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Hosoya Polynomials and Wiener Indices of Semifull Graph for Some Special Graphs

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ABSTRACT

Let G be a finite simple connected graph of order p, size q, and block $B = B(G)$. The semifull graph of G is a graph that has a vertex set $V(G) \cup E(G) \cup B(G)$. So that any two vertices in the semifull graph of G are adjacent if any two vertices are adjacent in G, or any two edges are adjacent in G, or any two blocks are adjacent in G, or any vertex incident on edge or block. It was found that Hosoya polynomial, Wiener index, and average distance of a semifull graph are for some special graphs such as star, wheel, path, and cycle graphs, in addition to specifying the diameter for each of the resulting graphs.

Keywords: Hosoya polynomial, Wiener Index, Semifull graph, certain graphs.

متعددات حدود هوسويا وأدلة وينر للبيان شبه الكامل لبعض البيانات الخاصة

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الخلاصه

ليكن G بيانا بسيطا متصلا من الرتبة p والحجم q والكتلة $B = B(G)$. البيان الشبه الكامل لـ G هو بيان له مجموعة رؤوس $V(G) \cup E(G) \cup B(G)$ بحيث يكون أي رأسين في البيان الشبه الكامل لـ G متجاورين إذا كان أي رأسين متجاورين في G أو أي حافتين متجاورتين في G أو أي كتلتين متجاورتين في Gأو أي رأس يقع على حافة أو على كتلة. تم أيجاد متعددات هوسويا ودليل وينر ومعدل المسافة للبيان الشبه الكامل لبعض البيانات الخاصة مثل: بيان النجمة، وبيان العجلة، وبيان الدرب، وبيان الدارة، بالإضافة إلى تحديد القطر لكل من البيانات الناتجة.

الكلمات المفتاحية: متعددة حدود هوسويا، دليل وينر، بيان الشبه كامل، بعض البيانات الخاصة.

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1. Introduction

Graph theory is one of the most important branches of mathematics, whose applications appeared widely and greatly in the last century. This prompted researchers to focus heavily on this branch by finding mathematical models that link chemical properties such as the boiling points and melting points for chemical structures. It also helps finding the invariant in graphs, such as topological indices, and the most important of these indices is Wiener index, which was found in 1947 by the chemist Wiener, [1]. Let G be a simple connected graph, [2], the concept of **distance** between two distinct vertices (say u and v) in G refers to the number of edges in the shortest path between them and is denoted by $d(u,v)$. If $u=v$, then $d(u,v)=0$. "**Wiener index**" of G is defined as the sum of distances between all pairs of vertices of a graph G and is denoted by $W(G)$, see [1]; that is,
$W(G)=\sum_{\{u,v\}\subseteq V(G)} d(u,v)$.
The diameter of G is denoted by δ, and defined by
$\delta = max\,\{d(u,v): \forall\, u,v \in V(G)\}$,
where $V(G)$ is a set of vertices of a graph G.

Hosoya polynomial or (**Wiener polynomial**) of G is denoted by $H(G;x)$, [3], and defined as
$H(G;x)=\sum_{k=0}^{\delta} d(G,k)x^k$,
where $d(G,k)$ refers to the number of unordered pairs of distinct vertices (say, (u,v)) such that $d(u,v)=k$, $k=0,1,2,\dots,\delta$.
Many researchers have found Hosoya polynomials for special graphs, [4-6]. Moreover, Hosoya polynomial was found for subdivided caterpillar graphs [7], and for C4C8(R) and C4C8(S) Nanosheets [8]. These polynomials are valuable tools for studying and understanding the structural properties of graphs in graph theory. There are many other types of polynomials which are of great importance in graph theory such as schultz polynomial [9], detour polynomial [10], and other types [11-13]. For a broad overview of the previous concepts, see [15-16]. In addition to their important applications in many fields, you can see [16] to solve a class of time-fractional diffusion equations, and more applications in [17-19].

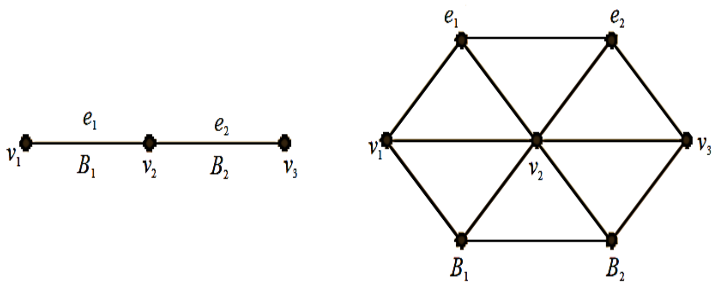
Furthermore, the **Wiener index** of a graph G can be written as follows:
$W(G)=\frac{d}{dx}H(G;x)|_{x=1}$.
Many topological indices depend on the distance, the degrees of the vertices, or both, [20-23]. The **average distance** $D(G)$ of a connected graph G of order p is defined as: the sum of distances between all pairs of vertices of a graph G ($W(G)$) over $\binom{p}{2}$,where $\binom{p}{2}$ is the number of unordered pairs distinct vertices in G, see [24].

A component of graph G is the largest connected subgraph which is not contained in any other connected subgraph, $c(G)$ is the number of components in G. Let v be a vertex of a graph G, then $G-v$ means that remove a vertex v and all edges $vu, u \in V(G)$ from that, if have $c(G-v) > c(G)$, then v is called **a cut-vertex** of G. Let G be a nontrivial connected graph G, the graph G is a non-separable graph if G contains no cut-vertices. A block is a maximal non-separable subgraph of G, [2]. The **semifull graph** $S_{full}(G)$ of a connected graph G, [25] is the graph whose vertex set $V(G)\cup E(G)\cup B(G)$ such that any two distinct vertices in $S_{full}(G)$ are adjacent if

- Any two vertices are adjacent in G .
- Any two edges are adjacent in G.
- Any two blocks are adjacent in G.
- Any vertex incident on edge in G.

- Any vertex incident on block in G.

To illustrate the previous concepts, we take the following example [25]. A semifull graph of the path graph $\mathcal{P}_3$, $\mathcal{S}_{full}(\mathcal{P}_3)$ is shown in the following Figure 1(a) and 1(b).



$\mathcal{P}_3$ (a) $\mathcal{S}_{full}(\mathcal{P}_3)$ (b)

Figure 1: The semifull graph of $\mathcal{P}_3$

2. Main results

In the following theorems, we find the Hosoya polynomial, Wiener index, and an average distance of semifull graph for certain graphs such as: $\mathcal{S}_{1,p}$, $\mathcal{P}_p$, $\mathcal{W}_{p+1}$ and $\mathcal{C}_p$ graphs, in addition to specifying the diameter for each of the resulting diameters.

2.1. Hosoya polynomial of $S_{full}(\mathcal{S}_{1,p})$

The graph $S_{full}(\mathcal{S}_{1,p})$ is shown in Figure 2, for all $p \geq 3$.

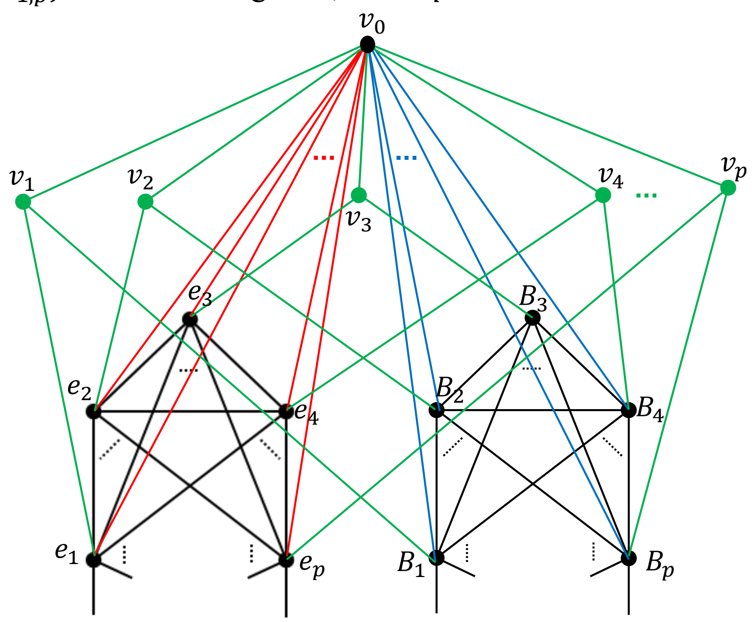


Figure 2: The $S_{full}(\mathcal{S}_{1,p})$ graph for all $p \geq 3$.

Clearly, the order and size of $S_{full}(\mathcal{S}_{1,p})$, $p \geq 3$ are $3p + 1$ and $p^2 + 4p$, respectively. Through the following proposition, we will determine the diameter of a graph $S_{full}(\mathcal{S}_{1,p})$, $p \geq 3$.

Proposition 2.1.1: For $p \geq 3$, the diameter of $diam\left(S_{full}(\mathcal{S}_{1,p})\right) = 2$.

Proof: Since the center vertex v_0 of a star graph $\mathcal{S}_{1,p}$ is adjacent to all vertices in $V\left(S_{full}(\mathcal{S}_{1,p})\right) - \{v_0\}$, where $V\left(S_{full}(\mathcal{S}_{1,p})\right) = \{v_0\} \cup \{v_i, e_i, B_i, 1 \leq i \leq p\}$ so that $v_i \in V(\mathcal{S}_{1,p})$, $e_i \in E(\mathcal{S}_{1,p})$ and B_i be a block of a star graph $\mathcal{S}_{1,p}$ for all $1 \leq i \leq p$. That is,
$d(v_0, u) = 1$, for all $u \in \{v_i, e_i, B_i, 1 \leq i \leq p\}$.
Also, from Figure. 2, we note that
$d(e_i, e_j) = 1$, for all $1 \leq i, j \leq p, i \neq j$,
$d(B_i, B_j) = 1$, for all $1 \leq i, j \leq p, i \neq j$.
Since any two vertices in $V(\mathcal{S}_{1,p}) - \{v_0\}$ is not adjacent, then
$d(v_i, v_j) = 2$ in $S_{full}(\mathcal{S}_{1,p})$, for all $1 \leq i, j \leq p, i \neq j$, (by definition of semifull graph).
Also, any vertex $e_i, 1 \leq i \leq p$ is not adjacent to any vertex $B_i, 1 \leq i \leq p$ in $S_{full}(\mathcal{S}_{1,p})$, then
$d(e_i, B_j) = 2$, for all $1 \leq i, j \leq p$.
Since the diameter is the maximum distance between any two distinct vertices, then $diam\left(S_{full}(\mathcal{S}_{1,p})\right) = 2$. ■

Theorem 2.1.2: For $p \geq 3$, then;

1. $H(S_{full}(\mathcal{S}_{1,p}); x) = 3p + 1 + (p^2 + 4p)x + \frac{1}{2}p(7p - 5)x^2$.
2. $W\left(S_{full}(\mathcal{S}_{1,p})\right) = p(8p - 1)$.
3. $D\left(S_{full}(\mathcal{S}_{1,p})\right) < \frac{16}{9}$.

Proof:

1. Since $\sum_{k=1}^{\delta} d(G, k) = \binom{|G|}{2}$, for any connected graph G of order p, then

$$d(S_{full}(\mathcal{S}_{1,p}), 1) + d(S_{full}(\mathcal{S}_{1,p}), 2) = \binom{3p+1}{2}$$

$$d(S_{full}(\mathcal{S}_{1,p}), 2) = \frac{3}{2}p(3p + 1) - (p^2 + 4p) = \frac{1}{2}p(7p - 5).$$

Hence,

$$H(S_{full}(\mathcal{S}_{1,p}); x) = 3p + 1 + (p^2 + 4p)x + \frac{1}{2}p(7p - 5)x^2.$$

2. $W\left(S_{full}(\mathcal{S}_{1,p})\right) = \frac{d}{dx}H(S_{full}(\mathcal{S}_{1,p}); x)|_{x=1} = p(8p - 1)$.
3. $D\left(S_{full}(\mathcal{S}_{1,p})\right) = \frac{W(S_{full}(\mathcal{S}_{1,p}))}{\binom{3p+1}{2}} = \frac{2(8p-1)}{3(3p+1)} = \frac{16}{9} - \frac{22}{3(3p+1)} < \frac{16}{9}$. ■

2.2. Hosoya polynomial of $S_{full}(\mathcal{W}_{p+1})$

Let $V(\mathcal{W}_{p+1}) = \{v_0, v_1, v_2, \dots, v_p\}$ and $E(\mathcal{W}_{p+1}) = \{e_1, e_2, e_3, \dots, e_{2p}\}$ where $e_i = v_0 v_i$, $1 \leq i \leq p$ and $e_{p+i} = v_i v_{i+1}$, $1 \leq i \leq p$, $v_{p+1} \equiv v_1$. See Figure 3.

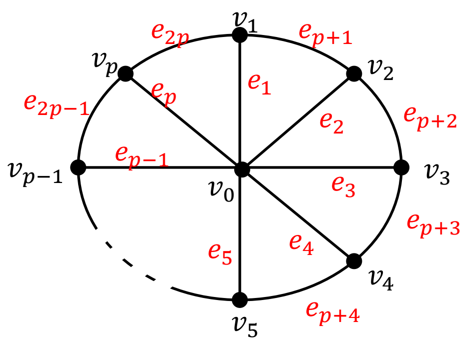


Figure 3: The $\mathcal{W}_{p+1}$ graph

Then, the graph $\boldsymbol{S_{full}(\mathcal{W}_{p+1})}$, $p \geq 3$ is shown in Figure 4.

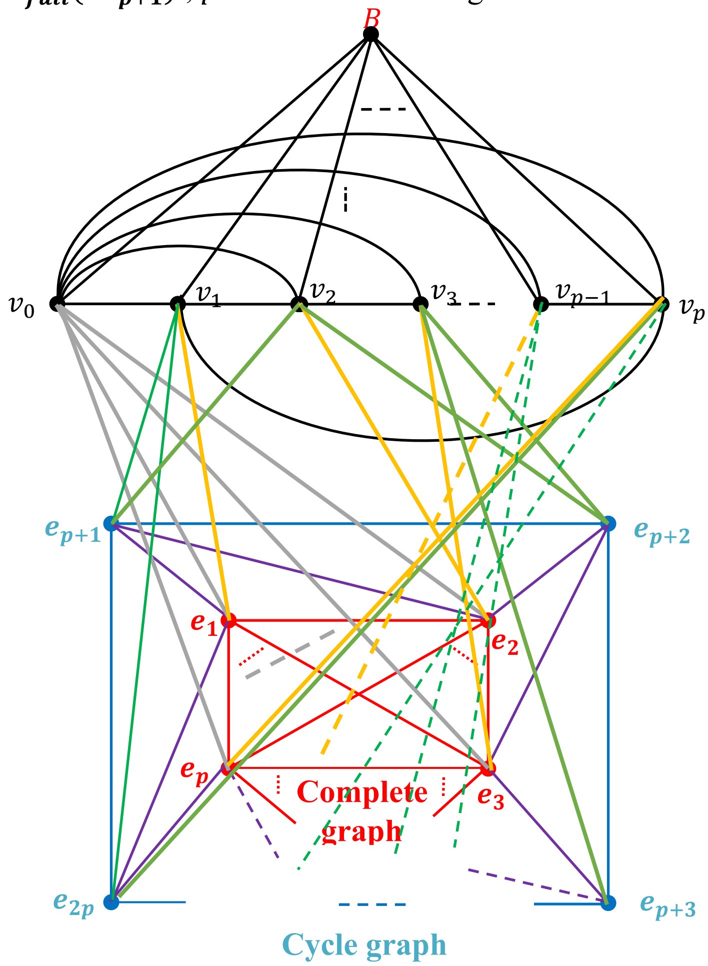


Figure 4: The $S_{full}(\mathcal{W}_{p+1})$ graph, $p \geq 3$.

That is, $p\left(S_{full}(\mathcal{W}_{p+1})\right) = 3p + 2$ and $q\left(S_{full}(\mathcal{W}_{p+1})\right) = \frac{1}{2}(p^2 + 19p + 2)$.

Proposition 2.2.1: For $p \geq 5$, $diam\left(S_{full}(\mathcal{W}_{p+1})\right) = 3$.

Proof: Let u be an arbitrary vertex belonging to $S_{full}(\mathcal{W}_{p+1})$, where $V(S_{full}(\mathcal{W}_{p+1})) = \{B, v_i, e_j: i = 0,1,2, \dots, p, j = 1,2, \dots, 2p\}$, then there are five cases.

Case 1: If $u = B$, then $d(u, w) \leq 2$, for all $w \in V(S_{full}(\mathcal{W}_{p+1})) - \{u\}$.

Case 2: If $u = v_0$, then $d(u, w) \leq 2$, for all $w \in V(S_{full}(\mathcal{W}_{p+1})) - \{u\}$.

Case 3: If $u = v_i$, $i = 1,2, \dots, p$, then $d(u, w) \leq 3$, for all $w \in V(S_{full}(\mathcal{W}_{p+1})) - \{u\}$.

Case 4: If $u = e_j$, $j = 1,2, \dots, p$, then $d(u, w) \leq 2$, for all $w \in V(S_{full}(\mathcal{W}_{p+1})) - \{u\}$.

Case 5: If $u = e_{p+j}$, $j = 1,2, \dots, p$, then $d(u, w) \leq 3$, for all $w \in V(S_{full}(\mathcal{W}_{p+1})) - \{u\}$.

From definition of the diameter, then $diam\left(S_{full}(\mathcal{W}_{p+1})\right) = 3$. ■

Theorem 2.2.2: For $p \geq 5$, then;

$H(S_{full}(\mathcal{W}_{p+1}); x) = 3p + 2 + \frac{1}{2}(p^2 + 19p + 2)x + \frac{1}{2}p(5p + 3)x^2 + \frac{1}{2}p(3p - 13)x^3$.

Proof: First, to calculate $d(S_{full}(\mathcal{W}_{p+1}), 2)$, we take three cases.

Case 1: If $u = B$, then there is one subcase $|\{(B, e_i): i = 1,2, \dots, 2p\}| = 2p$.

Case 2: If $u = v_i$, $i = 0,1,2, \dots, p$, then there are five subcases.

- $\left|\left\{\left(v_1, v_j\right): j = 3,4, \dots, p-1\right\}\right| = p-3$.
- $\left|\left\{\left(v_i, v_j\right): i = 2,3, \dots, p-2, j = i+2, \dots, p\right\}\right| = \frac{1}{2}(p-3)(p-2)$.
- $\left|\left\{\left(v_0, e_{p+j}\right): j = 1,2, \dots, p\right\}\right| = p$.
- $\left|\left\{\left(v_i, e_j\right): i,j = 1,2, \dots, p, \ j \neq i\right\}\right| = p(p-1)$.
- $\left|\left\{\left(v_i, e_j\right): i = 3,4, \dots, p-1, j = p+i-2, p+i+1\right\} \cup \left\{\left(v_i, e_j\right): i = 1,2, j = p+i+1, 2p+i-2\right\} \cup \left\{\left(v_p, e_j\right): j = p+1, 2p-2\right\}\right| = 2p$.

Case 3: If $u = e_i$, $i = 1,2, \dots, 2p$, then there are two subcases.

- $\left|\left\{\left(e_i, e_j\right): i = 2,3, \dots, p, j = p+1, p+2, \dots, 2p, j \neq p+i-1, p+i\right\} \cup \left\{\left(e_1, e_j\right): j = p+2, \dots, 2p-1\right\}\right| = p(p-2)$.
- $\left|\left\{\left(e_i, e_j\right): i = p+1, p+2, \dots, 2p-2, j = i+2\right\} \cup \left\{\left(e_{2p-1}, e_{p+1}\right), \left(e_{2p}, e_{p+2}\right)\right\}\right| = p$.

Hence,

$d\left(S_{full}(\mathcal{W}_{p+1}), 2\right) = 2p + \frac{1}{2}p(3p+1) + p(p-1) = \frac{1}{2}p(5p+3)$.

Since $d\left(S_{full}(\mathcal{W}_{p+1}), 1\right) = q\left(S_{full}(\mathcal{W}_{p+1}\right)$ and $\sum_{k=1}^{\delta=3} d\left(S_{full}(\mathcal{W}_{p+1}), k\right) = \binom{|S_{full}(\mathcal{W}_{p+1})|}{2} = \frac{1}{2}(9p^2 + 9p + 2)$, then

$d\left(S_{full}(\mathcal{W}_{p+1}), 3\right) = \frac{1}{2}(9p^2 + 9p + 2) - \left(\frac{1}{2}(p^2 + 19p + 2) + \frac{1}{2}p(5p+3)\right) = \frac{1}{2}p(3p-13)$.

Since $d\left(S_{full}(\mathcal{W}_{p+1}), 0\right) = 3p + 2$, then

$H\left(S_{full}(\mathcal{W}_{p+1}); x\right) = 3p + 2 + \frac{1}{2}(p^2 + 19p + 2)x + \frac{1}{2}p(5p+3)x^2 + \frac{1}{2}p(3p-13)x^3$. ■

Remark 2.2.3: For $p = 3,4$, then

- $H\left(S_{full}(\mathcal{W}_4); x\right) = 11 + 34x + 21x^2$.
- $H\left(S_{full}(\mathcal{W}_5); x\right) = 14 + 47x + 44x^2$.

Corollary 2.2.4: For $p \geq 5$, then

1. $W\left(S_{full}(\mathcal{W}_{p+1})\right) = 10p^2 - 7p + 1$.
2. $D\left(S_{full}(\mathcal{W}_{p+1})\right) < \frac{20}{9}$. ■

2.3. Hosoya Polynomial of $S_{full}(\mathcal{P}_p)$.

Figure 5 shows the semifull graph of $\mathcal{P}_p$, $p \geq 3$.

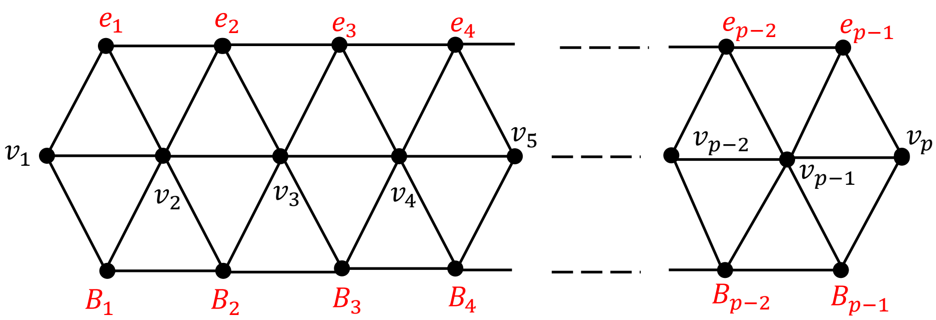


Figure 5: The $S_{full}(\mathcal{P}_p)$ graph

Note that for $p \geq 3$, $p\left(S_{full}(\mathcal{P}_p)\right) = 3p - 2$ and $q\left(S_{full}(\mathcal{P}_p)\right) = 7p - 9$.

Proposition 2.3.1: For $p \geq 3$, $diam\left(S_{full}(\mathcal{P}_p)\right) = p - 1$.

Proof: From Figure 5, we note that the maximum distance between the vertex of $\{e_1, v_1, B_1\}$ and the vertex of $\{e_{p-1}, v_p, B_{p-1}\}$ is $p-1$, that is

$$d(e_1, w) = \begin{cases} p-2, if\ w = e_{p-1}, \\ p-1, if\ w \in \{v_p, B_{p-1}\}. \end{cases}$$

$d(v_1, w) = p-1, if\ w \in \{e_{p-1}, v_p, B_{p-1}\}$.

$$d(B_1, w) = \begin{cases} p-2, if\ w = B_{p-1}, \\ p-1, if\ w \in \{e_{p-1}, v_p\}. \end{cases}$$

Hence, $diam\big(S_{full}(\mathcal{P}_p)\big) = p-1$. ■

Theorem 2.3.2: For $p \geq 5$, then;
$H\big(S_{full}(\mathcal{P}_p); x\big) = (3p-2) + (7p-9)x + (10p-21)x^2 + \sum_{k=3}^{p-2}(9p-9k-2)x^k + 7x^{p-1}$.

Proof: Let $V\Big(S_{full}(\mathcal{P}_p)\Big) = U \cup E \cup B$, where $U = \{v_i := 1,2,\dots,p\}$, $E = \{e_i : i = 1,2,\dots,p-1\}$ and $B = \{B_i : i = 1,2,\dots,p-1\}$, To find $H\big(S_{full}(\mathcal{P}_p); x\big)$, we take three cases:

Case 1: If the two vertices required to find the distance between them belong to the same subset, then there are three subcases, (i.e. $u, v \in U$ or $u, v \in E$, or $u, v \in B$).

- $H\big(U, S_{full}(\mathcal{P}_p); x\big) = H\big(\mathcal{P}_p; x\big) = \sum_{k=0}^{p-1}(p-k)x^k$.
- $H\big(E, S_{full}(\mathcal{P}_p); x\big) = H\big(\mathcal{P}_{p-1}; x\big) = \sum_{k=0}^{p-2}(p-1-k)x^k$.
- $H\big(B, S_{full}(\mathcal{P}_p); x\big) = H\big(\mathcal{P}_{p-1}; x\big) = \sum_{k=0}^{p-2}(p-1-k)x^k$.

Case 2: If the two vertices required to find the distance between them belong to the different subset, then there are also three subcases, (i.e. $u \in U$ and $v \in E$, or $u \in U$ and $v \in B$ or $u \in E$ and $v \in B$).

- If $u \in U$ and $v \in E$, then

➢ $d(v_i, e_j) = j-i+1, 1 \leq i \leq p-1, i \leq j \leq p-1$.
➢ $d(v_i, e_j) = i-j, 2 \leq i \leq p, 1 \leq j \leq i-1$.

Hence,

$$\begin{aligned} H\big(U \times E, S_{full}(\mathcal{P}_p); x\big) &= \sum_{i=1}^{p-1}\sum_{j=i}^{p-1} x^{j-i+1} + \sum_{i=2}^{p}\sum_{j=1}^{i-1} x^{i-j} \\ &= \sum_{j=1}^{p-1} x^j + \sum_{i=2}^{p-1}\sum_{j=i}^{p-1} x^{j-i+1} + \sum_{i=2}^{p-1}\sum_{j=1}^{i-1} x^{i-j} + \sum_{j=1}^{p-1} x^{p-j} \\ &= \sum_{j=1}^{p-1} x^j + \sum_{i=2}^{p-1}\sum_{j=1}^{p-i} x^j + \sum_{i=2}^{p-1}\sum_{j=1}^{i-1} x^j + \sum_{j=1}^{p-1} x^j \\ &= 2\sum_{j=1}^{p-1} x^j + \sum_{i=1}^{p-2}(p-i-1)x^i + \sum_{i=1}^{p-2}(p-i-1)x^i \\ &= 2\sum_{k=1}^{p-1} x^k + 2\sum_{k=1}^{p-2}(p-k-1)x^k \\ &= 2\sum_{k=1}^{p-2}(p-k)x^k + 2x^{p-1}. \end{aligned}$$

- If $u \in U$ and $v \in B$, then

➢ $d(v_i, B_j) = j-i+1, 1 \leq i \leq p-1, i \leq j \leq p-1$.
➢ $d(v_i, B_j) = i-j, 2 \leq i \leq p, 1 \leq j \leq i-1$.

Hence, with the same steps as the subcase above
$H\big(U \times B, S_{full}(\mathcal{P}_p); x\big) = 2\sum_{k=1}^{p-2}(p-k)x^k + 2x^{p-1}$.

- If $u \in E$ and $v \in B$, then

➢ $d(e_i, B_j) = j-i+1, 1 \leq i \leq p-2, i+1 \leq j \leq p-1$.
➢ $d(e_i, B_j) = 2, i = j, 1 \leq i \leq p-1$.

➢ $d(e_i, B_j) = i - j + 1, 2 \le i \le p-1, 1 \le j \le i-1.$

Hence,

$$H(E \times B, S_{full}(\mathcal{P}_p); x) = \sum_{i=1}^{p-2} \sum_{j=i+1}^{p-1} x^{j-i+1} + \sum_{i=1}^{p-1} x^2 + \sum_{i=2}^{p-1} \sum_{j=1}^{i-1} x^{i-j+1}$$
$$= \sum_{j=2}^{p-1} x^j + \sum_{i=2}^{p-2} \sum_{j=i+1}^{p-1} x^{j-i+1} + (p-1)x^2 + \sum_{i=2}^{p-2} \sum_{j=1}^{i-1} x^{i-j+1} + \sum_{j=1}^{p-2} x^{p-j}$$
$$= (p-1)x^2 + \sum_{j=2}^{p-1} x^j + \sum_{i=2}^{p-2} \sum_{j=2}^{p-i} x^j + \sum_{i=2}^{p-2} \sum_{j=2}^{i} x^j + \sum_{j=2}^{p-1} x^j$$
$$= (p-1)x^2 + 2\sum_{j=2}^{p-1} x^j + \sum_{i=2}^{p-2} (p-i-1)x^i + \sum_{i=2}^{p-2} (p-i-1)x^i$$
$$= (p-1)x^2 + 2\sum_{k=2}^{p-1} x^k + 2\sum_{k=2}^{p-2} (p-k-1)x^k$$
$$= (p-1)x^2 + 2\sum_{k=2}^{p-2} (p-k)x^k + 2x^{p-1}.$$

Then,

$$H(S_{full}(\mathcal{P}_p); x) = H(U, S_{full}(\mathcal{P}_p); x) + H(E, S_{full}(\mathcal{P}_p); x) + H(B, S_{full}(\mathcal{P}_p); x)$$
$$+ H(V \times E, S_{full}(\mathcal{P}_p); x) + H(V \times B, S_{full}(\mathcal{P}_p); x) + H(E \times B, S_{full}(\mathcal{P}_p); x)$$
$$= \sum_{k=0}^{p-1} (p-k)x^k + \sum_{k=0}^{p-2} (p-1-k)x^k + \sum_{k=0}^{p-2} (p-1-k)x^k$$
$$+ 4\sum_{k=1}^{p-2} (p-k)x^k + 4x^{p-1} + (p-1)x^2 + 2\sum_{k=2}^{p-2} (p-k)x^k + 2x^{p-1}$$
$$= p + (p-1)x + (p-2)x^2 + \sum_{k=3}^{p-2} (p-k)x^k + x^{p-1} + 2(p-1) +$$
$$2(p-2)x + 2(p-3)x^2 + 2\sum_{k=3}^{p-2} (p-1-k)x^k + 4(p-1)x + 4(p-2)x^2 +$$
$$4\sum_{k=3}^{p-2} (p-k)x^k + 4x^{p-1} + (p-1)x^2 + 2(p-2)x^2 + 2\sum_{k=3}^{p-2} (p-k)x^k + 2x^{p-1}$$
$$= (3p-2) + (7p-9)x + (10p-21)x^2 + \sum_{k=3}^{p-2} (9p-9k-2)x^k + 7x^{p-1}. \blacksquare$$

Remark 2.3.3:

- $H(S_{full}(\mathcal{P}_3); x) = 7 + 12x + 9x^2.$
- $H(S_{full}(\mathcal{P}_4); x) = 10 + 19x + 19x^2 + 7x^3.$

Corollary 2.3.4: For $p \ge 5$, then

1. $W\left(S_{full}(\mathcal{P}_p)\right) = \frac{1}{2}(3p^2 - 2p - 1).$
2. $D\left(S_{full}(\mathcal{P}_p)\right) < \frac{1}{3}(p+1).$ ■

2.4. Hosoya polynomial of $S_{full}(\mathcal{C}_p)$

The graph $S_{full}(\mathcal{C}_p)$, for all $p \ge 3$ is shown in Figure 6.

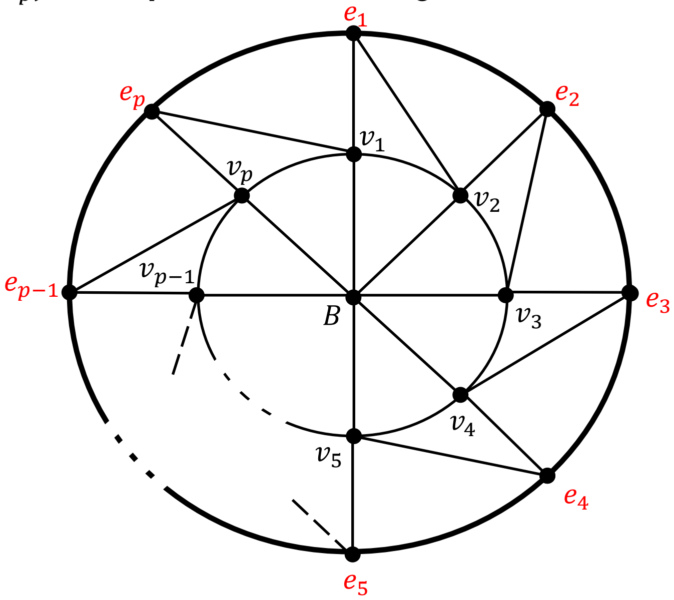


Figure 6: The $S_{full}(\mathcal{C}_p)$ graph**.**

Note that for $p \geq 3$, $p\left(S_{full}(\mathcal{C}_p)\right) = 2p + 1$ and $q\left(S_{full}(\mathcal{C}_p)\right) = 5p$.

Proposition 2.4.1: For $p \geq 7$, $diam\left(S_{full}(\mathcal{C}_p)\right) = 4$.

Proof: Let u be any vertex of $S_{full}(\mathcal{C}_p)$, where $V(S_{full}(\mathcal{C}_p)) = \{B, v_i, e_j : i = 1,2, \dots, p, j = 1,2, \dots, p\}$, then there are three cases.

Case 1: If $u = B$, then $d(u,w) \leq 2$, for all $w \in V(S_{full}(\mathcal{C}_p)) - \{u\}$.

Case 2: If $u = v_i, i = 1,2, \dots, p$, then $d(u,w) \leq 3$, for all $w \in V(S_{full}(\mathcal{C}_p)) - \{u\}$.

Case 3: If $u = e_j, j = 1,2, \dots, p$, then $d(u,w) \leq 4$, for all $w \in V(S_{full}(\mathcal{C}_p)) - \{u\}$.

From the definition of the diameter, then $diam\left(S_{full}(\mathcal{C}_p)\right) = 4$. ■

Theorem 2.4.2: For $p \geq 7$, then

$$H\left(S_{full}(\mathcal{C}_p); x\right) = 2p + 1 + 5px + \frac{1}{2}p(p+5)x^2 + p(p-3)x^3 + \frac{1}{2}p(p-7)x^4.$$

Proof: To find $H\left(S_{full}(\mathcal{C}_p); x\right)$, for all $p \geq 7$, we take four cases:

Case 1: If $k = 0,1$, clearly that

$$d\left(S_{full}(\mathcal{C}_p), 0\right) = |V\left(S_{full}(\mathcal{C}_p)\right)| = 2p + 1,$$

$$d\left(S_{full}(\mathcal{C}_p), 1\right) = |E\left(S_{full}(\mathcal{C}_p)\right)| = 5p.$$

Case 2: If $k = 2$, then there are five subcases.

- $|\{(e_i, B)\}| = |\{(e_i, B): 1 \leq i \leq p\}| = p.$
- $|\{(e_i, e_{i+2})\}| = \left|\{(e_i, e_{i+2}): 1 \leq i \leq p, e_{p+r} \equiv e_r, r = 1,2\}\right| = p.$
- $\left|\{(v_i, v_j): 2 \leq i \leq p-2, i+2 \leq j \leq p\} \cup \{(v_1, v_j): 3 \leq j \leq p-1\}\right| = \frac{1}{2}p(p-3).$
- $\left|\{(e_i, v_{i+2}): 1 \leq i \leq p, v_{p+r} \equiv v_r, r = 1,2\}\right| = p.$
- $\left|\{(e_i, v_{i-1}): 1 \leq i \leq p, v_0 \equiv v_p\}\right| = p.$

Hence,

$$d\left(S_{full}(\mathcal{C}_p), 2\right) = \frac{1}{2}p(p+5).$$

Case 3: If $k = 3$, then there are two subcases.

$\left|\{(e_i, e_{i+3}): 1 \leq i \leq p, e_{p+r} \equiv e_r, r = 1,2,3\}\right| = p.$

$\left|\{(e_i, v_j): 1 \leq i \leq p, i+3 \leq j \leq p-2+i, v_{p+r} \equiv v_r, r = 1,2, \dots, p-2\}\right| = p(p-4).$

Hence,

$$d\left(S_{full}(\mathcal{C}_p), 3\right) = p(p-3).$$

Case 4: If $k = 4$, then from $\sum_{k=1}^{\delta} d(G,k) = \binom{|G|}{2}$, for any connected graph G of order $|G|$ with diameter δ, we have

$$d\left(S_{full}(\mathcal{C}_p), 4\right) = p(2p+1) - 5p - \frac{1}{2}p(p+5) - p(p-3)) = \frac{1}{2}p(p-7).$$ ■

Remark 2.4.3:

1. $H\left(S_{full}(\mathcal{C}_3); x\right) = 7 + 15x + 6x^2.$
2. $H\left(S_{full}(\mathcal{C}_4); x\right) = 9 + 20x + 16x^2.$
3. $H\left(S_{full}(\mathcal{C}_5); x\right) = 11 + 25x + 25x^2 + 5x^3.$
4. $H\left(S_{full}(\mathcal{C}_6); x\right) = 13 + 30x + 33x^2 + 15x^3.$

Corollary 2.4.4: For $p \geq 7$, then

1. $W\left(S_{full}(\mathcal{C}_p)\right) = p(6p - 13).$
2. $D\left(S_{full}(\mathcal{C}_p)\right) < 3.$ ■

3. Conclusions:

From the previous results found in this research, the reader can note that the Wiener index of a semifull graph for some special graphs is directly proportional to the order of the graphs, but only in different proportions. This makes us find a relationship between Wiener topological indices for different graphs under a semifull graph, since the relation between them is direct directly proportional, see Figure 7.

Figure 7: Wiener Index of Semifull Graph for Star, Wheel, Path, and Cycle Graphs

REFERENCES:

[1] H. Wiener, "Structural determination of paraffin boiling points," *Journal of the American chemical society,* vol. 69, no. 1, pp. 17-20, 1947.

[2] G. Chartrand and L. Lesniak, "Graphs and Digraphs", 6th ed. Pacific Grove, CA, USA, Wadsworth and Brooks/ Cole, 2016.

[3] H. Hosoya, "On some counting polynomials in chemistry," *Discrete applied mathematics,* vol. 19, no. 1-3, pp. 239–257, 1988.

[4] C. Zhong-Lin, A. Ashaq, A. Haseeb, N. Asim and A. C. Maqbool, "Hosoya and Harary Polynomials of Hourglass and Rhombic Benzenoid Systems," *Journal of Chemistry*, vol. 2020, pp. 1-14, 2020.

[5] A. R. Bilal, A. Fawad, A. Suliman and N. Muhammad, " Hosoya polynomials of power graphs of certain finite groups," *Molecules*, vol. 27, no. 6081, pp. 1-19, 2022 .

[6] S. Das and S. Rai, "On The Hosoya Polynomial of The Third Type of The Chain Hex-Derived Network," *Discrete Mathematics and Mathematical Cybernetics,* vol.3, pp. 67-78, 2022.

[7] N. Muhammad, N. Aamra, A. Adnan and I. B. Saad, "Hosoya Polynomial for Subdivided Caterpillar Graphs," *Combinatorial Chemistry & High Throughput Screening.* vol.25, pp. 554 – 559, 2022.

[8] S. Umber, R. Sidra, O. Cenap and P. Richard, "On Hosoya Polynomial and Subsequent Indices of C4C8(R) and C4C8(S) Nanosheets," *Symmetry*, vol.14, no. 1349, pp.1-15, 2022.

[9] F.M. Reza, and W. Gao, "The Schultz index and Schultz polynomial of the Jahangir Graphs $J_{5,m}$," *Applied mathematics*, vol. 6, no. 14, pp. 2319-2325, 2015.

[10] S.R.Jalal, O. Khormali, and A. Mahmiani, "The polynomial of detour index for a graph," *World Applied Sciences Journal*, vol. 15, no. 10, pp. 1473-1483, 2011.

[11] S. Rai and S. Das, "M-polynomial and degree-based topological indices of subdivided chain hex-derived network of type 3," *International Conference on Advanced Network Technologies and Intelligent Computing*. Cham: Springer International Publishing, pp. 410–424, 2022.

[12] C. William and C. Logan, "A graph polynomial from chromatic symmetric functions," *Journal of Graph Theory*, vol. 105, no. 4, pp. 633-651, 2024.

[13] R. A. Mustafa, A. M. Ali and A. M. Khidhir, " Mn– Polynomials of Some Special Graphs," *Iraqi Journal of Science* , vol. 62, no. 6, pp. 1986-2021, 2022.

[14] S. Ramy, M. Suhail and A. Qays, "Hosoya, Schultz, and Gutman Polynomials of Generalized Petersen Graphs P(n, 1) and P(n, 2). Hindawi," *Journal of Mathematics*, vol. 2023, pp. 1-18, 2023.

[15] B. E. Sagan, Y. N. Yeh and P. Zhang, "The Wiener polynomial of a graphs," *International Journal of Quantum Chemistry*, vol. 60, no. 5, pp. 959-969, 1996.

[16] J. Hossein, M. G. Roghayeh, M. N. Sonali, K. Maluti and T. N. Van, "Application of hosoya polynomial to solve a class of time-fractional diffusion equations," *Fractals*, vol. 31, no. 4, pp. 1-12, 2023.

[17] G. Modjtaba, H. N. Mardjan and D. Matthias, "Novel results on partial hosoya polynomials: An application in chemistry," *Applied Mathematics and Computation*, vol. 433, no. 127379, pp. 1-15, 2022.

[18] W. Dejun, H. Ahmad and W. Nazeer, "Hosoya and Harary polynomials of TUC4 nanotube," *Mathematical Methods in the Applied Sciences*, vol. 6487, pp. 1–17, 2020.

[19] K. Martin and T. Niko, "A Method for Computing the Edge–Hosoya Polynomial with Application to Phenylenes," *Match: communications in mathematical and in computer chemistry*, vol. 89, pp. 605–629, 2023.

[20] A. Modabish, A. Alameri, M. S. Gumaan, and M. Alsharaf, "The second Hyper-Zagreb index of graph operations," *Journal of Mathematical and Computational Science*, vol. 11, no. 2, pp. 1455–1469, 2021.

[21] D. Sadik, D. Musa, S. C. Ahmet and N. C. Ismail, "On Omega Index and Average Degree of Graph," *Journal of Mathematics*, vol. 2021, pp. 1-5, 2021.

[22] J. Muhammad, I. Muhammad, A. Uzma and L. Jia-Bao, "Sharp Bounds of First Zagreb Coindex for F-Sum Graphs," *Journal of Mathematics*, vol. 2021, pp. 1- 19, 2021.

[23] B. K. Divyashree, "On Some Topological Indices of the Starphene Graph," *Global Journal of Pure and Applied Mathematics*, vol. 18, no.2, pp. 677-686, 2022.

[24] J. K. Doyle and J. E. Graver, "Mean distance in a graph," *Discrete Mathematics* , vol. 17, no. 2, pp. 147-154,1977.

[25] V. R. Kulli, "The Semifull Graph of a Graph," *Annals of Pure and Applied Mathematics,* vol. 10, vo.1, pp. 99-104, 2015.