية المغلقة النقية وتوسيع المقاسات النقية تم التعميم على المقاسات الجزئية المغلقة النقية وتوسيع المقاسات النقية باستخدام المقاسات الجزئية النقية. في هذا العمل، يتم النظر في بعض خصائص هذه المفاهيم.

1. Introduction

Let T to be a commutative ring with identity and D possess unity T –module (shortly, T-mod). A submodule (shortly, sub-mod) X of D is called an essential (or D is called an essential extension of X) ($X \leq_e D$), if $X \cap W \neq 0$, for every non-zero sub-mod W of D [1-3]. Ibrahim and Al-Mothafar in [4] defined a generalization of essential sub-mod, which is called pure-essential (Pr-essential), as follow a sub-mod Y is called Pr-essential in module D, (denoted by $Y \trianglelefteq_{pr} D$), if $Y \cap W = (0)$, implies that W is a pure sub-mod of D. Also, they introduced another concept called pure- relative complement and fully pure - essential.

A sub-mod X of D namely closed sub-mod (in short $X \leq_c D$), if X has no proper essential extensions in D, i.e., if $X \leq_e Y \leq D$ then $X = Y$ [1] and [2]. In this paper we study a new concept, named pure-closed (Pr-closed), which is more powerful than the notion of closed sub-mod, in which a sub-mod A of an T-mod D is said to be pure closed sub-mod of D, briefly (Pr-closed), if A has no proper Pr- essential extensions in D, i.e., if $X \trianglelefteq_{pr} Y \leq D$, implies that $X = Y$.

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Moreover, we present an idea, named Pr-closed sub-mod, that is stronger than the closed sub-mod, with many remarks and propositions. Harada and Muller [5] introduced the concept of extending-mod, where a T-mod D is said to be extending-mod if each sub-mod of D is essential in direct summand, it was also mentioned in [6-8]. In our work we provide a broad overview for extending-mod as follows: a T-mod D is called pure extending-mod (Pr-extending) for each sub-mod of D is Pr-essentiel in direct summand. Finally, we study pure extending-mod (Pr- extending-mod) by giving some results analogue to the known results on extending-mod.

2.Pure - closed sub-modules

This section introduces a new type of sub-modules called a pure closed sub-mod.

Definition 2.1:

A sub-mod X of a T-mod D is said to be pure closed sub-mod of D, briefly (Pr-closed), denoted by ($X \leq_{prc} D$), if X has no proper Pr- essential extensions in D, i.e., if $X \trianglelefteq_{pr} Y \leq D$, implies that $X = Y$.

Remarks and examples 2.2:

1. Obviously, all of the Pr- closed sub-mod of D is closed sub-mod . Thus ($\overline{18}$) is Pr-closed in Z_{36} as a Z- mod, hence it is closed sub-mod in Z_{36} as a Z- mod .

To see that:

Let $X \leq D$ such that $X \leq_e Y \leq D$. Since each essential sub-mod is a Pr - essential [4]. Then $X \trianglelefteq_{pr} Y \leq D$. But X is a Pr- closed, implies that $X = Y$.

2. Generally, the opposite of (1) may not be correct. As an illustration: in Z_{36} as Z - module, ($\overline{9}$) is a closed sub-module of Z_{36}, since $Z_{36} = (\overline{9}) \oplus (\overline{4})$, but $(\overline{9})$ is not a Pr- closed in Z_{36} since $(\overline{9}) \trianglelefteq_{pr} Z_{36}$, see [4].

3. The (0) may not be Pr- closed sub-mod of any T-mod D. For example: ($\overline{0}$) is not a Pr-closed in Z_2 as a Z- mod.

4. A direct summand of a T-mod D need not be a Pr-closed sub-mod of D. For example: in Z_{12} as Z - mod, $Z_{12} = (\overline{3}) \oplus (\overline{4})$, then $(\overline{4})$ is a direct summand of Z_{12} but not a Pr- closed in Z_{12}. Since $(\overline{4})$ is a Pr- essential in Z_{12}. Since $(\overline{4})$ is a direct summand of Z_{12} [4]

Proposition 2.3:

Let A and B be two sub-mods of a T-mod D, with $A \leq B \leq D$. If B is a Pr- closed sub-mod of D, then $\frac{B}{A}$ is a Pr- closed in $\frac{D}{A}$.

Proof:

Let $L \leq D$ such that $\frac{B}{A} \trianglelefteq_{pr} \frac{L}{A} \leq \frac{D}{A}$. By [4] $B \trianglelefteq_{pr} L$, since B is a Pr- closed in D, this implies $B = L$. As $A \leq B$, hence $\frac{B}{A} = \frac{L}{A}$. Thus $\frac{B}{A}$ is a Pr- closed in $\frac{D}{A}$.

Proposition 2.4:

Let A and B be two sub-mods of a T-mod D, with $A \leq B \leq D$. If A is a Pr- closed sub-mod of D, then A is a Pr- closed in B.

Proof:

Let $L \leq B$ such that $A \trianglelefteq_{pr} L \leq B$. Since A is a Pr- closed in D and $L \leq D$, implies that $A = L$. Thus A is a Pr- closed in B.

Corollary 2.5:

If A and B are Pr- closed in D, then A and B are Pr- closed in $A + B$.

Proof:
Since $A \leq A + B \leq D$, then by Proposition 2.4, A and B are Pr- closed in $A + B$.
Corollary 2.6:
Let X and Y be two sub-mods of a T-mod D. If $Y \cap X$ is a Pr- closed in D, then $Y \cap X$ is a Pr- closed in X and Y.
Proof:
By Proposition 2.4, $Y \cap X$ is a Pr- closed in X and Y.
Proposition 2.7:
Let A and B be two sub-mods of a T-mod D. If A is a Pr- closed sub-mod of B and B is a Pr- closed sub-mod of D, then A is a Pr- closed sub-mod of D. Where B contained in (or containing) any pure essential extension of A.
Proof:
Let $L \leq D$ such that $A \trianglelefteq_{pr} L \leq D$. By hypothesis we have two cases: Case (1) if $L \leq B$, since A is a Pr- closed sub-mod of B, then we have $A = L$, hence A is a Pr- closed sub-mod of D. Case(2) if $B \leq L$ since $A \trianglelefteq_{pr} L$, then by [4] $B \trianglelefteq_{pr} L$. But B is a Pr- closed sub-mod in D. Thus $B = L$, implies that $A \trianglelefteq_{pr} B$. Also, A is an r- closed sub-mod of B, so $A = B$ and hence $A = L$. Thus, A is aPr- closed sub-mod of D.

Recall that a T-mod D is called chained if for each sub-mod X and Y of D either $X \leq Y$, or $Y \leq X$ [9].
Corollary 2.8:
Let D be a chain mod and let A and B be sub-mods of D such that $A \leq B \leq D$. If A is a Pr- closed sub-mod in B and B is a Pr- closed sub-mod in D, then A is a Pr- closed sub-mod in D.
Proof:
Let $L \leq D$ such that $A \trianglelefteq_{pr} L \leq M$. Since D be a chain-mod either $L \leq B$ or $B \leq L$, hence the result follows by Proposition 2.7.
By the same argument of closed sub-mod we can prove the following
Proposition 2.9:
Let B be a Pr- closed sub-mod of a T-mod D. If $B \leq K \trianglelefteq_{pr} D$, then $\frac{K}{B} \trianglelefteq_{pr} \frac{D}{B}$.
Proof:
Suppose that $\frac{L}{B} \leq \frac{D}{B}$, such that $\frac{K}{B} \cap \frac{L}{B} = (0)$. Then $K \cap L = B$, since $K \trianglelefteq_{pr} D$ and $D \trianglelefteq_{pr} D$, so by [4] $K \cap L \trianglelefteq_{pr} D \cap L$, then $B \trianglelefteq_{pr} L$, but B be a Pr- closed in D, this gives $B = L$, hence $\frac{L}{B} = (\bar{0})$ which is a pure in $\frac{D}{B}$. Therefore, $\frac{L}{B} \leq_{p} \frac{D}{B}$, then $\frac{K}{B} \trianglelefteq_{pr} \frac{D}{B}$.
Remember that a non-zero T-mod D is namely fully Pr -essential if each non-zero Pr-essential sub-mod of D is an essential sub-mod of D, [4].
Proposition 2.10:
Let N be a non-zero closed sub-mod of a fully Pr-essential T-mod D, then N is Pr- closed.
Proof:
Let N be a closed sub-mod of D and let $L \leq D$ such that $N \trianglelefteq_{pr} L \leq D$, as L is fully Pr-essential, then $N \leq_{e} L$. But $N \leq_{c} D$, hence $N = L$.
Proposition 2.11:
Let A, B be a non-zero sub-mod of a fully Pr-essential T-mod D, with $A \leq B \leq D$. If $A \leq_{prc} B$ also $B \leq_{prc} D$. Thus $A \leq_{prc} D$.
Proof:
Since $A \leq_{prc} B$ and $B \leq_{prc} D$, so by Remark 2.2, $A \leq_{c} B$ and $B \leq_{c} D$, hence by [1] $A \leq_{c} D$ and by Proposition 2.10 we obtain $A \leq_{prc} D$.

Proposition 2.12:
Let A and B be Pr- closed sub-mods of a T-mod D, then $A \cap B$ is a Pr- closed in D.
Proof:
Let $L \leq D$ such that $A \cap B \trianglelefteq_{pr} L \leq D$. Then by [4] $A \trianglelefteq_{pr} L \leq D$ and $B \trianglelefteq_{pr} L \leq D$. Since A and B are Pr- closed in D, so $A = L = B$, hence $A \cap B = L$. Thus $A \cap B$ is a Pr- closed in D.
Remark 2.13:
Let A and B be sub-mods of a T-mod D, with $A \leq B \leq D$. If A is a Pr- closed in D, then B is not necessary be a Pr- closed of D. As an illustration: $A = (0)$ in Z as Z - mod is a Pr- closed, since (0) has no Pr- essential extension in Z. But $2Z$ is not a Pr- closed in Z, since $2Z$ is a Pr- essential in Z as Z - mod.
Proposition 2.14:
If every sub-mod of a T- mod is a Pr- closed in D, then D is a semi simple.
Proof:
Since every Pr- closed is closed by Remarks 2.2, then every sub-mod of D is closed, hence the result follow by [10].

Let X and Y be sub-mods of a T-mod D, Y is called pure- relative complemnt (Pr-relative complemnt) of X in D. If Y is a pure and maximal sub-mod with respect the property $Y \cap X = (0)$, [4].

Recall that a T-mod D namely pure-simple, if the only pure sub-mod of D are (0) and D, [11] and [12].
Proposition 2.15:
Let N and K be two sub-mods of a T-mod D, in which every sub-mod of D is a pure simple. If K is a Pr- relative complement for some N, then $K \leq_{prc} D$.
Proof:
Let $L \leq D$ such that $K \trianglelefteq_{pr} L \leq D$. Since K is Pr- relative complement for some N, then $K \cap N = (0)$, so $L \cap (K \cap N) = (0)$, then $(L \cap N) \cap K = (0)$. As K is a Pr- relative complement of N, then $L \cap N$ is a pure in D. Since $L \cap N \neq D$, so $L \cap N = (0)$, as $K \leq L$ and K is a Pr- relative complement, then $K = L$.
Remark 2.16:
If N is a Pr- closed sub-mod of D. Then $[N:_T D]$ is not necessary be Pr- closed in T. For example: in Z_{36} as Z-mod. $(\overline{18}) \leq_{prc} Z_{36}$, but $[(\overline{18}) :_Z Z_{36}] = 18Z$ is not a Pr- closed in Z. Since $18Z \trianglelefteq_{pr} 2Z \leq Z$, but $18Z \neq 2Z$.

A T-mod D is known as multiplication, if every sub-mod X of D, there is an ideal S of T, where $X = SD$ [13-16].
Theorem 2.17:
Let D be a finitely generated faithful multiplication T-mod. Then SD Pr-closed in D if and only if S is a Pr-closed in T, where S an ideal of a ring T.
Proof: $\Longrightarrow$)
Take $J \leq T$ such that $S \trianglelefteq_{pr} J \leq T$. By [4] $SD \trianglelefteq_{pr} JD \leq TD = D$. Since $SD \leq_{prc} D$, hence $SD=JD$, then by [12] we obtain $S=J$. Thus, $S \leq_{prc} T$.
$\Longleftarrow$) Let $W \leq D$ such that $SD \trianglelefteq_{pr} W \leq D$. Since D is a multiplication $\boldsymbol{T}$-mod, put $W = JD$, where $J \leq T$. Then $SD \trianglelefteq_{pr} JD \leq D$. Since D is a finitely generated faithful multiplication T-mod, then by [4] we have $S \trianglelefteq_{pr} T$. But $S \leq_{prc} T$, then $S=J$, hence $SD=JD = W$. Thus $SD \leq_{prc} D$.

Corollary 2.18:
Let D be a finitely generated faithful multiplication T-mod, and let $X \leq D$, therefore the next claims are equivalent:

1- X is Pr-closed in D;
2- [$X : D$] is Pr-closed in T;
3- $X = SD$, for some $S \leq_{prc} T$.

Proof:
1) $\Longrightarrow$ 2) Suppose that $X \leq_{prc} D$, since D is multiplcation then X = [$X : D$] D [13]. But $X \leq_{prc} D$, put [$X : D$] $D = SD$. By Theorem 2.20 we have [$X : D$] is a Pr-closed in T.
2) $\Longrightarrow$ 3) Since D is multiplicaton then X = [$X : D$] D [13]. Then we obtain our result.
3) $\Longrightarrow$ 1) Since $S \leq_{prc} T$. By Theorem 2.20, we have $SD = X \leq_{prc} D$.

3. Pure- extending modules:

Within this part we present pure-extending modules, which are generalization of extending-mod. We give some properties about this concept.

First, we give the following concept:

Definition 3.1:
Let D be a T-mod, D is said to be a pure-extending (Pr- extending) module if every sub-mod of D is a Pr-essential in a direct summand.

Examples and remarks 3.2:
1. Each extending-mod is a Pr-extending.
2. If D is a pure simple T-mod, Pr-extending and extending-mod are coincide.
3. It is easy to show that every semi simple T-mod is a Pr-extending-mod. For example: Z_6 as Z-mod.
4. Every uniform T-mod is a Pr-extending. For example: Z_{p^2} as Z-mod. In fact every Pr-uniform T-mod is a Pr-extending, since by [4], a non-zero an R- Mod M is said to be pure-uniform, briefly (Pr-uniform), if all non-zero sub module in M is a Pr-essential.

Lemma 3.3:
For any sub-mod A of a T-mod D, there exists a Pr-closed sub-mod H of D such that A is a Pr-essentiel in H.

Proof:
Let V= { K : K is a sub-mod of D, such that $A \trianglelefteq_{pr} K$ }. Evidently $V \neq \emptyset$. By Zorn's Lemma, V has, let's say, a maximal element H. To exhibit H is a Pr-closed in D, let $H \trianglelefteq_{pr} B \leq D$. Since $A \trianglelefteq_{pr} H$ and $H \trianglelefteq_{pr} B$, then by [4], we have $A \trianglelefteq_{pr} B$. However, this runs counter to the maximization of H .Thus H=B, therefore, H is a Pr-closed in D.

The theorem that follows provides a description of pr- extending modules.

Theorem 3.4:
Suppose that D be an T-mod, then D is a Pr-extending-mod if and only if each Pr-closed sub-mod of D is a direct summand.

Proof:
Assume that D is a Pr-extending-mod and Y be a Pr-closed sub-mod of D. Since D is a Pr-extending, then there exists a direct summand W of D such that Y is Pr-essential in W. But Y is a Pr-closed in D. Therefore, Y= W.
Conversely, let X be any sub-mod of D, by Lemma 3.3 there is a Pr-closed sub-mod H of D such that X is Pr-essential in H. Since H is Pr-closed in D, hence H is a direct summand of D.

Corollary 3.5:
If D is a Pr-extending T-mod, then every Pr-closed sub-mod of D is pure in D.
Proof:
Since every direct summand is pure [12, Remark 1.4, P.31], hence the result follows by Theorem 3.4.
Proposition 3.6:
Let D be an indecomposable T-mod. If D is a Pr-extending T-mod, then D is a Pr-uniform.
Proof:
Let $0 \neq X \leq D$ and let Y be a Pr-relativ complment of X in D, then by [4], $X \oplus Y \trianglelefteq_{pr} D$. By Proposition 2.15, Y is a Pr-closed in D. But D is a Pr-extending, then Y is direct summand of D. Since D is indecomposable and $Y \neq D$. Thus X is a Pr-essentail in D.
Proposition 3.7:
If a T-mod D is a Pr-extending, then every Pr-closed sub-mod of D is a Pr-extending.
Proof:
Let $N \leq_{prc} D$ and let $A \leq_{prc} N$, by Proposition 2.7, we have $A \leq_{prc} D$. Since D is Pr-extending, hence A is a direct summand of D, but $A \leq N$, then A is a direct summand of N.
Proposition 3.8:
If D is a Pr-extending T-mod and let N is a Pr-closed sub-mod of D, then $\frac{D}{N}$ is a Pr-extending.
Proof:
Suppose that $\frac{K}{N} \leq \frac{D}{N}$. Since D is a Pr-extending, then there exists a direct summand A of D such that $K \trianglelefteq_{pr} A$. Since $N \leq K$, and N is a Pr-closed of D, then by [4], we obtain $\frac{K}{N} \trianglelefteq_{pr} \frac{A}{N}$. But A is a direct sumand of D, hence by [7], we have $\frac{A}{N}$ is a direct sumand of $\frac{D}{N}$.
Remark 3.9:
If D is a Pr-extending T-mod, then it is not necessary that every pure sub-mod D is a Pr-extending or every Pr-essential sub-mod of D is a Pr-extending. For example: Z_6 as Z-mod is Pr-extending and $(\bar{2})$ is pure sub-mod of Z_6, but $(\bar{2})$ is not Pr-extending since (0) is not Pr-essential sub-mod of $(\bar{2})$.

Recall that D is said to have the summand intersection property (briefly SIP) if the intersection of any two direct summands is again a direct summand. This can be shown in [12], [17], [18].
Proposition 3.10:
Let D be a Pr-extending-mod and let N be summand of D. If D has the SIP, then N is a Pr-extending.
Proof:
Let $A \leq_{prc} N$, then by Lemma 3.3 there exists a Pr-closed sub-mod B of D such that $A \trianglelefteq_{pr} B$. Since $N \trianglelefteq_{pr} N$, then by [4], we have $A = A \cap B$ is Pr-essential in $B \cap N \leq N$. But $A \leq_{prc} N$, hence $A = B \cap N$, this is because D is a pr-extending and B is Pr-closed in D, then B is a direct summand of D. But N is a direct summand of D and D has SIP, so $A = B \cap N$ which is a summand of D, this means A is a summand of N. Thus N is a Pr-extending.
Corollary 3.11:
Let D be a multiplcation T-mod and let X be a diract summend of D. If D is a Pr-extending, then X is a Pr-extending.

Proof:
Since each multiplcation T-mod has the SIP [12], then by pervious theorem we are done.

Corollary 3.12:
Let D be a cyclic T-mod and let N be a diract summend of D. If D is a Pr-extending, then N is a Pr-extending.
Proof:
Since every cyclic is multiplication [14], then the outcome is as Corollary 3.11.
Corollary 3.13:
Suppose that T is a commutative ring with identity, also S be an ideal of T. If T is a Pr-extending, then S is a Pr-extending.
The next theorem provides another description for Pr-extending-mod.
Theorem 3.14:
Let D be a T-mod, the next claims are equivalent:
1. D is Pr-extending;
2. Each Pr-closed sub-module of D is a summand of D;
3. If A is a summand of the injective hull $E(D)$ of D, then $A \cap D$ is a summand of D. Where "A monomorphism η:$M \longrightarrow Q$ is called an injective hull of M :$\Leftrightarrow$ Q is injective and η is a large monomorphism". [10, P.124]
Proof:
1) $\Longrightarrow$ 2) Clear by Proposition 3.7.
2) $\Longrightarrow$ 3) Let $E(D) = A \oplus B$ where $B \leq E(D)$. To prove that $A \cap D \leq_{prc} D$. Let $A \cap D$ be a Pr-essential in H, where $H \leq D$ and let $h \in H$, then $h = a + b$; $a \in A$, $b \in B$. If $h \notin A$, then $b \neq 0$. But D is essential in $E(D)$, so by [1] there exists $t \in T$ such that $0 \neq tb \in D$. Now, $th = ta + tb$ hence $ta = th - tb \in D \cap A \leq H$. Thus, $tb = th - ta \in B \cap H$. Since $A \cap D \trianglelefteq_{pr} H$ and (0)=($A \cap D$)$\cap$ B is a Pr-essential in $B \cap H$ [4], implies $H \cap B$ = (0). Thus $t\,b$ = 0 which is a contradiction. Thus $A \cap D$ is Pr-closed in D, hence by (2) $A \cap D$ is a summand of D.
3) $\Longrightarrow$ 1) Let A be a sub-mod of D and let B a relative complemnt of A, then by [1] $A \oplus B$ is essential sub-mod of D, since D is essential in $E(D)$, therefore $A \oplus B$ is essential of $E(D)$. Thus by [1] $E(A) \oplus E(B) = E(D)$. Since $E(A)$ is a direct summand of $E(D)$, then assumption $E(A) \cap D$ is a summand of D. Now, A is essential in $E(A)$ and D is essential in M. Therefore, by [1] $A = A \cap D$ is a summand of D. So, A is essentiel in a diract summand. Thus D is an extending-mod, hence by Remarks and example 3.2, D is an Pr-extending.

4. Conclusions:
In this work, pure closed submodules and pure extending modules, which are generalization of closed submodules and pure extending modules respectively. We also show some of the following results:

- If B is a Pr- closed sub-mod of D, then $\frac{B}{A}$ is a Pr- closed in $\frac{D}{A}$, with $A \leq B \leq D$.
- If A is a Pr- closed sub-mod of D, then A is a Pr- closed in B, with $A \leq B \leq D$.
- If every sub-mod of an T- mod is Pr- closed in D, then D is a semi simple.
- If D is Pr-extending T-mod and let N is Pr-closed sub-mod of D, then D/N is Pr-extending.

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