DOI: 10.24996/ijs.2025.66.5.20

ISSN: 0067-2904

Modeling and Analysis of A Prey-Predator-Scavenger System Encompassing Fear, Hunting Cooperation, and Harvesting

Lubna Riyadh Ibrahim, Dahlia Khaled Bahlool *

Department of Mathematics, College of science, University of Baghdad, Baghdad, Iraq

Received: 19/3/2024 Accepted: 9/6/2024 Published: 30/5/2025

Abstract

In this research, a model was taken to explore the relationship between the prey, the predator, and the scavenger in the diet to study dynamic behavior. This was done by forming a system of ordinary differential equations, and critical points were found for the model that match biologically and mathematically to our study. Their existence was proven, and theories were created for the model. By applying local and global stability methods, also we were able determine the bifurcation occurring at particular sites. After that, we observed the result of cooperation in hunting, which greatly affects both prey and predators, playing a vital and influential role. Everything we mentioned above was also proven practically, using Mathematica 13.2 to prove the validity of what we mentioned practically.

Keywords: Stability, prey refuge, Food web, persistence, bifurcation.

نمذجة وتحليل نظام الفريسة والمفترس والزبال الذي يشمل الخوف , التعاون في الصيد والحصاد

لبنى رياض ابراهيم , داليا خالد بهلول *

قسم الرياضيات , كلية العلوم, جامعة بغداد, بغداد, العراق

الخلاصة

في هذا البحث تم اخذ مودل يستكشف علاقه بين كل من الفريسة ،المفترس والزبال في النظام الغذائي لدراسة سلوكية الديناميكية وتم ذلك من خلال تكوين نظام من المعادلات التفاضلية الاعتيادية وتم ايجاد نقاط الحرجة للموديل تطابق بايلوجيا ورياضيا لدراستنا ووجودها ايضا تم التحقق منه وتم ايضا وضع نظريات لكل من الاستقرايه المحلية والشامله لايجاد التشعب في تلك النقاط وبعد ذلك لحظنا تاثير تعاون في الصيد مايوثر على الفرائس والمفترسات وما له من دور مهم وموثر وتم اثبات كل ماذكرناه اعلاه ايضا عمليا باستخدام ماثماتكا13.2 لاثبات صحه ماذكرنا عمليا.

1. Introduction

The resilience and stability of ecosystems are greatly influenced by the prey, predator and scavenger relationships within intricate ecological web. Goldsmith and Sutherland [1] indicated in their article that a lot of focus has been put by ecologists and conservationists in grasping these mechanisms to sustain biodiversity; while maintaining the functions of ecosystem under human induced stresses and environmental changes. Predators are vital parts

* Email : dahlia.khaled@sc.uobaghdad.edu.iq

of the ecology due to their effective facilitation of energy flow across ecosystems. These interactions go beyond mere predation as envisaged by this theory. A few other drivers encompass hunting by human beings, methods of hunting and reaction to fear factors. In most cases, these types of ecological models oversimplify this complexity and ignore the characteristics of real ecosystems. By developing our understanding of stability and resilience in predator-prey dynamics, Freedman [2] argues that it is a fundamental element for the management and conservation of ecosystems. The survival rates for both predators and prey are very high for longer periods as indicated by empirical investigations and mathematical simulations. Borofsky and Freedman [3] have revealed that social learning, predator cooperation/lack thereof and prey availability can all interact powerfully to determine the nature of an ecosystem. Currently, the issue of how animals cooperate when they hunt preys versus those who do not is being considered essential in many scientific studies on predation/fear in ecology [4 - 6]. There are several studies on the effect of fear, including a study of the effect of fear in a model of a food chain consisting of three species that includes the Beddington-Deangelis functional response, where the growth rate in the first and second levels decreased as a result of the presence of the predator in the upper level [7]. Some researchers have studied the effect of fear on the system in the presence of cooperative hunting [8]. Besides, predator cooperative hunting is added to a layer of complexity that significantly alters the population dynamics and community structure [9]. Nonlinear dynamics can lead to shifts in scavenger, prey and predator populations distributions and abundances whenever all these predators combine their actions to surpass the aggregate result of their individual interactions [10]. Another research showed how prey refuge and hunting attempts affected equilibrium density values. In conclusion, the stable prey refuges have been found not to be affected by harvesting as far as final densities are concerned. However, when more efforts are concentrated on predator species through harvests and as efforts towards prey species increase, then there will be a decrease in the density of predator species; which is an astonishing finding [11]. The authors shed light on how prey refuge and harvesting attempts influence equilibrium density estimates. To determine how hunting cooperation impacts fear in preys and predation in predators. Some researchers have studied the effect of harvesting on the model of stage structure consisting of juvenile and adult prey population and one predator [12]. Other researchers studied the bifurcation of the prey-predator-scavenger model, with the effects of both harvest and toxicity as in [13]. One of the researchers also studied the prey and predator model, but he used functional response square root and noticed the effect of the fear parameter on the system as shown in [14]. [15] analyzed a modified Leslie-Gower predator-prey model with non-linear harvesting and fright impact Some researchers tried to study the effect of the disease, and an environmental-mathematical epidemiological model was formulated and studied, consisting of a system that includes fear and disease in the prey community [16]. Through all of these previous studies that were reviewed, we tried to know the effect of variables, therefore in this study, we tried to know the effect of variables such as fear, cooperative hunting, and harvesting on the system theoretically and practically through simulated the model as well as conducted quantitative analysis to understand issues within system stability through simulated experiments.

2. Mathematical Model Formulation

This section presents a food web model that includes three species: prey, predator, and scavenger. The mathematical representation of a food web in the real world is constructed applying the functional response of the Lotka-Volterra equations to describe the model. The mathematical representation of a food web system consists of three non-linear, independent ordinary differential equations:

$$\frac{d\mathcal{X}}{d\mathcal{T}} = \frac{r\mathcal{X}}{1+f(\mathcal{Y}+\mathcal{Z})} - b\mathcal{X}^2 - (a_1 + b_1\mathcal{Y})\mathcal{X}\mathcal{Y} - (a_2 + b_2\mathcal{Z})\mathcal{X}\mathcal{Z},$$
$$\frac{d\mathcal{Y}}{d\mathcal{T}} = e_1(a_1 + b_1\mathcal{Y})\mathcal{X}\mathcal{Y} - d_1\mathcal{Y} - q_1E_1\mathcal{Y}^2, \quad (1)$$
$$\frac{d\mathcal{Z}}{d\mathcal{T}} = e_2(a_2 + b_2\mathcal{Z})\mathcal{X}\mathcal{Z} + a_3\mathcal{Y}\mathcal{Z} - d_2\mathcal{Z} - q_2E_2\mathcal{Z}^2.$$

Given that $\mathcal{X}(0) \geq 0$, $\mathcal{Y}(0) \geq 0$, and $\mathcal{Z}(0) \geq 0$.The prey population at a given time is represented by $\mathcal{X}(\mathcal{T})$. The predator population at time $\mathcal{T}$ is represented by $\mathcal{Y}(\mathcal{T})$, and $\mathcal{Z}(\mathcal{T})$ denotes the scavenger population at time $\mathcal{T}$, according to the Lotka-Volterra model of functional response, the predator and scavenger feed on the prey. The scavenger survives itself by consuming the remains of decaying creatures, either on a permanent or temporary basis. Ultimately, it is observed that the population of prey increases in a logistic manner when predators and scavengers are absent, whereas the population of predators and scavengers decreases exponentially in the absence of prey. The parameters can be described in the following table 1.

Table 1: Description of parameters

Parameters	Description
r	The natural growth rate
f	Fear rate of prey
b	Intraspecific rivalry
a_1, a_2	The rates at which predator and scavenger attack, subsequently
b_1, b_2	The hunting cooperation level for both predator and scavenger, subsequently
e_1, e_2	The conversion rates of prey biomass into predator and scavenger subsequently
a_3	The conversion rate of predator's carcasses biomass to scavengers
d_1, d_2	The mortality rates of predators and scavenger subsequently
q_1, q_2	The catchability parameters of predator and scavenger subsequently
E_1, E_2	The harvesting attempts of predator and scavenger subsequently

In order to decrease the quantity of parameters, we employed a process of non-dimensionalized on the model system, utilizing the following scaling technique:

$$t = r\mathcal{T}, x = \frac{b}{r}\mathcal{X}, y = \frac{a_1}{r}\mathcal{Y}, z = \frac{a_2}{r}\mathcal{Z}$$
$$w_1 = f\frac{r}{a_1}, w_2 = \frac{a_1}{a_2}, w_3 = \frac{rb_1}{a_1^2}, w_4 = \frac{rb_2}{a_2^2}, w_5 = \frac{e_1a_1}{b},$$
$$w_6 = \frac{d_1}{r}, w_7 = \frac{q_1E_1}{a_1}, w_8 = \frac{e_2a_2}{b}, w_9 = \frac{a_3}{a_1}, w_{10} = \frac{d_2}{r}, w_{11} = \frac{q_2E_2}{a_2}.$$

Next, a dimensionless system is derived:

$$\frac{dx}{dt} = x\left[\frac{1}{1+w_1(y+w_2z)} - x - (w_3y+1)y - (w_4z+1)z\right] = xf_1(x,y,\mathrm{z}),$$
$$\frac{dy}{dt} = y[w_5(1+w_3y)x - w_6 - w_7y] = yf_2(x,y,\mathrm{z}), \quad (2)$$
$$\frac{dz}{dt} = \mathrm{z}[w_8(1+w_4z)x + w_9y - w_{10} - w_{11}z] = zf_3(x,y,\mathrm{z}).$$

We notice that the parameters of Eq. (1) decreased from what they used to be, as their number was sixteen and became eleven in Eq. (2).

2. Existences, Boundedness

The functions xf_1, yf_2, zf_3 exhibit continuity and possess continuous partial derivatives on $R_+^3 = \{(x,y,z) \in R^3 : x \geq 0, y \geq 0 \;\&\; z \geq 0\}$ in Eq. (2). Therefore, they can be classified as

Lipschitz functions. Therefore, the Eq. (2) exhibits both the property of having a solution and unique solution.

Theorem 1. Every population in Eq. (2) that initiate within R_+^3 have uniformly bounded solutions.

Proof. Set $(x(t), y(t), z(t))$ to be one of the solutions of Eq. (2), we get uniformly bounded from the first equation.

$\frac{dx}{dt} \leq x\left[\frac{1}{1+w_1(y+w_2 z)} - x\right] \leq x(1-x).$

Hence, we get $x(t) \leq 1$ as $t \to \infty$. Consider that $N_1(t) = x(t) + \frac{y(t)}{w_5}$, then after time derivative along the solutions of Eq. (2) we have:

$\frac{dN_1}{dt} \leq x - \frac{w_6}{w_5} y \leq (1+w_6)x - w_6\left(x + \frac{y}{w_5}\right).$

Therefore, it is getting the following

$\frac{dN_1}{dt} + w_6 N_1 \leq (1+w_6).$

Then by solving the differential inequality mentioned above, we get that $y \leq \frac{w_5(1+w_6)}{w_6} = \beta_1$ as $t \to \infty$.

Now, consider that $N_2(t) = x(t) + \frac{y(t)}{w_5} + \frac{z(t)}{w_8}$, then

$\frac{dN_2}{dt} \leq 2x - x - \frac{w_6}{w_5} y - z\left(\frac{w_{10} - w_9\beta_1}{w_8}\right) \leq 2 - \delta\left(x + \frac{y}{w_5} + \frac{z}{w_8}\right),$

Where $\delta = \min\{1, w_6, w_{10} - w_9\beta_1\}$. Thus, it is observed that

$\frac{dN_2}{dt} + \delta N_2 \leq 2$ gives $N_2 \leq \frac{2}{\delta}$ as $t \to \infty$.

As a result, all solution in Eq. (2) has a uniform boundary.

3. Equilibrium and their existence

This section establishes the existing conditions and conducts local stability analyses of equilibrium points for Eq. (2). It's note that there can be a maximum of five non-negative equilibrium points for Eq. (2), which are detailed below:

- The extinction equilibrium points $E_0 = (0,0,0)$ and axial point $E_1 = (1,0,0)$ always exists.
- The equilibrium in the absence of scavenger $E_{xy} = (\breve{x}, \breve{y}, 0)$, where

$$\breve{x} = \frac{w_6 + w_7\breve{y}}{w_5(1+w_3\breve{y})}, \tag{3}$$

with $\breve{y}$ is a solution that is positive for the fourth-order polynomial equation:

$R_1 y^4 + R_2 y^3 + R_3 y^2 + R_4 y + R_5 = 0.$

Where:

$R_1 = -w_1 w_3^2 w_5 < 0,$

$R_2 = -w_3 w_5(2w_1 + w_3) < 0,$

$R_3 = -w_5(w_1 + 2w_3) - w_1 w_7 < 0,$

$R_4 = -w_5 + w_3 w_5 - w_1 w_6 - w_7,$

$R_5 = w_5 - w_6.$

Clearly, E_{xy} exists uniquely within the positive quarter of xy-plane's interior if and only if the following sufficient condition is hold.

$$w_6 < w_5. \tag{4}$$

- The equilibrium in the absence of predator $E_{xz} = (\tilde{x}, 0, \tilde{z})$, where

$$\tilde{x} = \frac{w_{10} + w_{11}\tilde{z}}{w_8(1+w_4\tilde{z})}, \tag{5}$$

where $\tilde{z}$ is a positive root of the polynomial equation of order four:

$N_1 z^4 + N_2 z^3 + N_3 z^2 + N_4 z + N_5 = 0.$
Where
$N_1 = -w_1 w_2 w_4^2 w_8 < 0,$
$N_2 = -w_4 w_8 (2w_1 w_2 + w_4) < 0,$
$N_3 = -w_1 w_2 (w_{10} + w_{11}) - 2w_4 w_8 < 0,$
$N_4 = w_4 w_8 - w_1 w_2 w_{10} - w_{11} - w_8,$
$N_5 = w_8 - w_{10}.$
Obviously, E_{xz} exists uniquely within the positive quarter of xz-plane's interior if and only if the following sufficient condition is satisfied.

$$w_{10} < w_8. \tag{6}$$

- The cohabitation equilibrium point $E_{xyz} = (\bar{x}, \bar{y}, \bar{z})$, where

$$\left.\begin{aligned} \bar{x} &= \frac{w_6 + w_7 \bar{y}}{w_5 (1 + w_3 \bar{y})} \\ \bar{z} &= \frac{-w_{10} w_5 + w_6 w_8 - w_{10} w_3 w_5 \bar{y} + w_7 w_8 \bar{y} + w_5 w_9 \bar{y} + w_3 w_5 w_9 \bar{y}^2}{w_{11} w_5 - w_4 w_6 w_8 + w_{11} w_3 w_5 \bar{y} - w_4 w_7 w_8 \bar{y}} \end{aligned}\right\}, \tag{7a}$$

where $\bar{y}$ is a positive root of the polynomial equation of order seven.

$$\mu_1 y^7 + \mu_2 y^6 + \mu_3 y^5 + \mu_4 y^4 + \mu_5 y^3 + \mu_6 y^2 + \mu_7 y + \mu_8 = 0, \tag{7b}$$

where the coefficients $\mu_i, i = 1,2,\ldots,8$ are determined using Mathematica program. Due to their large and intricate forms, they will not be provided here. However, direct computation demonstrates that the cohabitation equilibrium point E_{xyz} can only be found inside octant's interior, if there is a singular positive root for the Eq. (7b). Consequently, Eq. (7b) has at least one positive root provided that one of the following sets of conditions holds.

$$\left.\begin{aligned} \mu_1 < 0, \mu_8 > 0 \\ or \\ \mu_1 > 0, \mu_8 < 0 \end{aligned}\right\}. \tag{8a}$$

Moreover, for the positivity of $\bar{z}$, it is necessary to have one set of the following sets of conditions:

$$\left.\begin{aligned} -w_{10} w_5 + w_6 w_8 - w_{10} w_3 w_5 \bar{y} + w_7 w_8 \bar{y} + w_5 w_9 \bar{y} + w_3 w_5 w_9 \bar{y}^2 > 0 \\ w_{11} w_5 - w_4 w_6 w_8 + w_{11} w_3 w_5 \bar{y} - w_4 w_7 w_8 \bar{y} > 0 \\ \text{or} \\ -w_{10} w_5 + w_6 w_8 - w_{10} w_3 w_5 \bar{y} + w_7 w_8 \bar{y} + w_5 w_9 \bar{y} + w_3 w_5 w_9 \bar{y}^2 < 0 \\ w_{11} w_5 - w_4 w_6 w_8 + w_{11} w_3 w_5 \bar{y} - w_4 w_7 w_8 \bar{y} < 0 \end{aligned}\right\}. \tag{8b}$$

Accordingly, E_{xyz} can only be found uniquely in interior of R_+^3 if and only if the above sufficient conditions are hold.

4. LOCAL STABILITY ANALYSIS

The stability of the preceding equilibrium points is examined by calculating the Jacobian matrix denoted as $L(x,y,z)$ at the point (x,y,z) Subsequently, the eigenvalues are computed. The Jacobian matrix is defined as follows:

$$L(x,y,z) = \begin{bmatrix} x\frac{\partial f_1}{\partial x} + f_1 & x\frac{\partial f_1}{\partial y} & x\frac{\partial f_1}{\partial z} \\ y\frac{\partial f_2}{\partial x} & y\frac{\partial f_2}{\partial y} + f_2 & y\frac{\partial f_2}{\partial z} \\ z\frac{\partial f_3}{\partial x} & z\frac{\partial f_3}{\partial y} & z\frac{\partial f_3}{\partial z} + f_3 \end{bmatrix}, \tag{9}$$

where

$$\frac{\partial f_1}{\partial x} = -1, \quad \frac{\partial f_1}{\partial y} = -\frac{w_1}{(1 + w_1(y + w_2 z))^2} - (1 + 2w_3 y),$$
$$\frac{\partial f_1}{\partial z} = -\frac{w_1 w_2}{(1 + w_1(y + w_2 z))^2} - (1 + 2w_4 z),$$
$$\frac{\partial f_2}{\partial x} = w_5 + w_3 w_5 y, \quad \frac{\partial f_2}{\partial y} = w_3 w_5 x - w_7, \quad \frac{\partial f_2}{\partial z} = 0,$$
$$\frac{\partial f_3}{\partial x} = w_8 + w_4 w_8 z, \quad \frac{\partial f_3}{\partial y} = w_9, \quad \frac{\partial f_3}{\partial z} = w_4 w_8 x - w_{11}.$$

The extinction equilibrium points E_0, the Jacobian matrix is expressed as follows:

$$L(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -w_6 & 0 \\ 0 & 0 & -w_{10} \end{bmatrix}. \tag{10a}$$

Then, the eigenvalues of the matrix $L(E_0)$, are given by:
$\lambda_{01} = 1 > 0$, $\lambda_{02} = -w_6 < 0$, and $\lambda_{03} = -w_{10} < 0$.
(10b)
As a result, the extinction equilibrium point E_0 is a saddle point.
For axial equilibrium E_1, the formula for the Jacobian matrix is:

$$L(E_1) = \begin{bmatrix} -1 & -(1 + w_1) & -(1 + w_1 w_2) \\ 0 & w_5 - w_6 & 0 \\ 0 & 0 & w_8 - w_{10} \end{bmatrix}. \tag{11a}$$

Thus, the eigenvalues of the matrix $L(E_1)$, are determined by:

$$\lambda_{11} = -1 < 0, \quad \lambda_{12} = w_5 - w_6, \quad \text{and} \quad \lambda_{13} = w_8 - w_{10}. \tag{11b}$$

Clearly, all the above eigenvalues are negative if and only if the following conditions are satisfied, E_1 is locally asymptotically stable:

$$w_5 < w_{6,} \tag{11c}$$
$$w_8 < w_{10}. \tag{11d}$$

For the equilibrium in the absence of scavenger $E_{xy} = (\breve{x}, \breve{y}, 0)$, the formula for the Jacobian matrix is:

$$L(E_{xy}) = \begin{bmatrix} -\breve{x} & -\breve{x}(\frac{w_1}{(1+w_1\breve{y})^2} + 1 + 2w_3\breve{y}) & -\breve{x}(\frac{w_1 w_2}{(1+w_1\breve{y})^2} + 1) \\ \breve{y}(w_5 + w_3 w_5 \breve{y}) & \breve{y}(w_3 w_5 \breve{x} - w_7) & 0 \\ 0 & 0 & w_8\breve{x} + w_9\breve{y} - w_{10} \end{bmatrix}. \tag{12a}$$

The following can be used to express the characteristic equation of $L(E_{xy})$:

Clearly, the characteristic equation of $L(E_{xy})$ can be written as follows:

$$[\lambda^2 - T_{xy}\lambda + D_{xy}][w_8\breve{x} + w_9\breve{y} - w_{10} - \lambda] = 0, \tag{12b}$$

$T_{xy} = -\breve{x} + \breve{y}(w_3 w_5 \breve{x} - w_7)$,

$$D_{xy} = -\breve{x}\breve{y}(w_3 w_5 \breve{x} - w_7) + \breve{x}\breve{y}\left(\frac{w_1}{(1 + w_1\breve{y})^2} + 1 + 2w_3\breve{y}\right)(w_5 + w_3 w_5 \breve{y}).$$

Therefore, the eigenvalues of $L(E_{xy})$ are obtained as:
$\lambda_{2i} = \frac{T_{xy}}{2} \pm \sqrt{T_{xy}^2 - 4D_{xy}}$, for $i = 1,2$ and $\lambda_{23} = w_8\breve{x} + w_9\breve{y} - w_{10}$.
Obviously, all the eigenvalues have negative real parts and hence $E_{xy} = (\breve{x}, \breve{y}, 0)$ is locally asymptotical provided that the following conditions hold.

$$\breve{x} < \frac{w_{10} - w_9\breve{y}}{w_8}, \tag{12c}$$
$$\breve{y} < \frac{\breve{x}}{w_3 w_5 \breve{x} - w_7}. \tag{12d}$$

the Jacobian matrix for the equilibrium $E_{xz} = (\tilde{x}, 0, \tilde{z})$, is as follows:

$$L(E_{xz}) = \begin{bmatrix} -\tilde{x} & -\tilde{x}(\frac{w_1}{(1+w_1w_2\tilde{z})^2}+1) & -\tilde{x}(\frac{w_1w_2}{(1+w_1w_2\tilde{z})^2}+1+2w_4\tilde{z}) \\ 0 & w_5\tilde{x}-w_6 & 0 \\ \tilde{z}(w_8+w_4w_8\tilde{z}) & \tilde{z}w_9 & \tilde{z}(w_4w_8\tilde{x}-w_{11}) \end{bmatrix}. \quad (13a)$$

The following can be used to express the characteristic equation of $L(E_{xz})$:

$$[\lambda^2 - T_{xz}\lambda + D_{xz}][w_5\tilde{x} - w_6 - \lambda] = 0, \quad (13b)$$

where

$T_{xz} = -\tilde{x} + \tilde{z}(w_4w_8\tilde{x} - w_{11}),$

$D_{xz} = -\tilde{x}\tilde{z}(w_4w_8\tilde{x} - w_{11}) + \tilde{x}\tilde{z}\left(\frac{w_1w_2}{(1+w_1w_2\tilde{z})^2} + 1 + 2w_4\tilde{z}\right)(w_8 + w_4w_8\tilde{z}).$

Therefore, the eigenvalues of $L(E_{xz})$ are obtained as:

$\lambda_{3i} = \frac{T_{xz}}{2} \pm \sqrt{T_{xz}^2 - 4D_{xz}}$, for $i = 1,3$ and $\lambda_{32} = w_5\tilde{x} - w_6$.

As a consequence, the real components for every eigenvalue are negative, hence $E_{xz} = (\tilde{x}, 0, \tilde{z})$ is locally asymptotical provided that the following conditions hold.

$\tilde{x} < \frac{w_6}{w_5},$

(13c)

$$\tilde{z} < \frac{\tilde{x}}{w_4w_8\tilde{x} - w_{11}}. \quad (13d)$$

Theorem 2. Assume that the cohabitation equilibrium point $E_{xyz} = (\bar{x}, \bar{y}, \bar{z})$ of Eq. (2) exists, if the subsequent sufficient conditions are hold then it is locally asymptotically stable

$\bar{x} < \min\left\{\frac{w_7}{w_3w_5}, \frac{w_{11}}{w_4w_8}\right\},$

(14a)

$$l_{31}(l_{11} + l_{22} + l_{33}) + (l_{21}l_{32} - l_{22}l_{31}) < 0, \quad (14b)$$

where $l_{ij}; i,j = 1,2,3$ are the Jacobian matrix elements at E_{xyz}.

Proof. At cohabitation equilibrium point $E_{xyz} = (\bar{x}, \bar{y}, \bar{z})$ of Eq. (2), The Jacobian can be expressed as:

$$L(E_{xyz}) = [l_{ij}]_{3\times3}, \quad (15a)$$

where

$l_{11} = -\bar{x},\ l_{12} = -\bar{x}(\frac{w_1}{(1+w_1(\bar{y}+w_2\bar{z}))^2} + 1 + 2w_3\bar{y}),$

$l_{13} = -\bar{x}\left(\frac{w_1w_2}{(1+w_1(\bar{y}+w_2\bar{z}))^2} + 1 + 2w_4\bar{z}\right),$

$l_{21} = \bar{y}(w_5 + w_3w_5\bar{y}),\ l_{22} = \bar{y}(w_3w_5\bar{x} - w_7),\ l_{23} = 0,$

$l_{31} = \bar{z}(w_8 + w_4w_8\bar{z}),\ l_{32} = w_9\bar{z},\ l_{33} = \bar{z}(w_4w_8\bar{x} - w_{11}).$

Then, the characteristic equations of $L(E_{xyz})$ is given by:

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0, \quad (15b)$$

where

$A = -(l_{11} + l_{22} + l_{33}),\ \ B = l_{11}l_{22} - l_{12}l_{21} + l_{11}l_{33} - l_{13}l_{31} + l_{22}l_{33},$

$C = -(l_{11}l_{22}l_{33} + l_{13}l_{21}l_{32} - l_{13}l_{22}l_{31} - l_{12}l_{21}l_{33}).$

And

$\Delta = AB - C = -(l_{11} + l_{22})[l_{11}l_{22} - l_{12}l_{21}]$

$-(l_{11} + l_{33})[l_{11}l_{33} - l_{13}l_{31}] - (l_{22}l_{33})[l_{11} + l_{22} + l_{33}]$

$-l_{11}l_{22}l_{33} + l_{13}l_{21}l_{32}.$

$$\Delta = AB - C = -(l_{11} + l_{22})[l_{11}l_{22} - l_{12}l_{21}] - l_{11}l_{33}(l_{11} + l_{33}) - l_{22}l_{33}(l_{22} + l_{33})$$
$$-2l_{11}l_{22}l_{33} + l_{13}l_{31}(l_{11} + l_{22} + l_{33}) + l_{13}(l_{21}l_{32} - l_{22}l_{31}).$$

Clearly, the conditions (14a)-(14b) verification $A > 0,\ C > 0,$ and $\Delta > 0$. By using the Routh-Hurwitz criterion, all the eigenvalues of Eq. (15b) have negative real parts. Therefore, E_{xyz} is locally asymptotically stable.

5. Persistence

The persistence of Eq. (2) is obtained in this section. The Eq. (2) is considered be persistent if and only if none of their species become extinct. This means there is no omega limit set determined by the bounds of R_+^3 for Eq. (1). The existence of periodic dynamics in the border planes must first be confirmed. It appears simply mentioning that the Eq. (2) provides two subsystems can be obtained from Eq.(2), which belongs to xy −plane and xz −plane consequently. The following is a technique to write these subsystems:

$$\begin{aligned} \frac{dx}{dt} &= x\left[\frac{1}{1+w_1y} - x - (1+w_3y)y\right] = h_1(x,y) \\ \frac{dy}{dt} &= y[w_5(1+w_3y)x - w_6 - w_7y] = h_2(x,y). \end{aligned} \tag{16}$$

And

$$\begin{aligned} \frac{dx}{dt} &= x\left[\frac{1}{1+w_1w_2z} - x - (1+w_4z)z\right] = g_1(x,z) \\ \frac{dz}{dt} &= z[w_8(1+w_4z)x - w_{10} - w_{11}z] = g_2(x,z). \end{aligned} \tag{17}$$

Obviously, subsystems (16) and (17) lie in the interior of xy −plane and xz −plane respectively. Now, define the Dulac function as $Q_1(x,y) = \frac{1}{xy}$, that satisfies $Q_1(x,y) > 0$, and C^1 function in the $int\ R_+^2$ of xy −plane. Then

$$\Delta(x,y) = \frac{\partial(Q_1h_1)}{\partial x} + \frac{\partial(Q_1h_2)}{\partial y} = -\frac{1}{y} + w_3w_5 - \frac{w_7}{x}.$$

Since, $\Delta(x,y)$ does not identically zero and does not change the sign in the $int\ R_+^2$ of the xy −plane provided that the following condition holds:

$$\left.\begin{aligned} w_3w_5 &> \frac{1}{y} + \frac{w_7}{x} \\ &or \\ w_3w_5 &< \frac{1}{y} + \frac{w_7}{x} \end{aligned}\right\}. \tag{18}$$

Thus, by using the Dulic-Bendixon's criterion, the $int\ R_+^2$ has no closed curves of the xy −plane. Therefore, depending on the Poincare-Bendixson theorem states that wherever it exists, the unique equilibrium points in the $int\ R_+^2$ of the xy −plane that given by $(\check{x},\check{y})$will be both locally and globally asymptotically stable.
Similarly, by using C^1 function in the $int\ R_+^2$ of xz −plane, which define by using Dulac function as $Q_2(x,z) = \frac{1}{xz}$. It is observed that there is no periodic dynamics in the $int\ R_+^2$ of the xz −plane that given by $(\tilde{x},\tilde{z})$ provided that the following condition holds:

$$\left.\begin{aligned} w_4w_8 &> \frac{1}{z} + \frac{w_{11}}{x} \\ &or \\ w_4w_8 &< \frac{1}{z} + \frac{w_{11}}{x} \end{aligned}\right\}. \tag{19}$$

Consequently, the only equilibrium points $int\ R_+^2$ of the xz −plane shall be both locally and globally asymptotically stable whenever it occurs.

Theorem 3. If conditions (4), (6) with the following conditions have been satisfied, Eq. (2) is uniformly persistent.

$$\breve{x} > \frac{w_{10} - w_9 \breve{y}}{w_8}, \tag{20a}$$
$$\tilde{x} > \frac{w_6}{w_5}. \tag{20b}$$

Proof. Define the following function $\sigma(x, y, z) = x^{p_1} y^{p_2} z^{p_3}$ as an average Lyapunov function, where p_1, p_2, p_3 are positive constants. Obviously, $\sigma(x, y, z)$ is a C^1 positive function defined in $int\, R_+^3$, and if $x \to 0$ or $y \to 0$ or $z \to 0$, as result $\sigma(x, y, z) \to 0$. Accordingly, the following is obtained:

$$\Omega(x, y, z) = \frac{\sigma'(x, y, z)}{\sigma(x, y, z)} = p_1 \left[\frac{1}{w_1(y + w_2 z) + 1} - x - (w_3 y + 1)y - (1 + w_4 z)z\right]$$
$$+ p_2[w_5(1 + w_3 y)x - w_6 - w_7 y] + p_3[w_8(1 + w_4 z)x + w_9 y - w_{10} - w_{11} z].$$

Therefore, according to technique of the average Lyapunov function that proposed by Gard, the proof is complete if $\Omega(x, y, z)$ for any boundary equilibrium points is positive, for a suitable selection of the positive constant p_i, $i = 1,2,3$. Since

$$\Omega(E_0) = p_1 - p_2 w_6 - p_3 w_{10},$$
$$\Omega(E_1) = p_2(w_5 - w_6) + p_3(w_8 - w_{10}),$$
$$\Omega(E_{xy}) = p_1 \left[\frac{1}{1 + w_1 \breve{y}} - \breve{x} - (1 + w_3 \breve{y})\breve{y}\right] + p_2[w_5(1 + w_3 \breve{y})\breve{x} - w_6 - w_7 \breve{y}] + p_3[w_8 \breve{x} + w_9 \breve{y} - w_{10}],$$
$$\Omega(E_{xz}) = p_1 \left[\frac{1}{1 + w_1 w_2 \tilde{z}} - \tilde{x} - (1 + w_4 \tilde{z})\tilde{z}\right] + p_2[w_5 \tilde{x} - w_6] + p_3[w_8(1 + w_4 \tilde{z})\tilde{x} - w_{10} - w_{11} \tilde{z}].$$

Consequently, by choosing p_1 to be a significant enough value to get $\Omega(E_0)$ greater then zero in regards to p_2 and p_3. However, $\Omega(E_1), \Omega(E_{xy})$ as and $\Omega(E_{xz})$ positive as long as required conditions (20a), (20b), (4), and (6) holds subsequently.

6. Global stability nalysis

In this section, the global stability for each equilibrium point is determined with help of the method of Lyapunov function as shown in the next theorems.

Theorem (4): If we assume that E_1 is locally asymptotically stable, then it can be said to be globally-asymptotically stable under the following conditions:

$$\frac{w_6}{w_5} > w_1 + 1 + w_3 y, \tag{21a}$$

$$\frac{w_{10}}{w_8} > w_1 w_2 + 1 + w_4 z + \frac{w_9}{w_8}\eta. \tag{21b}$$

Proof: Consider the real-valued function $U_1 = k_1\left(x - \hat{x} - \hat{x}\, ln\frac{x}{\hat{x}}\right) + k_2 y + k_3 z$, where the constants $k_i > 0; i = 1,2,3$, to be determined. Clearly, the positive define function U_1 shows that $U_1: R_+^3 \to R$, so that $U_1(E_1) = 0$ and $U_1(x, y, z)$, greater than zero $\forall\{(x, y, z) \in R_+^3: x > 0, y \text{ and } z \geq 0, (x, y, z) \neq E_1\}$. Then, $\frac{dU_1}{dt}$ is given by:

$$\frac{dU_1}{dt} = k_1\left(\frac{x - \hat{x}}{x}\right)\frac{dx}{dt} + k_2\frac{dy}{dt} + k_3\frac{dz}{dt}.$$
$$\frac{dU_1}{dt} \leq -k_1(x - \hat{x})^2 - k_1\frac{w_1(y + w_2 z)x}{1 + w_1(y + w_2 z)} + k_1\frac{w_1(y + w_2 z)\hat{x}}{1 + w_1(y + w_2 z)}$$
$$-(k_1 - k_2 w_5)(1 + w_3 y)xy + k_1(1 + w_3 y)\hat{x}y - (k_1 - k_3 w_8)(1 + w_4 z)xz$$
$$+k_1(1 + w_4 z)\hat{x}z - k_2 w_6 y + k_3 w_9 yz - k_3 w_{10} z.$$

Choosing the positive constant values as $k_1 = 1, k_2 = \frac{1}{w_5}$, and $k_3 = \frac{1}{w_8}$, and by using the maximize concept with the upper bound constant η, we obtained that:

$$\frac{dU_1}{dt} \leq -(x-1)^2 - \left[\frac{w_6}{w_5} - w_1 - (1+w_3 y)\right] y - \left[\frac{w_{10}}{w_8} - w_1 w_2 - (1+w_4 z) - \frac{w_9}{w_8}\eta\right] z.$$

Hence, conditions (21a) and (21b) lead to $\frac{dU_1}{dt} < 0$. So that E_1 is globally-asymptotically stable.

Theorem 5. If E_{xy} is assumed to be locally asymptotically stable, then it can be declared globally asymptotically stable as long as the following conditions are hold:

$$\frac{w_{10}}{w_8} > \frac{w_9}{w_8}\eta + \breve{x}(b_{44}+1), \quad (22a)$$

$$\breve{x} < min\{\frac{w_7}{w_3 w_5}, \frac{w_{11}}{w_4 w_8}\}. \quad (22b)$$

Proof. Consider the real-valued function $U_2 = ɣ_1\left(-\breve{x} + x - \breve{x}\, ln\frac{x}{\breve{x}}\right) + ɣ_2\left(-\breve{y} + y - \breve{y}\, ln\frac{y}{\breve{y}}\right) + ɣ_3 z$, where the constants $ɣ_1, ɣ_2, ɣ_3$ greater than zero to be selected. Clearly, the positive define function U_2 shows that $U_2 : R_+^3 \to R$, so that $U_2(E_{xy}) = 0$ and $U_2(x,y,z)$ positive for all $\{(x,y,z) \in R_+^3 : x > 0, y > 0 \;\&\; z \geq 0, (x,y,z) \neq E_{xy}\}$. Then, $\frac{dU_2}{dt}$ is given by:

$$\frac{dU_2}{dt} = ɣ_1\left(\frac{x-\breve{x}}{x}\right)\frac{dx}{dt} + ɣ_2\left(\frac{y-\breve{y}}{y}\right)\frac{dy}{dt} + ɣ_3\frac{dz}{dt}.$$

$$\frac{dU_2}{dt} \leq -ɣ_1(x-\breve{x})^2 - [ɣ_2 w_7 - ɣ_2 \breve{x} w_3 w_5](y-\breve{y})^2$$
$$-\left[\frac{ɣ_1 w_1}{\big(1+w_1(y+w_2 z)\big)(1+w_1\breve{y})} + ɣ_1 + ɣ_1 w_3 \breve{y} - ɣ_2 w_5\right](y-\breve{y})(x-\breve{x})$$
$$-[ɣ_1 w_3 - ɣ_2 w_3 w_5](y-\breve{y})(x-\breve{x})y - [ɣ_1 w_4 - ɣ_3 w_4 w_8]xz^2$$
$$-\left[\frac{ɣ_1 w_1 w_2}{\big(1+w_1(y+w_2 z)\big)(1+w_1\breve{y})} + ɣ_1 - ɣ_3 w_8\right] xz$$
$$-\left[ɣ_3 w_{10} - ɣ_3 w_9 \eta - ɣ_1 \breve{x} - \frac{ɣ_1 \breve{x}\, w_1 w_2}{\big(1+w_1(y+w_2 z)\big)(1+w_1\breve{y})}\right] z$$
$$-[ɣ_3 w_{11} - ɣ_1 w_4 \breve{x}]z^2.$$

Choosing the positive constant values as $ɣ_1 = 1, ɣ_2 = \frac{1}{w_5}$, and $ɣ_3 = \frac{1}{w_8}$,and by using the maximize concept with the upper bound constant η, we obtained that:

$$\frac{dU_2}{dt} \leq -(x-\breve{x})^2 - [b_{11} + w_3\breve{y}](y-\breve{y})(x-\breve{x}) - b_{22}(y-\breve{y})^2$$
$$-b_{33}xz - \left[\frac{w_{10}}{w_8} - \frac{w_9}{w_8}\eta - \breve{x}(b_{44}+1)\right] z - \left[\frac{w_{11}}{w_8} - w_4\breve{x}\right] z^2.$$

Then by using the above conditions (22a) and (22b), we obtain that:

$$\frac{dU_2}{dt} \leq -[(x-\breve{x}) - \sqrt{b_{22}}(y-\breve{y})]^2 - b_{33}xz$$
$$-\left[\frac{w_{10}}{w_8} - \frac{w_9}{w_8}\eta - \breve{x}(b_{44}+1)\right] z - \left[\frac{w_{11}}{w_8} - w_4\breve{x}\right] z^2.$$

Where $b_{11} = \frac{w_1}{(1+w_1(y+w_2z))(1+w_1\breve{y})}$, $b_{22} = \left[\frac{w_7}{w_5} - \breve{x}w_3\right]$, $b_{33} = \frac{w_1w_2}{(1+w_1(y+w_2z))(1+w_1\breve{y})}$, and $b_{44} = \frac{w_1w_2}{(1+w_1(y+w_2z))(1+w_1\breve{y})}$. According to, conditions (22a) and (22b) lead to $\frac{dU_2}{dt} < 0$. So that E_{xy} is globally-asymptotically stable.

Theorem 6. If E_{xz} is assumed to be locally asymptotically stable, then it can be declared globally-asymptotically stable as long as the following conditions are hold:

$$\tilde{x} < min\{\frac{w_7}{w_3w_5}, \frac{w_{11}}{w_4w_8}\} \quad (23a)$$

$$\frac{w_6}{w_5} > \tilde{x}(c_{44}+1) + \frac{w_9}{w_8}(z-\tilde{z}). \quad (23b)$$

Proof: Consider the real-valued function $U_3 = \rho_1\left(x - \tilde{x} - \tilde{x}\ln\frac{x}{\tilde{x}}\right) + \rho_2 y + \rho_3(z - \tilde{z} - \tilde{z}\ln\frac{z}{\tilde{z}})$. Where the constants $\rho_i > 0; i = 1,2,3$, to be determined. Clearly, the positive define function U_3 shows that $U_3: R_+^3 \to R$, so that $U_3(E_{xz}) = 0$ and $U_3(x,y,z)\ positive\ \forall\{(x,y,z) \in R_+^3: x > 0, y \geq 0, z > 0, (x,y,z) \neq E_{xz}\}$. Then, $\frac{dU_3}{dt}$ is given by:

$$\frac{dU_3}{dt} = \rho_1\left(\frac{x-\tilde{x}}{x}\right)\frac{dx}{dt} + \rho_2\frac{dy}{dt} + \rho_3\left(\frac{z-\tilde{z}}{z}\right)\frac{dz}{dt}.$$

$$\frac{dU_3}{dt} \leq -\rho_1(x-\tilde{x})^2 - [\rho_3w_{11} - \rho_3\tilde{x}\,w_4w_8](z-\tilde{z})^2$$
$$-\left[\frac{\rho_1\,w_1w_2}{\big(1+w_1(y+w_2z)\big)(1+w_1w_2\tilde{z})} + \rho_1 + \rho_1\tilde{z}\,w_4 - \rho_3w_8\right](x-\tilde{x})(z-\tilde{z})$$
$$-[\rho_1w_4 - \rho_3w_4w_8](x-\tilde{x})(z-\tilde{z})z - [\rho_1w_3 - \rho_2w_3w_5]xy^2$$
$$-\left[\frac{\rho_1\,w_1}{\big(1+w_1(y+w_2z)\big)(1+w_1w_2\tilde{z})} + \rho_1 - \rho_2w_5\right]xy - [\rho_2w_7 - \rho_1\tilde{x}\,w_3]y^2$$
$$-\left[\rho_2w_6 - \frac{\rho_1\tilde{x}\,w_1}{\big(1+w_1(y+w_2z)\big)(1+w_1w_2\tilde{z})} - \rho_1\tilde{x} - \rho_3(z-\tilde{z})w_9\right]y.$$

Choosing the positive constant values as $\rho_1 = 1, \rho_2 = \frac{1}{w_5}$, and $\rho_3 = \frac{1}{w_8}$, then we obtained that:

$$\frac{dU_3}{dt} \leq -(x-\tilde{x})^2 - [c_{11} + \tilde{z}\,w_4](z-\tilde{z})(x-\tilde{x}) - c_{22}(z-\tilde{z})^2$$
$$-c_{33}xy - \left[\frac{w_7}{w_5} - \tilde{x}\,w_3\right]y^2 - \left[\frac{w_6}{w_5} - \tilde{x}(c_{44}+1) - \frac{w_9}{w_8}(z-\tilde{z})\right]y.$$

Then by using the above conditions (23a) and (23b), we obtain that:

$$\frac{dU_3}{dt} \leq -[(x-\tilde{x}) + \sqrt{c_{22}}(z-\tilde{z})]^2 - c_{33}xy$$
$$-\left[\frac{w_7}{w_5} - \tilde{x}\,w_3\right]y^2 - \left[\frac{w_6}{w_5} - \tilde{x}(c_{44}+1) - \frac{w_9}{w_8}(z-\tilde{z})\right]y.$$

Where $c_{11} = \frac{w_1w_2}{(1+w_1(y+w_2z))(1+w_1w_2\tilde{z})}$, $c_{22} = \left[\frac{w_{11}}{w_8} - \tilde{x}w_4\right]$, $c_{33} = \frac{w_1}{(1+w_1(y+w_2z))(1+w_1w_2\tilde{z})}$, and $c_{44} = \frac{w_1}{(1+w_1(y+w_2z))(1+w_1w_2\tilde{z})}$.Clearly, the conditions (23a) and (23b) lead to $\frac{dU_3}{dt} < 0$. So that E_{xz} is globally-asymptotically stable.

Theorem 7. Assume that E_{xyz} is locally asymptotically stable, then it is globally asymptotically stable provided that the following conditions hold:

$$(B_1)^2 < B_2, \quad (24a)$$

$(B_4)^2 < B_3,$ (24b)
$(B_5)^2 < B_2 B_3.$ (24c)
$\bar{x} < min\{\frac{w_{11}}{w_4 w_8} \ , \ \frac{w_7}{w_3 w_5}\}.$ (24d)

Proof. Let the real-valued function $U_4 = q_1\left(x - \bar{x} - \bar{x}\, ln\frac{x}{\bar{x}}\right) + q_2(y - \bar{y} - \bar{y}\, ln\frac{y}{\bar{y}}) + q_3(z - \bar{z} - \bar{z}\, ln\frac{z}{\bar{z}})$. Where the constants $q_i > 0; i = 1,2,3$, to be determined. Clearly, the positive define function $U_4: R_+^3 \to R$, so that $U_4(E_{xyz}) = 0$ and $U_4(x,y,z) > 0$, for all $\{(x,y,z) \in R_+^3 : x > 0, y > 0, z > 0, (x,y,z) \neq E_{xyz}\}$. Then, $\frac{dU_4}{dt}$ is given by:

$$\frac{dU_4}{dt} = q_1\left(\frac{x-\bar{x}}{x}\right)\frac{dx}{dt} + q_2\left(\frac{y-\bar{y}}{y}\right)\frac{dy}{dt} + q_3\left(\frac{z-\bar{z}}{z}\right)\frac{dz}{dt}.$$

$$\frac{dU_4}{dt} = -(x-\bar{x})^2 - \left[\frac{w_7}{w_5} - \bar{x}w_3\right](y-\bar{y})^2 - \left[\frac{w_{11}}{w_8} - \bar{x}w_4\right](z-\bar{z})^2$$
$$-\left[\frac{w_1}{T\bar{T}} + \bar{y}\, w_3\right](y-\bar{y})(x-\bar{x})$$
$$-\left[\frac{w_1 w_2}{T\bar{T}} + \bar{z}\, w_4\right](z-\bar{z})(x-\bar{x})$$
$$+\frac{w_9}{w_8}(z-\bar{z})(y-\bar{y}).$$

Where $T = (1 + w_1(y + w_2 z))$, and $\bar{T} = (1 + w_1(\bar{y} + w_2\bar{z}))$. Therefore, by choosing the positive constant values as $q_1 = 1, q_2 = \frac{1}{w_5}$, and $q_3 = \frac{1}{w_8}$, then after some algebraic computation, we obtained:

$$\frac{dU_4}{dt} = -\frac{1}{2}(x-\bar{x})^2 - B_1(y-\bar{y})(x-\bar{x}) - \frac{B_2}{2}(y-\bar{y})^2$$
$$-\frac{1}{2}(x-\bar{x})^2 - B_4(z-\bar{z})(x-\bar{x}) - \frac{B_3}{2}(z-\bar{z})^2$$
$$-\frac{B_2}{2}(y-\bar{y})^2 - B_5(z-\bar{z})(y-\bar{y}) - \frac{B_3}{2}(z-\bar{z})^2.$$

Then by using the above conditions (24a), (24b), and (24c), we obtain that:

$$\frac{dU_4}{dt} \leq -\frac{1}{2}[(x-\bar{x}) + \sqrt{B_2}(y-\bar{y})]^2 - \frac{1}{2}[(x-\bar{x}) + \sqrt{B_3}(z-\bar{z})]^2$$
$$-\frac{1}{2}[\sqrt{B_2}(y-\bar{y}) + \sqrt{B_3}(z-\bar{z})]^2.$$

Where $B_1 = \frac{w_1}{(1+w_1(y+w_2 z))(1+w_1(\bar{y}+w_2\bar{z}))} + \bar{y}\, w_3$, $B_2 = \frac{w_7}{w_5} - \bar{x}w_3$, $B_3 = \frac{w_{11}}{w_8} - \bar{x}w_4$, $B_4 = \frac{w_1 w_2}{(1+w_1(y+w_2 z))(1+w_1(\bar{y}+w_2\bar{z}))} + \bar{z}\, w_4$ and $B_5 = \frac{w_9}{w_8}$. Clearly, the conditions (24a), (24b), and (24c) lead to $\frac{dU_4}{dt} < 0$. So that E_{xyz} is globally asymptotically stable.

7. Local bifurcation analysis

In this section, the occurrence of local bifurcation is determined utilizing Sotomayor's theorem. In a dynamical system the possible bifurcation parameter is specified, such that, for a given value of that parameter, the equilibrium point is not hyperbolic, because the presence of a non-hyperbolic equilibrium point represents a necessary but not sufficient condition for local bifurcation to occur. Hence, Eq. (2) can be rewritten in the vector form:

$$\frac{dX}{dt} = F(X), \quad (25)$$

Where $X = (x,y,z)^T$, and $F(x) = (xf_1, yf_2, zf_3)^T$.

Now, for any vector $V=(v_1,v_2,v_3)^T$, the second directional derivative of Eq. (25) with concerning X can be obtained as:

$$D^2F(X)(V,V)=[\Lambda_{ij}]_{3\times 1}. \quad (26)$$

Where

$$\Lambda_{11}=-2v_1^2-2\left[1+2w_3y+\frac{w_1}{(1+w_1(y+w_2z))^2}\right]v_1v_2$$
$$-2\left[1+2w_4z+\frac{w_1w_2}{(1+w_1(y+w_2z))^2}\right]v_1v_3+2\left[-w_3x+\frac{{w_1}^2x}{(1+w_1(y+w_2z))^3}\right]v_2^2$$
$$+4\left[\frac{{w_1}^2w_2x}{(1+w_1(y+w_2z))^3}\right]v_2v_3+2\left[-w_4x+\frac{{w_1}^2w_2^2x}{(1+w_1(y+w_2z))^3}\right]v_3^2.$$
$$\Lambda_{21}=2[w_5+2w_3w_5y]v_1v_2+2[-w_7+w_3w_5x]v_2^2.$$
$$\Lambda_{31}=2[w_8+2w_4w_8z]v_1v_3+2w_9v_2v_3+2[-w_{11}+w_4w_8x]v_3^2.$$

Theorem 8. Suppose that condition (11c) holds, then the Eq.(2) around the equilibrium in the absence of predation E_1 undergoes Transcritical bifurcation at w_8 is equal to $w_{10}=w_{10}^*$ with the following condition

$$w_{11}>w_8(\alpha_1+1). \quad (27)$$

Proof. It is easy to verify that the Jacobian matrix that is given by (11a), with $w_{10}=w_{10}^*$ can be written as the following form:

$$L(E_1,w_{10}^*)=\begin{pmatrix}-1 & -w_1-1 & -w_1w_2-1\\ 0 & w_5-w_6 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Where $\lambda_{11}=-1,\lambda_{12}=w_5-w_6<0$ are eigenvalues of $L(E_1,w_{10}^*)$ under conditions (11c) and $\lambda_{13}^*=0$. So that, the equilibrium in the absence of predation E_1 becomes a non-hyperbolic point.

Let the vector $V_1=(v_{11},v_{12},v_{13})^T$ be the eigenvector of $L(E_1,w_{10}^*)$ associated with eigenvalue $\lambda_{13}^*=0$. Therefore, the straightforward calculation obtained that $V_1=(\alpha_1v_{13},0,v_{13})^T$, where v_{13} be any real number not equal zero, and $\alpha_1=-(w_1w_2+1)<0$.

Let the vector $\Psi_1=(\Psi_{11},\Psi_{12},\Psi_{13})^T$ be the eigenvector of $L(E_1,w_{10}^*)^T$ associated with the zero eigenvalue $\lambda_{13}^*=0$. Therefore, the direct computation obtained that $\Psi_1=(0,0,\Psi_{13})^T$, where Ψ_{13} be any real number not equal zero.

Accordingly, $\frac{\partial F}{\partial w_{10}}=F_{w_{10}}=(0,0,-z)^T$, hence it gives that $F_{w_{10}}(E_1,w_{10}^*)=(0,0,0)^T$, that leads to $\Psi_1^T\left[F_{w_{10}}(E_1,w_{10}^*)\right]=0$.

Therefore, when $w_{10}=w_{10}^*$, thus there is no saddle-node bifurcation in Eq. (2) at E_1.

Moreover, the direct calculation gives that:

$$DF_{w_{10}}(E_1,w_{10}^*)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&-1\end{pmatrix}\Longrightarrow DF_{w_{10}}(E_1,w_{10}^*)V_1=(0,0,-v_{13})^T.$$

Then, $\Psi_1^T\left[DF_{w_{10}}(E_1,w_{10}^*)V_1\right]=-\Psi_{13}v_{13}\neq 0$. Now, by using Eq. (26) with V_1 at (E_1,w_{10}^*), then it gives that:

$$D^2[F(E_1,w_{10}^*)(V_1,V_1)]=$$
$$\begin{pmatrix}-2\alpha_1^2v_{13}^2+2v_{13}^2w_1^2w_2^2+\alpha_1v_{13}(-2v_{13}-2v_{13}w_1w_2)-2v_{13}^2w_4\\ 0\\ -2v_{13}^2w_{11}+2\alpha_1v_{13}^2w_8+2v_{13}^2w_4w_8\end{pmatrix}.$$

Then

$$\Psi_1^T[D^2F(E_1,w_{10}^*)(V_1,V_1)]=-2v_{13}^2\Psi_{13}(w_{11}-w_8(\alpha_1+w_4)\neq 0.$$

Thus, the Eq. (2) experiences a bifurcation that is Transcritical in Concept of Sotomayor's theorem at E_1 due to (27).

Theorem 9. Suppose that condition (12d) holds, then the Eq.(2) around the equilibrium in the absence of scavenger $E_{xy} = (\breve{x}, \breve{y}, 0)$, undergoes Transcritical bifurcation when the parameter w_{10} is equal to $w_{10}^* = w_8\breve{x} + w_9\breve{y}$with satisfying condition:

$$2\Psi_{23}(v_{21})^2[(-w_{11} + w_4w_8\breve{x})(\alpha_3)^2 + w_8\alpha_3 + w_9\alpha_2\alpha_3] \neq 0. \quad (28)$$

Proof. It is straight to establish that the Jacobian matrix gave by condition (12a), with $w_{10}^* = w_8\breve{x} + w_9\breve{y}$. Can be written as the following form:

$$L(E_{xy}, w_{10}^*) = \begin{pmatrix} -\breve{x} & -\breve{x}(\frac{w_1}{(1+w_1\breve{y})^2} + 1 + 2w_3\breve{y}) & -\breve{x}(\frac{w_1w_2}{(1+w_1\breve{y})^2} + 1) \\ \breve{y}(w_5 + w_3w_5\breve{y}) & \breve{y}(w_3w_5\breve{x} - w_7) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= (m_{ij}).$$

Obviously, the Jacobian matrix $L(E_{xy}, w_{10}^*)$ under condition(12d) with $w_{10}^* = w_8\breve{x} + w_9\breve{y}$ has two negative real part, whilst $\lambda_{23}^* = 0$. So, the equilibrium in the absence of scavenger $E_{xy}(\breve{x}, \breve{y}, 0)$ becomes a non-hyperbolic point.

Let the vector $V_2 = (v_{21}, v_{22}, v_{23})^T$ be the eigenvector of $L(E_{xy}, w_{10}^*)$ associated with eigenvalue $\lambda_{23}^* = 0$. Therefore, Simple calculations obtained $V_2 = (v_{21}, \alpha_2v_{21}, \alpha_3v_{21})^T$, where v_{21} be any real number not equal zero, and $\alpha_2 = -\frac{m_{21}}{m_{22}} < 0$, $\alpha_3 = \frac{m_{21}m_{12} - m_{11}m_{22}}{m_{22}m_{13}}$.

Let the vector $\Psi_2 = (\Psi_{21}, \Psi_{22}, \Psi_{23})^T$ be the eigenvector of $L(E_{xy}, w_{10}^*)^T$ associated with eigenvalue $\lambda_{23}^* = 0$. Therefore, the direct computation obtained that $\Psi_2 = (0,0,\Psi_{23})^T$, where Ψ_{23} be any real number not equal zero.

Furthermore, $\frac{\partial F}{\partial w_{10}} = F_{w_{10}} = (0,0,-z)^T$, hence it gives that $F_{w_{10}}(E_{xy}, w_{10}^*) = (0,0,0)^T$, that leads to $\Psi_2^T[F_{w_{10}}(E_{xy}, w_{10}^*)] = 0$.

Therefore, when $w_{10} = w_{10}^*$, thus, there no saddle-node bifurcation at E_{xy}.

Moreover, the direct computation gives that:

$$DF_{w_{10}}(E_{xy}, w_{10}^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Longrightarrow DF_{w_{10}}(E_{xy}, w_{10}^*)V_2 = (0,0,-\alpha_3v_{21})^T.$$

Then, $\Psi_2^T[DF_{w_{10}}(E_{xy}, w_{10}^*)V_2] = -\Psi_{23}\alpha_3v_{21} \neq 0$. Now, by using eq.(26) with V_2 at (E_{xy}, w_{10}^*), then it gives that:

$D^2[F(E_{xy}, w_{10}^*)(V_2, V_2)] = [\bar{\bar{\Lambda}}_{i1}]_{3\times1}$.

Where

$$\bar{\bar{\Lambda}}_{11} = -2v_{21}^2 + 2\left[-w_3\breve{x} + \frac{w_1^2\breve{x}}{(1+w_1\breve{y})^3}\right](\alpha_2v_{21})^2 + 4\left[\frac{w_1^2w_2\breve{x}}{(1+w_1\breve{y})^3}\right]\alpha_2\alpha_3(v_{21})^2$$
$$+2\left[-w_4\breve{x} + \frac{w_1^2w_2^2\breve{x}}{(1+w_1\breve{y})^3}\right](\alpha_3v_{21})^2 - 2\left[1 + 2w_3\breve{y} + \frac{w_1}{(1+w_1\breve{y})^2}\right]\alpha_2(v_{21})^2$$
$$-2\left[1 + \frac{w_1w_2}{(1+w_1\breve{y})^2}\right]\alpha_3(v_{21})^2.$$
$$\bar{\bar{\Lambda}}_{21} = 2[-w_7 + w_3w_5\breve{x}](\alpha_2v_{21})^2 + 2[w_5 + 2w_3w_5\breve{y}]\alpha_2(v_{21})^2$$
$$\bar{\bar{\Lambda}}_{31} = 2[-w_{11} + w_4w_8\breve{x}](\alpha_3v_{21})^2 + 2w_8\alpha_3(v_{21})^2 + 2w_9\alpha_2\alpha_3(v_{21})^2.$$

Then $\Psi_2^T[D^2F(E_{xy}, w_{10}^*)(V_2, V_2)] = 2\Psi_{23}(v_{21})^2[(-w_{11} + w_4w_8\breve{x})(\alpha_3)^2 + w_8\alpha_3 + w_9\alpha_2\alpha_3]$.

It is clear that $\Psi_2^T\left[D^2F\left(E_{xy},w_{10}^*\right)(V_2,V_2)\right] \neq 0$, consequently, the Eq. (2) experiences a Transcritical bifurcation in the concept of Sotomayor's theorem at E_{xy} due to (28).

Theorem 10. Suppose that condition(13d) holds, then the Eq. (2) around the equilibrium in the absence of predator $E_{xz} = (\tilde{x},0,\tilde{z})$, undergoes Transcritical bifurcation when the parameter w_6 is equal to $w_6^* = w_5\tilde{x}$ with the following condition

$$2\Psi_{32}(v_{32})^2[(-w_7 + w_3w_5\tilde{x}) + w_5\alpha_5] \neq 0. \tag{29}$$

Proof. It is simple to establish that the Jacobian matrix that is given by condition (13a), with $w_6 = w_6^*$ can be written as the following form:

$$L(E_{xz},w_6^*) = \begin{pmatrix} -\tilde{x} & -\tilde{x}(\frac{w_1}{(1+w_1w_2\tilde{z})^2}+1) & -\tilde{x}(\frac{w_1w_2}{(1+w_1w_2\tilde{z})^2}+1+2w_4\tilde{z}) \\ 0 & 0 & 0 \\ \tilde{z}(w_8+w_4w_8\tilde{z}) & \tilde{z}w_9 & \tilde{z}(w_4w_8\tilde{x}-w_{11}) \end{pmatrix}$$
$$= \left(r_{ij}\right).$$

Obviously, the Jacobian matrix $L(E_{xz},w_6^*)$ under condition (13d) with $w_6^* = w_5\tilde{x}$ has two negative real part, whilst $\lambda_{32}^* = 0$. So, the equilibrium in the absence of predator $E_{xz} = (\tilde{x},0,\tilde{z})$ becomes a non-hyperbolic point.

Let the vector $V_3 = (v_{31},v_{32},v_{33})^T$ be the eigenvector of $L(E_{xz},w_6^*)$ associated with $\lambda_{32}^* = 0$. Therefore, the direct computation obtained that $V_3 = (\alpha_5v_{32},v_{32},\alpha_4v_{32})^T$, where v_{32} be any real number not equal zero, and $\alpha_4 = -\frac{r_{11}r_{32}-r_{31}r_{21}}{r_{11}r_{33}-r_{31}r_{13}} > 0$, $\alpha_5 = -\frac{(r_{12}+\alpha_4r_{13})}{r_{11}} < 0$.

Let the vector $\Psi_3 = (\Psi_{31},\Psi_{32},\Psi_{33})^T$ be the eigenvector of $L(E_{xz},w_6^*)^T$ associated with the zero eigenvalue $\lambda_{32}^* = 0$.Therefore, the straightforward calculation acquired that $\Psi_3 = (0,\Psi_{32},0)^T$, where Ψ_{32} be any real number not equal zero.

Furthermore, $\frac{\partial F}{\partial w_6} = F_{w_6} = (0,-y,0)^T$, hence it gives that $F_{w_6}(E_{xz},w_6^*) = (0,0,0)^T$, that leads to $\Psi_3^T\left[F_{w_6}(E_{xz},w_6^*)\right] = 0$.

Therefore, when $w_6 = w_6^*$, Thus, there is no saddle-node bifurcation at E_{xz}in Eq. (2).

Moreover, the direct computation gives that:

$$DF_{w_6}(E_{xz},w_6^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow DF_{w_6}(E_{xz},w_6^*)V_3 = (0,-v_{32},0)^T.$$

Then, $\Psi_3^T\left[DF_{w_6}(E_{xz},w_6^*)V_3\right] = -\Psi_{32}v_{32} \neq 0$. Now, by using eq. (26) with V_3 at (E_{xz},w_6^*), then it gives that:

$D^2[F(E_{xz},w_6^*)(V_3,V_3)] = [\tilde{\Lambda}_{i1}]_{3\times1}$.

Where

$$\tilde{\Lambda}_{11} = 2\left[-w_3\tilde{x} + \frac{w_1^2\tilde{x}}{(1+w_1w_2\tilde{z})^3}\right](v_{32})^2 + 2\left[-w_4\tilde{x} + \frac{w_1^2w_2^2\tilde{x}}{(1+w_1w_2\tilde{z})^3}\right](\alpha_4v_{32})^2$$
$$-2(\alpha_5v_{32})^2 + 4\left[\frac{w_1^2w_2\tilde{x}}{(1+w_1w_2\tilde{z})^3}\right]\alpha_4(v_{32})^2 - 2\left[1 + \frac{w_1}{(1+w_1w_2\tilde{z})^2}\right]\alpha_5(v_{32})^2$$
$$-2\left[1 + 2w_4\tilde{z} + \frac{w_1w_2}{(1+w_1w_2\tilde{z})^2}\right]\alpha_4\alpha_5(v_{32})^2.$$

$\tilde{\Lambda}_{21} = 2[-w_7 + w_3w_5\tilde{x}](v_{32})^2 + 2w_5\alpha_5(v_{32})^2$.

$\tilde{\Lambda}_{31} = 2[-w_{11} + w_4w_8\tilde{x}](\alpha_4v_{32})^2 + 2w_9\alpha_4(v_{32})^2 + 2[w_8 + 2w_4w_8\tilde{z}]\alpha_4\alpha_5(v_{32})^2$.

Then $\Psi_3^T\left[D^2F_{w_6}(E_{xz},w_6^*)(V_3,V_3)\right] = 2\Psi_{32}(v_{32})^2[(-w_7 + w_3w_5\tilde{x}) + w_5\alpha_5]$.

It is clear that $\Psi_3^T\left[D^2F_{w_6}(E_{xz},w_6^*)(V_3,V_3)\right] \neq 0$, thus, the Eq. (2) goes through a Transcritical bifurcation in the sense of Sotomayor's theorem at E_{xz} due to (29).

Theorem 11. The Eq. (2) around the coexistence equilibrium point $E_{xyz} = (\bar{x}, \bar{y}, \bar{z})$ undergoes a saddle-node bifurcation when the parameter w_7 is equal to positive value w_7^* provided that the following conditions hold:

$$l_{11}l_{33} - l_{13}l_{31} \neq 0. \quad (30a)$$

$$l_{11}l_{22}^* - l_{12}l_{21} \neq 0. \quad (30b)$$

$$\alpha_8 \Lambda_{11}^* + \alpha_9 \Lambda_{21}^* + \Lambda_{31}^* \neq 0. \quad (30c)$$

Where $l_{ij}, \forall\, i, j = 1,2,3$ with $l_{22}^* = l_{22}(w_7^*)$ are the elements of $L(E_{xyz})$, with the value of w_7^* that given by $w_7^* = \frac{(l_{11}l_{33} - l_{13}l_{31})w_3 w_5 \bar{x}\bar{y} + l_{13}l_{21}l_{32} - l_{12}l_{21}l_{33}}{(l_{11}l_{33} - l_{13}l_{31})\bar{y}}$.

Proof. The Jacobian matrix that is given by condition (15a), with $w_7 = w_7^*$ can be written in the following form:

$$L(E_{xyz}, w_7^*) = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22}^* & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}.$$

Substituting the value directly shows that $w_7 = w_7^*$ makes C in Eq. (15b) has the value $C = 0$. As a result, the characteristic equation can be calculated as follows, $L(E_{xyz}, w_7^*)$ has a zero eigenvalue that is represented by $\lambda^* = 0$, which means E_{xyz} is a non-hyperbolic point when $w_7 = w_7^*$.

Let the vector $V_4 = (v_{41}, v_{42}, v_{43})^T$ be the eigenvector of $L(E_{xyz}, w_7^*)$ with eigenvalue $\lambda^* = 0$. Consequently, the basic calculation obtained that $V_4 = (\alpha_6 v_{43}, \alpha_7 v_{43}, v_{43})^T$, where v_{41} be any real number, not equal zero, and $\alpha_6 = -\frac{l_{13}l_{22}^*}{l_{11}l_{22}^* - l_{12}l_{21}}$ and $\alpha_7 = \frac{l_{13}l_{21}}{l_{11}l_{22}^* - l_{12}l_{21}}$.

Let's consider the vector $\Psi_4 = (\Psi_{41}, \Psi_{42}, \Psi_{43})^T$ be the eigenvector of $L(E_{xyz}, w_7^*)^T$ associated with the zero eigenvalue $\lambda^* = 0$. Therefore, the basic calculation gives that $\Psi_4 = (\alpha_8 \Psi_{43}, \alpha_9 \Psi_{43}, \Psi_{43})^T$, where Ψ_{43} be any real number, not equal zero, and $\alpha_8 = \frac{l_{21}l_{32} - l_{22}^* l_{31}}{l_{11}l_{22}^* - l_{12}l_{21}}$, $\alpha_9 = \frac{l_{12}l_{31} - l_{11}l_{32}}{l_{11}l_{22}^* - l_{12}l_{21}}$.

Furthermore, $\frac{\partial F}{\partial w_7} = F_{w_7} = (0, -y^2, 0)^T$, hence it gives that $F_{w_7}(E_{xyz}, w_7^*) = (0, -\bar{y}^2, 0)^T$, that leads to $\Psi_4^T [F_{w_7}(E_{xyz}, w_7^*)] = -\alpha_9 \Psi_{43} \bar{y} \neq 0$. So, the first condition of saddle-node bifurcation holds according to Sotomayor's theorem. Now, by using Eq. (26) with V_4 at (E_{xyz}, w_7^*), it gives that:

$$D^2 F(E_{xyz}, w_7^*)(V_4, V_4) = [\Lambda_{ij}^*]_{3\times 1}.$$

Where

$$\begin{aligned} \Lambda_{11}^* = {} & -2(\alpha_6 v_{43})^2 - 2\left[1 + 2w_3\bar{y} + \frac{w_1}{(1 + w_1(\bar{y} + w_2\bar{z}))^2}\right]\alpha_6\alpha_7 {v_{43}}^2 \\ & -2\left[1 + 2w_4\bar{z} + \frac{w_1 w_2}{(1 + w_1(\bar{y} + w_2\bar{z}))^2}\right]\alpha_6 {v_{43}}^2 \\ & +2\left[-w_3\bar{x} + \frac{{w_1}^2\bar{x}}{(1 + w_1(\bar{y} + w_2\bar{z}))^3}\right](\alpha_7 v_{43})^2 \\ & +4\left[\frac{{w_1}^2 w_2\bar{x}}{(1 + w_1(\bar{y} + w_2\bar{z}))^3}\right]\alpha_7 {v_{43}}^2 + 2\left[-w_4\bar{x} + \frac{{w_1}^2 w_2^2\bar{x}}{(1 + w_1(\bar{y} + w_2\bar{z}))^3}\right]{v_{43}}^2. \end{aligned}$$

$\Lambda_{21}^* = 2[w_5 + 2w_3 w_5\bar{y}]\alpha_6\alpha_7 {v_{43}}^2 + 2[-w_7 + w_3 w_5\bar{x}](\alpha_7 v_{43})^2$.

$\Lambda_{31}^* = 2[w_8 + 2w_4 w_8\bar{z}]\alpha_6 {v_{43}}^2 + 2w_9\alpha_7 {v_{43}}^2 + 2[-w_{11} + w_4 w_8\bar{x}]{v_{43}}^2$.

Then, using condition (30c), it is obtained that:

$$\Psi_4^T\left[D^2F_{w_7}\left(E_{xyz},w_7^*\right)(V_4,V_4)\right]=(\alpha_8\Lambda_{11}^{\ *}+\alpha_9\Lambda_{21}^{\ *}+\Lambda_{31}^{\ *})\Psi_{43}\neq 0.$$

Thus, Eq. (2) experiences a bifurcation of saddle nodes in the sense of Sotomayor's theorem at E_{xyz} when $w_7=w_7^*$.

8. Numerical Simulation

In the previous explanation and theories of existence, we will now try to represent them using numerical application and through drawings, where the effect of each of the 11 parameters present in Eq. (2) was studied. This is through use Mathematica-13.2. This was done by imposing data for each of the eleven parameters that were carefully selected to obtain the required results and are recorded in the table 2:

Table 2: Data of parameter values.

w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}
0.5	0.6	0.6	0.6	0.5	0.2	0.3	0.4	0.4	0.2	0.3

By substituting parameters into the system, a positive point $E_{xyz}=(0.451\ ,0.156\ ,0.223\ )$ was obtained, and the drawing was illustrated in figure 1. Later, several drawings were drawn once for the prey, the predator, and then the scavenger, but from more than one starting point. We chose 5 points and then combined them into one drawing. All these drawings are placed in the figure 2.

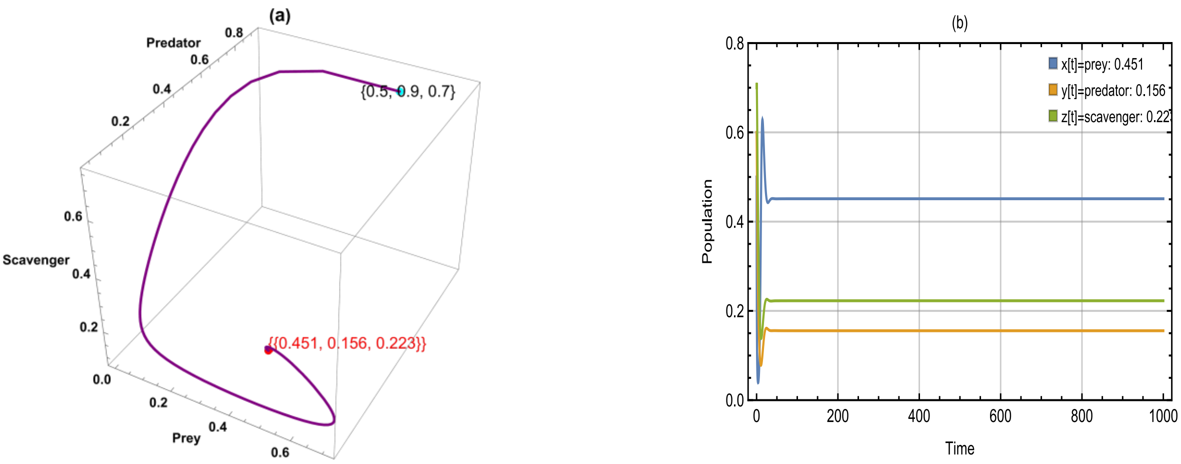


Figure 1: (a) 3$\mathcal{D}$ -Phase portrait of the Eq. (2), (b)The time series of the Eq. (2) by utilizing table (2), the trajectories of three species demonstrate an asymptotic positive convergence towards $E_{xyz}=(0.451\ ,0.156\ ,0.223\ )$.

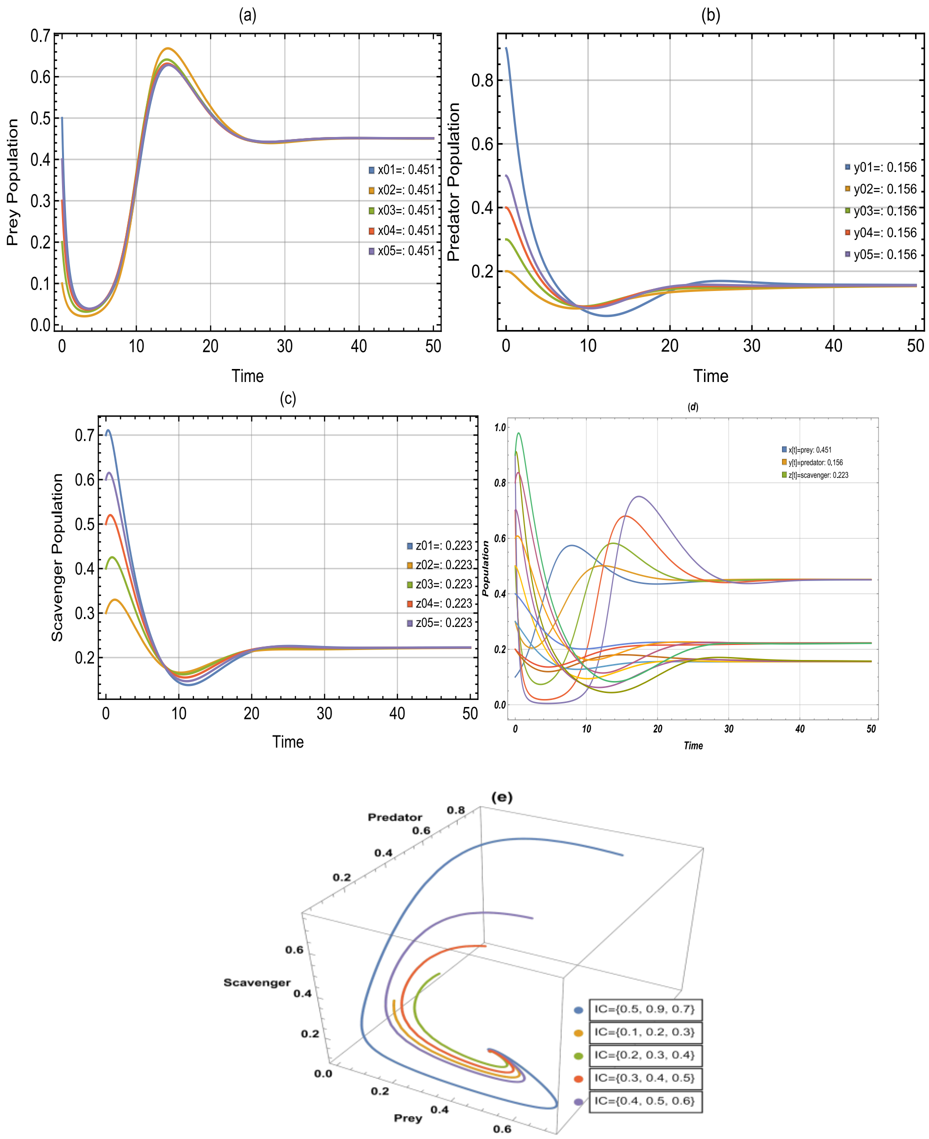


Figure 2: (a) The prey population starting from different initial points. (b) The depredator population starting from different initial points. (c) The scavenger population starting from different initial points. (d) The time series exhibits the trajectories of the Eq. (2), for population from five different initial start. (e) 3$\mathcal{D}$-Phase portrait of the Eq. (2), for five different initial convergence towards $E_{xyz} = (0.451\ , 0.156\ , 0.223\ )$.

In figure 3, we observe the following: if the parameter $w_1 \epsilon (0,14.5)$, it converges towards E_{xyz}, but it converges towards E_{xy} when $w_1 \epsilon [14.5,20)$

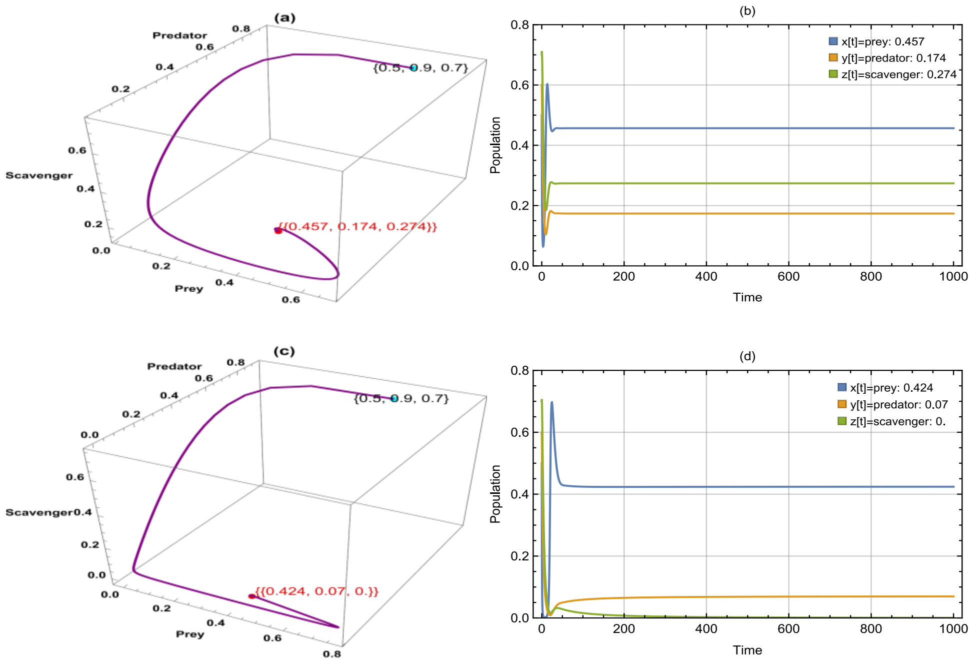


Figure 3: (a) Trajectories of system converge asymptotically to E_{xyz}, (b). Time series of the Eq. (2) converge asymptotically to $E_{xyz} = (0.457, 0.174\ ,0.274)$ for $w_1 = 0.1$. (c). Trajectories of system converge asymptotically to E_{xy}, (d). Time series of the Eq. (2), converge asymptotically to $E_{xy} = (0.424, 0.07\ ,0)$ for $w_1 = 14.5$.

The effect of parameters $w_2, w_3, w_4, w_{7,}w_{11}$ is quantitative impact, so we will discuss only the parameters that effect the system as we noted parameter w_5 approaches positive point in the period (0.326,1), while in interval (0,0.326] impact of w_5 happened on the predator. Thus approach happen to E_{xz} as in figure 4.

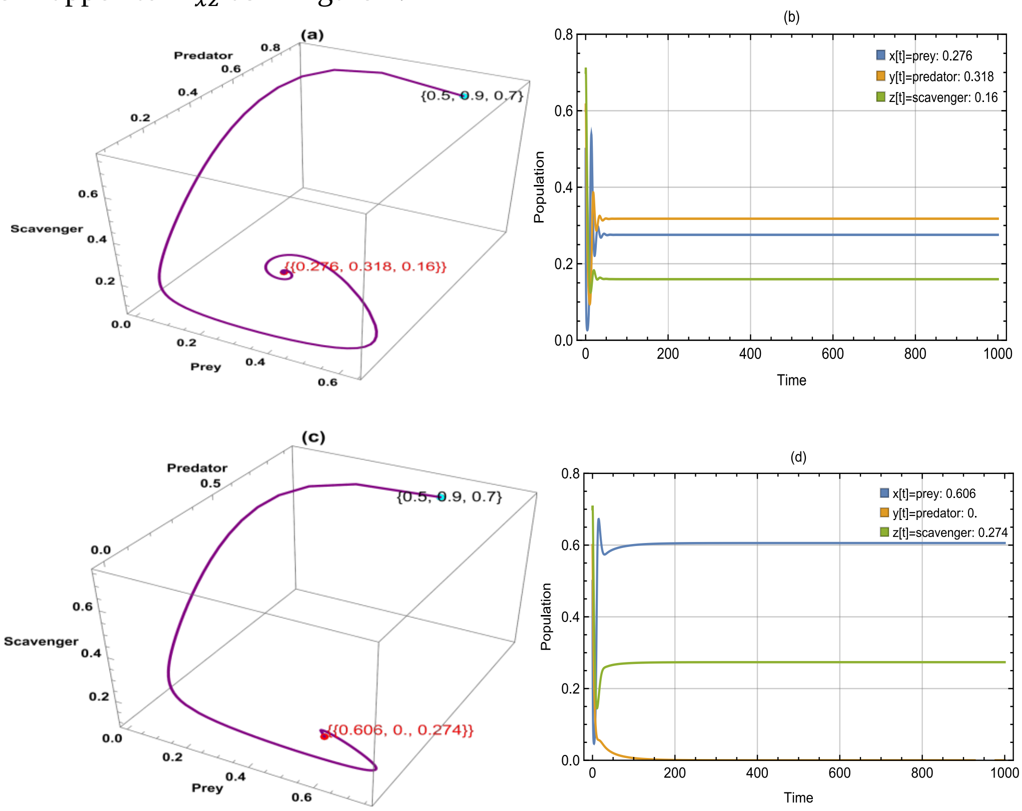


Figure 4: (a) Trajectories of system converge asymptotically to E_{xyz}, (b). Time series of the Eq. (2) converge asymptotically to $E_{xyz} = (0.276, 0.318\ ,0.16)$ for $w_5 = 0.9$. (c).

Trajectories of system converge asymptotically to E_{xz}, (d). Time series of the Eq. (2), converge asymptotically to $E_{xz} = (0.606, 0 ,0.274)$ for $w_5 = 0.3$.

When we put the value of w_6 in specific intervals, we get the following results: As w_6 increases from 0 to 0.305, it approaches to E_{xyz}. Nevertheless, as $w_6$continues to rise within the range of 0.305 to 1, the trajectories gradually converge toward E_{xz}, as in Figure 5.

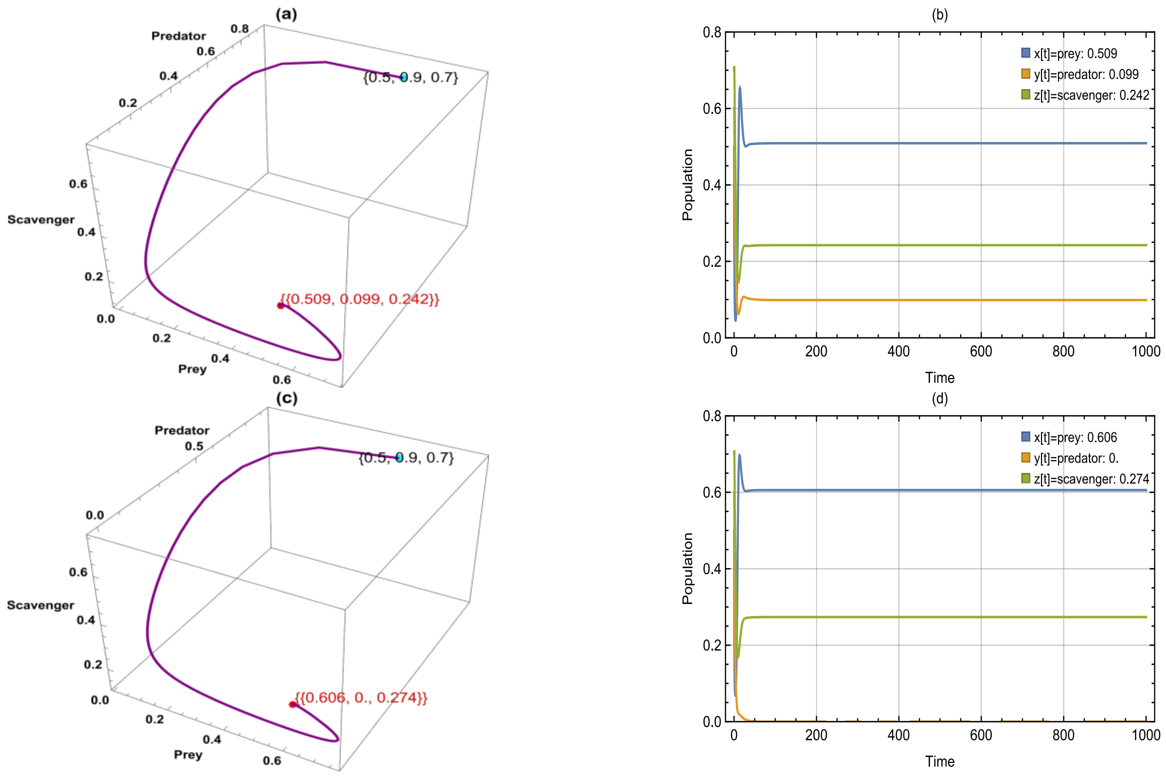


Figure. 5 (a) Trajectories of system converge asymptotically to E_{xyz}, (b) Time series of the Eq. (2) converge asymptotically to $E_{xyz} = (0.509, 0.099 ,0.242)$ for $w_6 = 0.24$. (c) Trajectories of system converge asymptotically to E_{xz}, (d) Time series of the Eq. (2), approach asymptotically to $E_{xz} = (0.606, 0 ,0.274)$ for $w_6 = 0.4$.

When plotting the outcome of changing the w_8, three cases appear during certain periods. The first case is approaching the point E_{xyz} during the period $0.147 < w_8 < 0.654$, the second case is approaching E_{xz}during the period $0.654 \leq w_8 < 1$, and final case is approaching the point E_{xy} from the period $0 < w_8 \leq 0.147$ as shown in Figure 6.

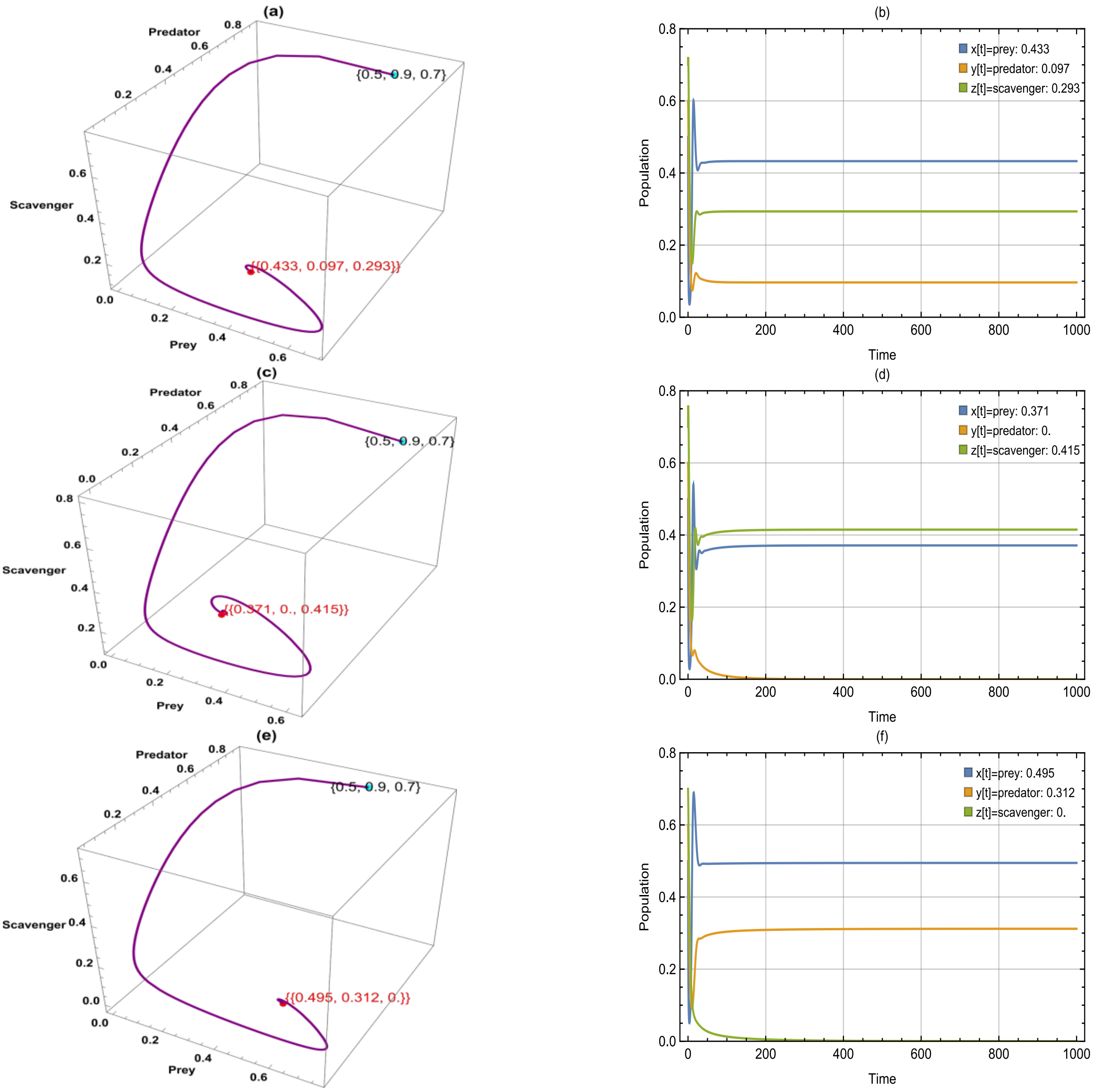


Figure 6: (a) Trajectories of system converge asymptotically to E_{xyz}, (b) Time series of the Eq. (2) converge asymptotically to $E_{xyz} = (0.433, 0.097, 0.293)$ for $w_8 = 0.49$. (c) Trajectories of system approach asymptotically to E_{xz}, (d). Time series of the Eq. (2), approach asymptotically to $E_{xz} = (0.371, 0 , 0.415)$ for $w_8 = 0.7$. (e) Trajectories of system converge asymptotically to E_{xy}, (f) Time series of the Eq.(2) approach asymptotically to $E_{xy} = (0.495, 0.312, 0)$ for $w_8 = 0.142$.

Next, the outcome of changing w_9 within the interval period $0.007 < w_9 < 1$, also its movement towards the point E_{xy} within the range of $0 < w_9 \leq 0.007$, we observe that in Figure 7.

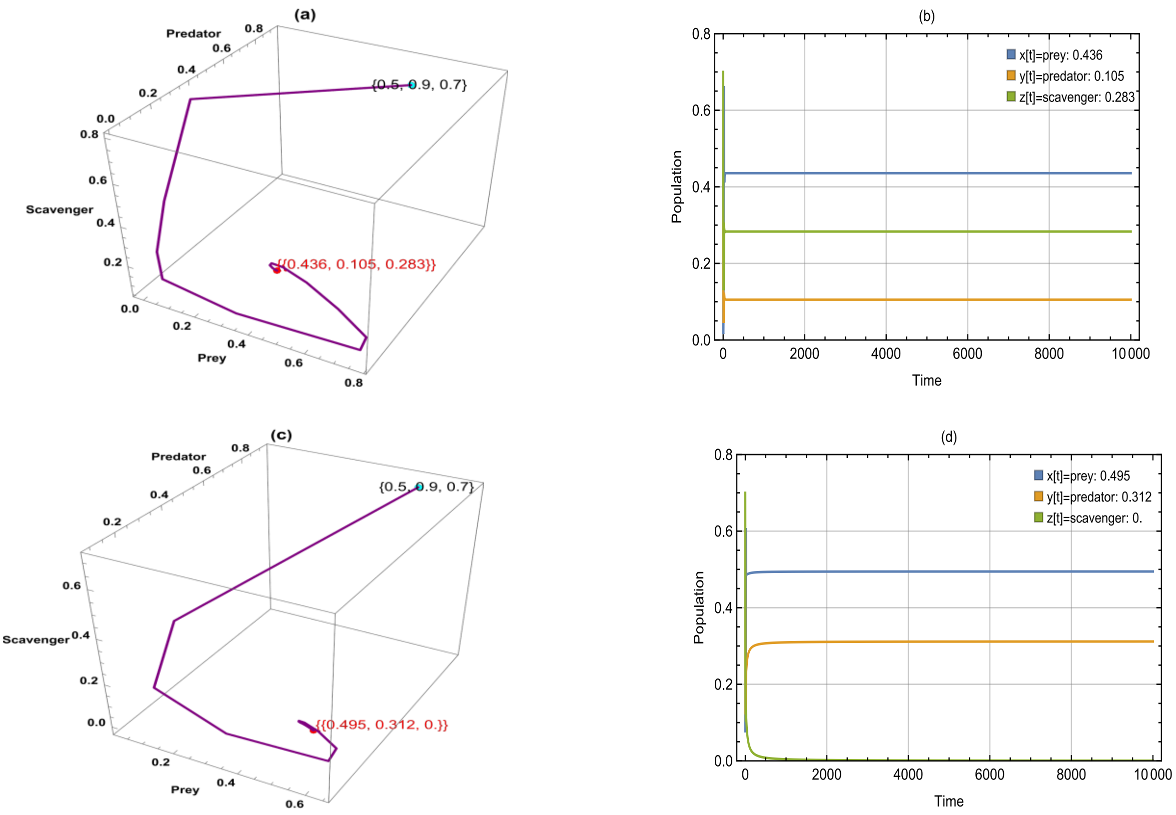
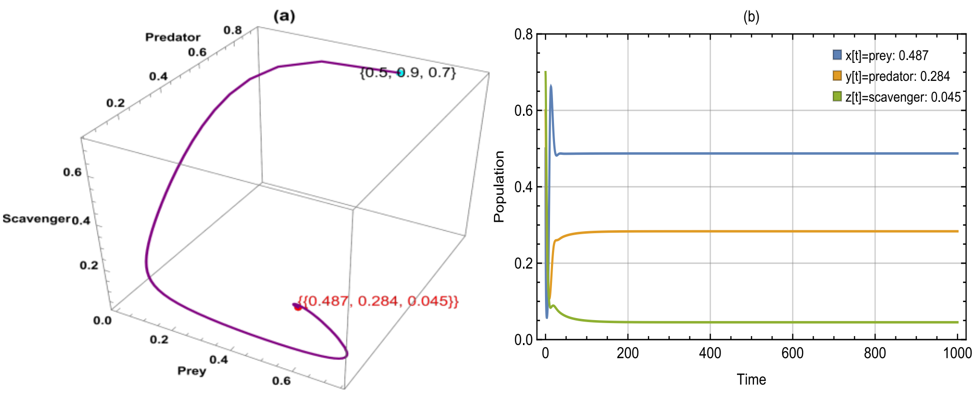


Figure 7: (a) Trajectories of system converge asymptotically to E_{xyz}, (b) Time series of the Eq. (2) approach asymptotically to $E_{xyz} = (0.436, 0.105, 0.283)$ for $w_9 = 0.77$. (c) Trajectories of system converge asymptotically to E_{xy}, (d) Time series of the Eq. (2), converge asymptotically to $E_{xy} = (0.495, 0.312, 0)$ for $w_9 = 0.007$.

Finally, changing in w_{10} approached happen in three cases once to E_{xyz} during the period $0.079 < w_{10} < 0.323$, second approaches to E_{xz}during the period $0 < w_{10} \leq 0.079$, and third case approaches to E_{xy} appeared during the period $0.323 \leq w_{10} < 1$ as in figure (8).



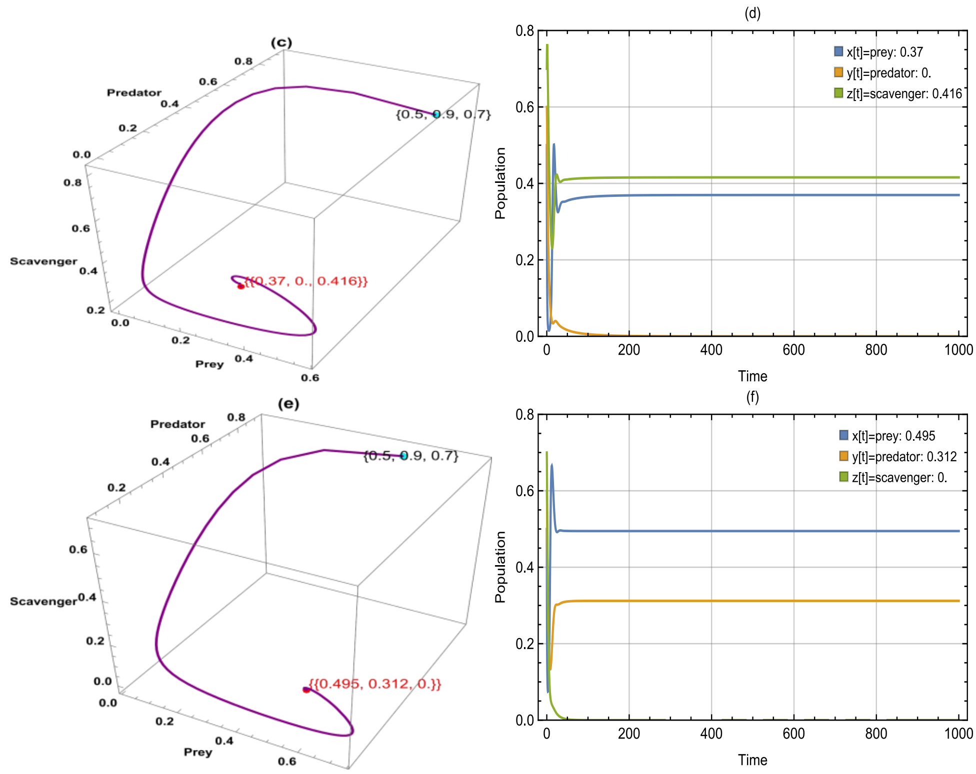


Figure 8: (a) Trajectories of system converge asymptotically to E_{xyz}, (b) Time series of the Eq. (2) approach asymptotically to $E_{xyz} = (0.487, 0.284, 0.045)$ for $w_8 = 0.3$. (c) Trajectories of system converge asymptotically to E_{xz}, (d) Time series of the Eq. (2), approach asymptotically to $E_{xz} = (0.37, 0\,, 0.416)$ for $w_8 = 0.06$ (e) Trajectories of system converge asymptotically to E_{xy}, (f) Time series of the Eq.(2) converge asymptotically to $E_{xy} = (0.495, 0.312, 0)$ for $w_8 = 0.4$.

9. Conclusion

Understanding the dynamics of ecological systems involves delving into the intriguing foraging behaviors of animals. Both predators and prey employ various strategies to bolster their populations. Predators collaborate in hunting endeavors to efficiently capture prey, enabling them to tackle larger or swifter animals, subsequently enhancing their success rates. In response, prey species develop anti-predator defenses to counteract the intensified predation pressure. This constitutes our area of study a focus on exploring this behavior that often involves coordinated efforts among predators. These efforts highlight strategic hunting techniques and the division of tasks, all directed towards maximizing hunting success. So, this research delved into a model describing interactions between prey and predators, with the prey population showing fear reactions. The predators comprised two different categories: active predators and scavengers, both dependent on the prey as their primary food source. The model also incorporated a parameter that defined the cooperative behavior among predators while hunting. The main aim is to assess the influence of collaborative hunting on the food web system dynamics, alongside studying the repercussions of harvesting within predator populations. The study initiated by scrutinizing solution constraints and verifying them, then proceeded to analyze stability points, encompassing both local and global stability assessments. Our research commenced by defining the variables for the prey, predator, and scavenger. Through five initial conditions, we reached the positive point in our study. So, we started with the first parameter w_1, and we obtained two drawings that were explained above, approaching two different points, E_{xyz} and E_{xy}. However, some parameters had a quantitative effect, remaining points within the positive point, but increasing them directly or inversely affects the variables of the system, and they were explained in detail Like all of the parameters $w_i, i = 2,3,4,7,11$. Regarding the remaining parameters, they demonstrated a tangible impact, yielding outcomes consistent with our study and findings. This alignment between the theoretical and numerical solutions was achieved through the mathematica 13.

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