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Sandwich Results for Multivalent Analytic Functions Associated with Borel Distribution and Ruscheweyh Derivative

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Abstract
In this work, we obtain some differential subordination and superordination results defined by Hadamard product for multivalent analytic functions with Borel distribution and Ruscheweyh derivative in the open unit disk. We applied these results and obtain sandwich results.

Keywords: Borel distribution, Ruscheweyh derivative, Analytic functions, Subordinant, Differential subordination, Dominant, Differential superordination.

نتائج الساندوج للدوال التحليلية المتعددة التكافؤ المرتبطة بتوزيع بوريل ومشتقة رشويه

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الخلاصة
في هذا العمل حصلنا على بعض نتائج التابعية والفوق التابعية التفاضلية المعرفة بواسطة ضرب هادمارد للدوال التحليلية المتعددة التكافؤ مع توزيع بوريل ومشتقة الرشويه في قرص الوحدة المفتوح. طبقنا هذه النتائج وحصلنا على نتائج الساندوج.

1. Introduction and Preliminaries

Suppose that $\mathcal{A}_p$ represents the set of functions that may be expressed in the following form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \ , \ \ p \in \mathbb{N} = \{1,2, \dots\}, \tag{1.1}$$

which are analytic and multivalent in the open unit disk $\mathfrak{D} = \{z \in \mathbb{C} \ : |z| < 1\}$. For simplicity, let $\mathcal{A}_1 = \mathcal{A}$.

Consider the class $\mathcal{H}$, which represents the class of analytic functions in the domain $\mathfrak{D}$. Denote $\mathcal{H}[a\ ,p]$ as the subclass of $\mathcal{H}$ that consists of functions with the following form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots \ , \ \ (a \in \mathbb{C}, p \in \mathbb{N}).$$

Let f and g be functions of the set $\mathcal{H}$. If there is a Schwarz function K that is analytic in $\mathfrak{D}$, with $K(0) = 0$ and $|K(z)| < 1$ ($z \in \mathfrak{D}$), such that $f(z) = g(K(z))$, then the function f is considered subordinate to g, or g is considered superordinate to f. This subordination is

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indicated by the notation $f \prec g$ or $f(\mathfrak{z}) \prec g(\mathfrak{z})(\mathfrak{z} \in \mathfrak{D})$. It is well known that, if the function g is univalent in $\mathfrak{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathfrak{D}) \subset g(\mathfrak{D})$.

Assume k and h represent elements of the class $\mathcal{H}$, and let $\psi(r, s, t; \mathfrak{z}): \mathbb{C}^3 \times \mathfrak{D} \longrightarrow \mathbb{C}$. If k and $(k(\mathfrak{z}), \mathfrak{z}k'(\mathfrak{z}), \mathfrak{z}^2 k''(\mathfrak{z}); \mathfrak{z})$ are univalent functions in U and if k fulfills the second-order differential superordination condition

$$h(\mathfrak{z}) \prec \psi(k(\mathfrak{z}), \mathfrak{z}k'(\mathfrak{z}), \mathfrak{z}^2 k''(\mathfrak{z}); \mathfrak{z}), \tag{1.2}$$

then k denote the solution of the differential superordination Equation (1.2). If f is considered subordinate to g, it can be said that g is superior or higher in rank to f. An analytic function q is referred to as a subordinate of (1.2) if q is strictly subordinate to every k that fulfills Equation (1.2). The best subordinant is a univalent subordinant $\tilde{q}$ that fulfills the condition $q \prec \tilde{q}$ for all subordinants q of Equation (1.2).
Let f be a function in $\mathcal{A}_p$, defined by Equation (1.1), and let g be a function in $\mathcal{A}_p$, defined as

$$g(\mathfrak{z}) = \mathfrak{z}^p + \sum_{n=p+1}^{\infty} b_n \mathfrak{z}^n \ , \ \ p \in \mathbb{N},$$

the Hadamard product (or convolution) of the functions f and g, denoted as $f * g$, is defined as follows:

$$(f * g)(\mathfrak{z}) = \mathfrak{z}^p + \sum_{n=p+1}^{\infty} a_n b_n \mathfrak{z}^n = (g * f)(\mathfrak{z}).$$

The elementary distributions, such as the Poisson, Pascal, Logarithmic, Binomial, and beta negative binomial, have been partially analyzed in Geometric Function Theory from a theoretical standpoint (see references $[1-5]$).

Wanas and Khuttar [6] recently introduced the Borel distribution (BD), which is characterized by a probability mass function

$$P(x = r) = \frac{(\alpha r)^{r-1} e^{-\alpha r}}{r!}, \ \ r = 1,2,3, \dots .$$

Wanas and Khuttar [6] presented a series in which the coefficients represent the probability of the Borel distribution (BD).

$$\mathcal{N}_p(\alpha; \mathfrak{z}) = \mathfrak{z}^p + \sum_{n=p+1}^{\infty} \frac{\big(\alpha(n-p)\big)^{n-p-1} e^{-\alpha(n-p)}}{(n-p)!} \mathfrak{z}^n,$$

where $0 < \alpha \le 1$.
For $\xi > -p$ and $f \in \mathcal{A}_p$. The Ruscheweyh derivative of order $\xi + p - 1$ (see [7]) is denoted by $D^{\xi+p-1} f$ and defined as following :

$$D^{\xi+p-1} f(\mathfrak{z}) = \frac{\mathfrak{z}^p}{(1-\mathfrak{z})^{p+\xi}} * f(\mathfrak{z}) = \mathfrak{z}^p + \sum_{n=p+1}^{\infty} \frac{\Gamma(\xi + n)}{\Gamma(\xi + p)(n-p)!} a_n \, \mathfrak{z}^n \ \ (\xi > -p) .$$

Now, by making of the convolution of two series, we define the operator $\Upsilon_\alpha^{\xi+p-1}: \mathcal{A}_{p \to} \mathcal{A}_p$ as follows:

$$\Upsilon_\alpha^{\xi+p-1} = \mathcal{N}_p(\alpha; \mathfrak{z}) * D^{\xi+p-1} f(\mathfrak{z})$$

$$= \mathfrak{z}^p + \sum_{n=p+1}^{\infty} \frac{\Gamma(\xi+n)\big(\alpha(n-p)\big)^{n-p-1} e^{-\alpha(n-p)}}{\Gamma(\xi+p)\big((n-p)!\big)^2} a_n \, \mathfrak{z}^n \ , \tag{1.3}$$

where $\xi > -p$ and $0 < \alpha \le 1$.
It can be easily confirmed from Equation (1.3) that

$$ \mathfrak{z}\left(\Upsilon_{\alpha}^{\xi+p-1}f(\mathfrak{z})\right)' = (\xi+p)\Upsilon_{\alpha}^{\xi+p-1}f(\mathfrak{z}) - \xi\Upsilon_{\alpha}^{\xi+p-1}f(\mathfrak{z}). \quad (1.4) $$

Bulboacă [8] (see also [9]) examined specific categories of first-order differential superordinations, along with integral operators that preserve superordination, utilizing the findings of Miller and Mocanu [10]. Several writers, including Shanmugam et al. [11], Goyal et al. [12], Murugusundaramoorthy and Magesh [13,14], Magesh et al. [15] and Ibrahim and Darus [16], have recently derived sandwich conclusions for specific categories of analytic functions. In addition, some authors ($[17-19]$) have deduced differential subordination and superordination conclusions by employing sandwich theorems.

The primary objective of this study is to establish adequate criteria for specific normalized analytic functions f in the unit disc $\mathfrak{D}$, such that $(f*\Psi)(\mathfrak{z}) \neq 0$ and f satisfies certain constraints

$$ q_1(\mathfrak{z}) \prec \left(\frac{\Upsilon_{\alpha}^{\xi+p}(f*\Phi)(\mathfrak{z})}{\Upsilon_{\alpha}^{\xi+p-1}(f*\Psi)(\mathfrak{z})}\right)^{\gamma} \prec q_2(\mathfrak{z}) $$

and

$$ q_1(\mathfrak{z}) \prec \left(\frac{t_1\Upsilon_{\alpha}^{\xi+p}(f*\Phi)(\mathfrak{z}) + t_2\Upsilon_{\alpha}^{\xi+p-1}(f*\Psi)(\mathfrak{z})}{(t_1+t_2)\mathfrak{z}^p}\right)^{\gamma} \prec q_2(\mathfrak{z}), $$

where q_1 and q_2 are given univalent functions in $\mathfrak{D}$ such that $q_1(0) = q_2(0) = 1$ with

$$ \Phi(\mathfrak{z}) = \mathfrak{z}^p + \sum_{n=p+1}^{\infty} r_n \mathfrak{z}^n, \quad \Psi(\mathfrak{z}) = \mathfrak{z}^p + \sum_{n=p+1}^{\infty} e_n \mathfrak{z}^n $$

are analytic functions in $\mathfrak{D}$ such that $r_n \geq 0, e_n \geq 0$.

In order to establish our primary findings, it is necessary to have the following definitions and lemmas.

Definition 1.1 [20]. Suppose that Q is the set of any functions f that are both analytic and injective on $\overline{\mathfrak{D}} \setminus E(f)$, with

$$ E(f) = \left\{\zeta \in \partial\mathfrak{D} : \lim_{\mathfrak{z}\to\zeta} f(\mathfrak{z}) = \infty\right\} $$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathfrak{D}\setminus E(f)$.

Lemma 1.2 [20]. Assume that q is a univalent in the unit disk $\mathfrak{D}$. Consider ϑ and $\mathfrak{X}$ as analytic functions defined in a domain D that contains $q(\mathfrak{D})$. Additionally, $\mathfrak{X}(w) \neq 0$ when $w \in q(\mathfrak{D})$. Set $Q(\mathfrak{z}) = \mathfrak{z}q'(\mathfrak{z})\mathfrak{X}(q(\mathfrak{z}))$ and $h(\mathfrak{z}) = \vartheta(q(\mathfrak{z})) + Q(\mathfrak{z})$. Assume that

(1) $Q(\mathfrak{z})$ is starlike univalent in $\mathfrak{D}$,

(2) $Re\left\{\frac{\mathfrak{z}h'(\mathfrak{z})}{Q(\mathfrak{z})}\right\} > 0$ for $\mathfrak{z} \in \mathfrak{D}$.

If k is an analytic function in $\mathfrak{D}$, satisfying the conditions $k(0) = q(0),\ k(\mathfrak{D}) \subset D$ with

$$ \vartheta(k(\mathfrak{z})) + \mathfrak{z}k'(\mathfrak{z})\mathfrak{X}(k(\mathfrak{z})) \prec \vartheta(q(\mathfrak{z})) + \mathfrak{z}q'(\mathfrak{z})\mathfrak{X}(q(\mathfrak{z})), \quad (1.5) $$

then $k \prec q$ and q is the best dominant of Equation (1.5).

Lemma 1.3 [8]. Consider q to be a convex and univalent function in the unit disk $\mathfrak{D}$. Additionally, consider ϑ and $\mathfrak{X}$ denote analytic in a domain D containing $q(\mathfrak{D})$. Assume that

(1) $Re\left\{\frac{\vartheta'(q(\mathfrak{z}))}{\mathfrak{X}(q(\mathfrak{z}))}\right\} > 0$ for $\mathfrak{z} \in \mathfrak{D}$,

(2) $Q(\mathfrak{z}) = \mathfrak{z}q'(\mathfrak{z})\mathfrak{X}(q(\mathfrak{z}))$ is starlike univalent in $\mathfrak{D}$.

If $k \in \mathcal{H}\,[q(0),1] \cap Q$, such that $k(\mathfrak{D}) \subset D, \vartheta(k(\mathfrak{z})) + \mathfrak{z}k'(\mathfrak{z})\mathfrak{X}(k(\mathfrak{z}))$ is univalent in $\mathfrak{D}$ and

$$ \vartheta(q(\mathfrak{z})) + \mathfrak{z}q'(\mathfrak{z})\mathfrak{X}(q(\mathfrak{z})) \prec \vartheta(k(\mathfrak{z})) + \mathfrak{z}k'(\mathfrak{z})\mathfrak{X}(k(\mathfrak{z})), \quad (1.6) $$

then $q \prec k$ and q is the best subordinant of Equation (1.6).

2. Subordination results

Theorem 2.1. Suppose that $\Phi,\Psi \in \mathcal{A}_p$, $\rho,\delta,\mu,\eta,\sigma,\tau,\gamma \in \mathbb{C}$, with $\gamma \neq 0$ and σ,τ are not simultaneously zero, q be a convex and univalent function in $\mathfrak{D}$ such that $q(0)=1$. We suppose that

$$Re\left\{\frac{1}{\sigma q(z)+\tau}\left(-\eta+\delta q^2(z)+2\mu q^3(z)-\sigma z q'(z)-\frac{2\tau z q'(z)}{q(z)}\right)+\frac{z q''(z)}{q'(z)}+1\right\}>0. \quad (2.1)$$

If $f \in \mathcal{A}_p$ fulfills the differential subordination

$$D_1(f,\Phi,\Psi,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,\xi,\alpha,p;z) \prec \rho+\delta q(z)+\mu q^2(z)+\frac{\eta}{q(z)}+\left(\frac{\sigma}{q(z)}+\frac{\tau}{q^2(z)}\right)z q'(z), \quad (2.2)$$

where

$$D_1(f,\Phi,\Psi,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,\xi,\alpha,p;z) = \rho+\left(\frac{\Upsilon_\alpha^{\xi+p}(f*\Phi)(z)}{\Upsilon_\alpha^{\xi+p-1}(f*\Psi)(z)}\right)^\gamma\left(\delta+\mu\left(\frac{\Upsilon_\alpha^{\xi+p}(f*\Phi)(z)}{\Upsilon_\alpha^{\xi+p-1}(f*\Psi)(z)}\right)^\gamma\right)+\eta\left(\frac{\Upsilon_\alpha^{\xi+p-1}(f*\Psi)(z)}{\Upsilon_\alpha^{\xi+p}(f*\Phi)(z)}\right)^\gamma+\gamma\left(\sigma+\tau\left(\frac{\Upsilon_\lambda^{\xi+p-1}(f*\Psi)(z)}{\Upsilon_\alpha^{\xi+p}(f*\Phi)(z)}\right)^\gamma\right)$$
$$\times\left((\xi+p+1)\frac{\Upsilon_\alpha^{\xi+p+1}(f*\Phi)(z)}{\Upsilon^{\xi+p}(f*\Phi)(z)}-(\xi+p)\frac{\Upsilon_\alpha^{\xi+p}(f*\Psi)(z)}{\Upsilon^{\xi+p-1}(f*\Psi)(z)}-1\right), \quad (2.3)$$

then

$$\left(\frac{\Upsilon_\alpha^{\xi+p}(f*\Phi)(z)}{\Upsilon_\alpha^{\xi+p-1}(f*\Psi)(z)}\right)^\gamma \prec q(z)$$

with q is the best dominant of Equation (2.2).

Proof. Suppose that the function k is indicated by

$$k(z)=\left(\frac{\Upsilon_\alpha^{\xi+p}(f*\Phi)(z)}{\Upsilon_\alpha^{\xi+p-1}(f*\Psi)(z)}\right)^\gamma, \quad (z \in \mathfrak{D}). \quad (2.4)$$

Thus, the function k is analytic in $\mathfrak{D}$ with $k(0)=1$.
Conducting an easy calculation using of Equation (2.4) gives

$$\frac{z k'(z)}{k(z)}=\gamma\left(\frac{z\left(\Upsilon_\alpha^{\xi+p}(f*\Phi)(z)\right)'}{\Upsilon_\alpha^{\xi+p}(f*\Phi)(z)}-\frac{z\left(\Upsilon_\alpha^{\xi+p-1}(f*\Psi)(z)\right)'}{\Upsilon_\alpha^{\xi+p-1}(f*\Psi)(z)}\right).$$

Considering Equation (1.4), we get

$$\frac{z k'(z)}{k(z)}=\gamma\left((\xi+p+1)\frac{\Upsilon_\alpha^{\xi+p+1}(f*\Phi)(z)}{\Upsilon_\alpha^{\xi+p}(f*\Phi)(z)}-(\xi+p)\frac{\Upsilon_\alpha^{\xi+p}(f*\Psi)(z)}{\Upsilon_\alpha^{\xi+p-1}(f*\Psi)(z)}-1\right).$$

Also, we find that

$$\rho+\delta k(z)+\mu k^2(z)+\frac{\eta}{k(z)}+\left(\frac{\sigma}{k(z)}+\frac{\tau}{k^2(z)}\right)z k'(z) = D_1(f,\Phi,\Psi,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,\xi,\alpha,p;z), \quad (2.5)$$

where $D_1(f,\Phi,\Psi,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,\alpha,\xi,p;z)$ is given by Equation (2.3).
By using Equation (2.5) in Equation (2.2), we have

$$\rho+\delta k(z)+\mu k^2(z)+\frac{\eta}{k(z)}+\left(\frac{\sigma}{k(z)}+\frac{\tau}{k^2(z)}\right)z k'(z) \prec \rho+\delta q(z)+\mu q^2(z)+\frac{\eta}{q(z)}+\left(\frac{\sigma}{q(z)}+\frac{\tau}{q^2(z)}\right)z q'(z).$$

By setting

$$\vartheta(w) = \rho + \delta w + \mu w^2 + \frac{\eta}{w} \quad and \quad \mathfrak{X}(w) = \frac{\sigma}{w} + \frac{\tau}{w^2},$$

It is evident that $\vartheta(w)$ and $\mathfrak{X}(w)$ are analytic in $\mathbb{C} \setminus \{0\}$
that $\mathfrak{X}(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$. Additionally, we obtain

$$Q(z) = zq'(z)\mathfrak{X}(q(z)) = \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right) zq'(z)$$

and

$$h(z) = \vartheta(q(z)) + Q(z) = \rho + \delta q(z) + \mu q^2(z) + \frac{\eta}{q(z)} + \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right) zq'(z).$$

Considering the hypothesis of Theorem 2.1, it is evident that $Q(z)$ is a starlike univalent function in the domain $\mathfrak{D}$ with

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{1}{\sigma q(z) + \tau}\left(-\eta + \delta q^2(z) + 2\mu q^3(z) - \sigma zq'(z) - \frac{2\tau zq'(z)}{q(z)}\right) + \frac{zq''(z)}{q'(z)} + 1\right\} > 0.$$

Therefore, the result may now be deduced by utilizing Lemma 1.2.
By substituting $\Phi(z) = \Psi(z) = \frac{z^p}{1-z}$ into theorem 2.1, we get the subsequent corollary:

Corollary 2.2. Suppose that $\rho, \delta, \mu, \eta, \sigma, \tau, \gamma \in \mathbb{C}$ with $\gamma \neq 0$ and σ, τ are not simultaneously zero, q be convex univalent in $\mathfrak{D}$ such that $q(0) = 1$. We also assume that condition (2.1) is satisfied. If $f \in \mathcal{A}_p$ and fulfills the differential subordination condition

$$D_2(f, \rho, \delta, \mu, \eta, \sigma, \tau, \gamma, \alpha, \xi, p; z) \prec \rho + \delta q(z) + \mu q^2(z) + \frac{\eta}{q(z)} + \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right) zq'(z), \quad (2.6)$$

where

$$D_2(f, \rho, \delta, \mu, \eta, \sigma, \tau, \gamma, \alpha, \xi, p; z) = \rho + \left(\frac{\Upsilon_\alpha^{\xi+p} f(z)}{\Upsilon_\alpha^{\xi+p-1} f(z)}\right)^\gamma \left(\delta + \mu\left(\frac{\Upsilon_\alpha^{\xi+p} f(z)}{\Upsilon_\alpha^{\xi+p-1} f(z)}\right)^\gamma\right) + \eta\left(\frac{\Upsilon_\alpha^{\xi+p-1} f(z)}{\Upsilon_\alpha^{\xi+p} f(z)}\right)^\gamma + \gamma\left(\sigma + \tau\left(\frac{\Upsilon_\alpha^{\xi+p-1} f(z)}{\Upsilon_\alpha^{\xi+p} f(z)}\right)^\gamma\right) \times \left((\xi + P + 1)\frac{\Upsilon_\alpha^{\xi+p+1} f(z)}{\Upsilon_\alpha^{\xi+p} f(z)} - (\xi + p)\frac{\Upsilon_\alpha^{\xi+p} f(z)}{\Upsilon_\alpha^{\xi+p-1} f(z)} - 1\right), \quad (2.7)$$

then

$$\left(\frac{\Upsilon_\alpha^{\xi+p} f(z)}{\Upsilon_\alpha^{\xi+p-1} f(z)}\right)^\gamma \prec q(z)$$

and q is the best dominant of (2.6).

Theorem 2.3. Suppose that $\Phi, \Psi \in \mathcal{A}_p$, $\rho, \delta, \mu, \eta, \sigma, \tau, t_1, t_2 \in \mathbb{C}$ with $t_1 + t_2 \neq 0$ and σ, τ are not simultaneously zero, q be convex univalent in $\mathfrak{D}$ such that $q(0) = 1$. Additionally, assume that condition (2.1) is satisfied. If $f \in \mathcal{A}_p$ and fulfills the differential subordination rule

$$D_3(f, \Phi, \Psi, \rho, \delta, \mu, \eta, \sigma, \tau, \gamma, t_1, t_2, \xi, \alpha, p; z) \prec \rho + \delta q(z) + \mu q^2(z) + \frac{\eta}{q(z)} + \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right) zq'(z), \quad (2.8)$$

where

$D_3(f, \Phi, \Psi, \rho, \delta, \mu, \eta, \sigma, \tau, \gamma, t_1, t_2, \xi, \alpha, p; z)$

$$= \rho + \left(\frac{t_1\,\Upsilon_\alpha^{\xi+p}(f*\Phi)(\mathfrak{z})+t_2\,\Upsilon^{\xi+p-1}(f*\Psi)(z)}{(t_1+t_2)\mathfrak{z}^p}\right)^\gamma\left(\delta+\mu\left(\frac{t_1\Upsilon_\alpha^{\xi+p}(f*\Phi)(\mathfrak{z})+t_2\,\Upsilon^{\xi+p-1}(f*\Psi)(z)}{(t_1+t_2)\mathfrak{z}^p}\right)^\gamma\right)$$
$$+\eta\left(\frac{(t_1+t_2)\mathfrak{z}^p}{t_1\Upsilon_\alpha^{\xi+p}(f*\Phi)(\mathfrak{z})+t_2\,\Upsilon^{\xi+p-1}(f*\Psi)(z)}\right)^\gamma+\gamma\left(\sigma+\tau\left(\frac{(t_1+t_2)\mathfrak{z}^p}{t_1\Upsilon_\alpha^{\xi+p}(f*\Phi)(\mathfrak{z})+t_2\,\Upsilon^{\xi+p}(f*\Psi)(\mathfrak{z})}\right)^\gamma\right)$$
$$\times\left(\frac{t_1(\xi+p+1)[\Upsilon_\alpha^{\xi+p+1}(f*\Phi)(z)-\Upsilon_\alpha^{\xi+p}(f*\Phi)(z)]}{t_1\Upsilon_\alpha^{\xi+p}(f*\Phi)(\mathfrak{z})+t_2\,\Upsilon^{\xi+p-1}(f*\Psi)(z)}+\frac{t_2\,(\xi+p)[\Upsilon_\alpha^{\xi+p}(f*\Psi)(\mathfrak{z})-\Upsilon_\alpha^{\xi+p-1}(f*\Psi)(\mathfrak{z})]}{t_1\Upsilon_\alpha^{\xi+p}(f*\Phi)(\mathfrak{z})+t_2\,\Upsilon^{\xi+p-1}(f*\Psi)(z)}\right), \tag{2.9}$$

then

$$\left(\frac{t_1\Upsilon_\alpha^{\xi+p}(f*\Phi)(\mathfrak{z})+t_2\,\Upsilon^{\xi+p-1}(f*\Psi)(z)}{(t_1+t_2)\mathfrak{z}^p}\right)^\gamma \prec q(\mathfrak{z})$$

with q is the best dominant of Equation (2.8).

Proof. Suppose that the function k is indicated by

$$k(\mathfrak{z}) = \left(\frac{t_1\Upsilon_\alpha^{\xi+p}(f*\Phi)(\mathfrak{z})+t_2\,\Upsilon^{\xi+p-1}(f*\Psi)(z)}{(t_1+t_2)\mathfrak{z}^p}\right)^\gamma, (\mathfrak{z}\in\mathfrak{D}). \tag{2.10}$$

Thus, the function k is analytic in $\mathfrak{D}$ with $k(0)=1$.

Thus, by utilizing Equations (2.10) and (1.4), we derive

$$\rho+\delta k(\mathfrak{z})+\mu k^2(\mathfrak{z})+\frac{\eta}{k(\mathfrak{z})}+\left(\frac{\sigma}{k(\mathfrak{z})}+\frac{\tau}{k^2(\mathfrak{z})}\right)\mathfrak{z}k'(\mathfrak{z})$$
$$= D_3(f,\Phi,\Psi,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,t_1,t_2,\alpha,p,;\mathfrak{z}), \quad (2.11)$$

where $D_3(f,\Phi,\Psi,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,t_1,t_2,\alpha,\xi,p;\mathfrak{z})$ is given by Equation (2.9).

Considering Equation (2.11), we can express the subordination (2.8) as follows:

$$\rho+\delta k(\mathfrak{z})+\mu k^2(\mathfrak{z})+\frac{\eta}{k(\mathfrak{z})}+\left(\frac{\sigma}{k(\mathfrak{z})}+\frac{\tau}{k^2(\mathfrak{z})}\right)\mathfrak{z}k'(\mathfrak{z})$$
$$\prec \rho+\delta q(\mathfrak{z})+\mu q^2(\mathfrak{z})+\frac{\eta}{q(\mathfrak{z})}+\left(\frac{\sigma}{q(\mathfrak{z})}+\frac{\tau}{q^2(\mathfrak{z})}\right)\mathfrak{z}q'(\mathfrak{z}).$$

By defining the functions $\vartheta(w)=\rho+\delta w+\mu w^2+\frac{\eta}{w}$ and $\mathfrak{X}(w)=\frac{\sigma}{w}+\frac{\tau}{w^2}$, it is evident that $\vartheta(w)$ and $\mathfrak{X}(w)$ are both analytic in $\mathbb{C}\setminus\{0\}$ and that $\mathfrak{X}(w)\neq 0, w\in\mathbb{C}\setminus\{0\}$. Therefore, we obtain the result.

By substituting $\Phi(\mathfrak{z})=\Psi(\mathfrak{z})=\frac{\mathfrak{z}^p}{1-\mathfrak{z}}$ in Theorem 2.3, we get the subsequent corollary:

Corollary 2.4. Assume that $\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,t_1,t_2\in\mathbb{C}$ such that $\gamma\neq 0$, $t_1+t_2\neq 0$. Additionally, σ,τ are not simultaneously zero, q be convex univalent in $\mathfrak{D}$ and $q(0)=1$. We assume that condition (2.1) is satisfied. If $f\in\mathcal{A}_p$ fulfills the differential subordination condition

$$D_4(f,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,t_1,t_2,\alpha,\xi,p;\mathfrak{z})$$
$$\prec \rho+\delta q(\mathfrak{z})+\mu q^2(\mathfrak{z})+\frac{\eta}{q(\mathfrak{z})}+\left(\frac{\sigma}{q(\mathfrak{z})}+\frac{\tau}{q^2(\mathfrak{z})}\right)\mathfrak{z}q'(\mathfrak{z}), \tag{2.12}$$

where

$$D_4(f,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,t_1,t_2,\alpha,\xi,p;\mathfrak{z}) = \rho+\left(\frac{t_1\Upsilon_\alpha^{\xi+p}f(\mathfrak{z})+t_2\Upsilon_\alpha^{\xi+p-1}f(\mathfrak{z})}{(t_1+t_2)\mathfrak{z}^p}\right)^\gamma$$
$$\times\left(\delta+\mu\left(\frac{t_1\Upsilon_\alpha^{\xi+p}f(\mathfrak{z})+t_2\Upsilon_\alpha^{\xi+p-1}f(\mathfrak{z})}{(t_1+t_2)\mathfrak{z}^p}\right)^\gamma\right)+\eta\left(\frac{(t_1+t_2)\mathfrak{z}^p}{t_1\Upsilon_\alpha^{\xi+p}f(\mathfrak{z})+t_2\Upsilon_\alpha^{\xi+p-1}f(\mathfrak{z})}\right)^\gamma+\gamma\left(\sigma+\right.$$
$$\left.\tau\left(\frac{(t_1+t_2)\mathfrak{z}^p}{t_1\Upsilon_\alpha^{\xi+p}f(\mathfrak{z})+t_2\Upsilon_\alpha^{\xi+p-1}f(\mathfrak{z})}\right)^\gamma\right)$$

$$\times\left(\frac{t_1(\xi+p+1)[\Upsilon_\alpha^{\xi+p+1}f(\mathfrak{z})-\Upsilon_\alpha^{\xi+p}f(\mathfrak{z})}{t_1\Upsilon_\alpha^{\xi+p}f(\mathfrak{z})+t_2\Upsilon_\alpha^{\xi+p-1}f(\mathfrak{z})}\right.$$
$$\left.+\frac{t_2(\xi+p)\left[\Upsilon_\alpha^{\xi+p}f(\mathfrak{z})+\Upsilon_\alpha^{\xi+p-1}f(\mathfrak{z})\right]}{t_1\Upsilon_\alpha^{\xi+p}f(\mathfrak{z})+t_2\Upsilon_\alpha^{\xi+p-1}f(\mathfrak{z})}\right), \tag{2.13}$$

then

$$\left(\frac{t_1\Upsilon_\alpha^{\xi+p}f(\mathfrak{z})+t_2\Upsilon_\alpha^{\xi+p-1}f(\mathfrak{z})}{(t_1+t_2)\mathfrak{z}^p}\right)^\gamma \prec q(\mathfrak{z})$$

and q is the best dominant of (2.12).

3. Superordination results

Theorem 3.1. Suppose that $\Phi,\Psi\in\mathcal{A}_p$, $\rho,\delta,\mu,\eta,\sigma,\tau,\gamma\in\mathbb{C}$ with $\gamma\neq 0$ and σ,τ are not simultaneously zero, q be convex univalent in $\mathfrak{D}$ with $q(0)=1$ and assume that

$$Re\left\{\frac{q'(\mathfrak{z})}{\sigma q(\mathfrak{z})+\tau}\left(\delta q^2(\mathfrak{z})+2\mu q^3(\mathfrak{z})-\eta\right)\right\}>0. \tag{3.1}$$

Let $f\in\mathcal{A}_p$,

$$\left(\frac{\Upsilon_\alpha^{\xi+p}(f*\Phi)(\mathfrak{z})}{\Upsilon_\alpha^{\xi+p-1}(f*\Psi)(\mathfrak{z})}\right)^\gamma\in\mathcal{H}[q(0),1]\cap Q$$

and $D_1(f,\Phi,\Psi,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,\alpha,\xi,p;\mathfrak{z})$ as defined by Equation (2.3) be univalent in $\mathfrak{D}$. If

$$\rho+\delta q(\mathfrak{z})+\mu q^2(\mathfrak{z})+\frac{\eta}{q(\mathfrak{z})}+\left(\frac{\sigma}{q(\mathfrak{z})}+\frac{\tau}{q^2(\mathfrak{z})}\right)\mathfrak{z}q'(\mathfrak{z})$$
$$\prec D_1(f,\Phi,\Psi,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,\alpha,\xi,p;\mathfrak{z}), \tag{3.2}$$

then

$$q(\mathfrak{z})\prec\left(\frac{\Upsilon_\alpha^{\xi+p}(f*\Phi)(\mathfrak{z})}{\Upsilon_\alpha^{\xi+p-1}(f*\Psi)(\mathfrak{z})}\right)^\gamma$$

and q is the best subordinant of Equation (3.2).

Proof . Suppose that the function k is indicated by Equation (2.4).
Considering Equation (1.4), the superordination (3.2) becomes

$$\rho+\delta q(\mathfrak{z})+\mu q^2(\mathfrak{z})+\frac{\eta}{q(\mathfrak{z})}+\left(\frac{\sigma}{q(\mathfrak{z})}+\frac{\tau}{q^2(\mathfrak{z})}\right)\mathfrak{z}q'(\mathfrak{z})$$
$$\prec\rho+\delta k(\mathfrak{z})+\mu k^2(\mathfrak{z})+\frac{\eta}{k(\mathfrak{z})}+\left(\frac{\sigma}{k(\mathfrak{z})}+\frac{\tau}{k^2(\mathfrak{z})}\right)\mathfrak{z}k'(\mathfrak{z}).$$

By setting $\vartheta(w)=\rho+\delta w+\mu w^2+\frac{\eta}{w}$ and $\mathfrak{X}(w)=\frac{\sigma}{w}+\frac{\tau}{w^2}$. It is evident that $\vartheta(w)$ and $\mathfrak{X}(w)$ are analytic in $\mathbb{C}\setminus\{0\}$ and that $\mathfrak{X}(w)\neq 0, w\in\mathbb{C}\setminus\{0\}$. Additionally, we obtain

$$Re\left\{\frac{\vartheta'(q(\mathfrak{z}))}{\mathfrak{X}(q(\mathfrak{z}))}\right\}=Re\left\{\frac{q'(\mathfrak{z})}{\sigma q(\mathfrak{z})+\tau}\left(\delta q^2(\mathfrak{z})+2\mu q^3(\mathfrak{z})-\eta\right)\right\}>0.$$

Now Theorem 3.1 follows by utilizing Lemma 1.3.

By substituting $\Phi(\mathfrak{z})=\Psi(\mathfrak{z})=\frac{\mathfrak{z}^p}{1-\mathfrak{z}}$ in Theorem 3.1, we get the subsequent corollary:

Corollary 3.2. Suppose that $\rho,\delta,\mu,\eta,\sigma,\tau,\gamma\in\mathbb{C}$ with $\gamma\neq 0$ and σ,τ are not simultaneously zero, q be convex univalent in $\mathfrak{D}$ with $q(0)=1$. We assume that condition (3.1) is satisfied. Consider that $f\in\mathcal{A}_p$,

$$\left(\frac{\Upsilon_{\alpha}^{\xi+p} f(z)}{\Upsilon_{\alpha}^{\xi+p-1} f(z)}\right)^{\gamma} \in \mathcal{H}[q(0),1] \cap Q$$

and $D_2(f,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,\alpha,\xi,p;z)$ as given by Equation (2.7) be univalent in $\mathfrak{D}$. Assuming

$$\rho + \delta q(z) + \mu q^2(z) + \frac{\eta}{q(z)} + \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right) z q'(z) \prec D_2(f,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,\xi,\alpha,p;z), \quad (3.3)$$

then

$$q(z) \prec \left(\frac{\Upsilon_{\alpha}^{\xi+p} f(z)}{\Upsilon_{\alpha}^{\xi+p-1} f(z)}\right)^{\gamma}$$

and q is the best subordinant of Equation (3.3).

Theorem 3.3. Suppose that $\Phi,\Psi \in \mathcal{A}_p$, $\rho,\delta,\mu,\eta,\sigma,\tau,t_1,t_2 \in \mathbb{C}$ with $t_1 + t_2 \neq 0$ and σ,τ are not simultaneously zero, q be convex univalent in $\mathfrak{D}$ and $q(0) = 1$. We assume that condition (3.1) is satisfied. Consider that $f \in \mathcal{A}_p$,

$$\left(\frac{t_1\Upsilon_{\alpha}^{\xi+p}(f*\Phi)(z) + t_2\Upsilon_{\alpha}^{\xi+p}(f*\Phi)(z)}{(t1+t_2)z^p}\right)^{\gamma} \in \mathcal{H}[q(0),1] \cap Q$$

and $D_3(f,\Phi,\Psi,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,\alpha,\xi,p;z)$ as defined by (2.9) be univalent in $\mathfrak{D}$. If

$$\rho + \delta q(z) + \mu q^2(z) + \frac{\eta}{q(z)} + \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right) z q'(z) \prec D_3(f,\Phi,\Psi,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,\alpha,\xi,p;z), \quad (3.4)$$

then

$$q(z) \prec \left(\frac{t_1\Upsilon_{\alpha}^{\xi+p}(f*\Phi)(z) + t_2\Upsilon_{\alpha}^{\xi+p}(f*\Phi)(z)}{(t_1+t_2)z^p}\right)^{\gamma}$$

and q is the best subordinant of (3.4).

For $k(z) = \left(\frac{t_1\Upsilon_{\alpha}^{\xi+p}(f*\Phi)(z)+t_2\Upsilon_{\alpha}^{\xi+p}(f*\Phi)(z)}{(t_1+t_2)z^p}\right)^{\gamma}$, the proof of Theorem 3.3 is a line that closely resembles the proof of Theorem 3.1, therefore we will skip it.

By substituting $\Phi(z) = \Psi(z) = \frac{z^p}{1-z}$ in Theorem 3.3, we derive the subsequent corollary:

Corollary 3.4. Suppose that $\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,t_1,t_2 \in \mathbb{C}$ where $\gamma \neq 0$, $t_1 + t_2 \neq 0$ and σ,τ are not simultaneously zero, let q be a convex univalent function in the unit disk $\mathfrak{D}$ such that $q(0) = 1$. We assume that condition (3.1) is satisfied. Assume that $f \in \mathcal{A}_p$,

$$\left(\frac{t_1\Upsilon_{\alpha}^{\xi+p}(z) + t_2\Upsilon_{\alpha}^{\xi+p}(z)}{(t_1+t_2)z^p}\right)^{\gamma} \in \mathcal{H}[q(0),1] \cap Q$$

and $D_4(f,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,t_1,t_2,\alpha,\xi,p;z)$ as defined by Equation (2.13) be univalent in $\mathfrak{D}$. If

$$\rho + \delta q(z) + \mu q^2(z) + \frac{\eta}{q(z)} + \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right) z q'(z) \prec D_4(f,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,t_1,t_2,,\alpha,\xi,p;z), \quad (3.5)$$

then

$$q(z) \prec \left(\frac{t_1\Upsilon_{\alpha}^{\xi+p}(z) + t_2\Upsilon_{\alpha}^{\xi+p}(z)}{(t_1+t_2)z^p}\right)^{\gamma}$$

and q is the best subordinant of Equation (3.5).

4. Sandwich results

As a summary of the findings of unequal subordination and superordination, we get the following "sandwich results".

Theorem 4.1. Suppose that q_1 and q_2 be convex univalent in $\mathfrak{D}$ with $q_1(0) = q_2(0) = 1$, $\rho, \delta, \mu, \eta, \sigma, \tau, \gamma \in \mathbb{C}$ with $\gamma \neq 0$ and σ, τ are not simultaneously zero. Assume that q_2 fulfills condition (2.1) and q_1 fulfills condition (3.1). For $f, \Phi, \Psi \in \mathcal{A}_p$, let

$$\left(\frac{\Upsilon_\alpha^{\xi+p}(f * \Phi)(\mathfrak{z})}{\Upsilon_\alpha^{\xi+p-1}(f * \Psi)(\mathfrak{z})}\right)^\gamma \in \mathcal{H}[1,1] \cap Q$$

and $D_1(f, \Phi, \Psi, \rho, \delta, \mu, \eta, \sigma, \tau, \gamma, \alpha, \xi, p; \mathfrak{z})$ as defined by (2.3) be univalent in $\mathfrak{D}$. If

$$\rho + \delta q_1(\mathfrak{z}) + \mu q_1^2(\mathfrak{z}) + \frac{\eta}{q_1(\mathfrak{z})} + \left(\frac{\sigma}{q_1(\mathfrak{z})} + \frac{\tau}{q_1^2(\mathfrak{z})}\right)\mathfrak{z}q_1'(\mathfrak{z})$$
$$\prec D_1(f, \Phi, \Psi, \rho, \delta, \mu, \eta, \sigma, \tau, \gamma, \alpha, \xi, p; \mathfrak{z})$$
$$\prec \rho + \delta q_2(\mathfrak{z}) + \mu q_2^2(\mathfrak{z}) + \frac{\eta}{q_2(\mathfrak{z})} + \left(\frac{\sigma}{q_2(\mathfrak{z})} + \frac{\tau}{q_2^2(\mathfrak{z})}\right)\mathfrak{z}q_2'(\mathfrak{z}),$$

then

$$q_1(\mathfrak{z}) \prec \left(\frac{t_1\Upsilon_\alpha^{\xi+p}(\mathfrak{z}) + t_2\Upsilon_\alpha^{\xi+p-1}(\mathfrak{z})}{(t_1+t_2)\mathfrak{z}^p}\right)^\gamma \prec q_2(\mathfrak{z})$$

and q_1, q_2 are represent the optimal subordinate and the optimal dominant, respectively.

Theorem 4.2. Suppose that q_1 and q_2 be convex univalent in $\mathfrak{D}$ with $q_1(0) = q_2(0) = 1$, $\rho, \delta, \mu, \eta, \sigma, \tau, \gamma, t_1, t_2 \in \mathbb{C}$ such that $\gamma \neq 0$, $t_1 + t_2 \neq 0$ and σ, τ are not simultaneously zero. Assume that q_2 fulfills condition (2.1) and q_1 fulfills condition (3.1). For $f, \in \mathcal{A}_p$, let

$$\left(\frac{t_1\Upsilon_\alpha^{\xi+p}(f * \Phi)(\mathfrak{z}) + t_2\Upsilon_\alpha^{\xi+p-1}(f * \Psi)(\mathfrak{z})}{(t_1 + t_2)\mathfrak{z}^p}\right)^\gamma \in \mathcal{H}[1,1] \cap Q$$

and $D_3(f, \Phi, \Psi, \rho, \delta, \mu, \eta, \sigma, \tau, \gamma, t_1, t_2, \xi, \alpha, p; \mathfrak{z})$ as defined by (2.9) be univalent in $\mathfrak{D}$. If

$$\rho + \delta q_1(\mathfrak{z}) + \mu q_1^2(\mathfrak{z}) + \frac{\eta}{q_1(\mathfrak{z})} + \left(\frac{\sigma}{q_1(\mathfrak{z})} + \frac{\tau}{q_1^2(\mathfrak{z})}\right)\mathfrak{z}q_1'(\mathfrak{z})$$
$$\prec D_3(f, \Phi, \Psi, \rho, \delta, \mu, \eta, \sigma, \tau, \gamma, t_1, t_2, \xi, \alpha, p; \mathfrak{z})$$
$$\prec \rho + \delta q_2(\mathfrak{z}) + \mu q_2^2(\mathfrak{z}) + \frac{\eta}{q_2(\mathfrak{z})} + \left(\frac{\sigma}{q_2(\mathfrak{z})} + \frac{\tau}{q_2^2(\mathfrak{z})}\right)\mathfrak{z}q_2'(\mathfrak{z}),$$

then

$$q_1(\mathfrak{z}) \prec \left(\frac{t_1\Upsilon_\alpha^{\xi+p}(f * \Phi)(\mathfrak{z}) + t_2\Upsilon_\alpha^{\xi+p-1}(f * \Psi)(\mathfrak{z})}{(t_1 + t_2)\mathfrak{z}^p}\right)^\gamma \prec q_2(\mathfrak{z})$$

and q_1, q_2 are respectively the best subordinant and the best dominant.

By using Corollaries 2.2 and 3.2, we derive the subsequent corollary:

Corollary 4.3. Consider q_1 and q_2 as convex univalent in $\mathfrak{D}$ with $q_1(0) = q_2(0) = 1$, $\rho, \delta, \mu, \eta, \sigma, \tau, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and σ, τ are not simultaneously zero. Assume that q_2 fulfills condition (2.1) and q_1 fulfills condition (3.1). For $f \in \mathcal{A}_p$, let

$$\left(\frac{\Upsilon_\alpha^{\xi+p}f(\mathfrak{z})}{\Upsilon_\alpha^{\xi+p-1}f(\mathfrak{z})}\right)^\gamma \in \mathcal{H}[1,1] \cap Q$$

and $D_2(f, \rho, \delta, \mu, \eta, \sigma, \tau, \gamma, \alpha, \xi, p; \mathfrak{z})$ as defined by (2.7) be univalent in $\mathfrak{D}$. If

$$\rho+\delta q_1(\mathfrak{z})+\mu q_1^2(\mathfrak{z})+\frac{\eta}{q_1(\mathfrak{z})}+\left(\frac{\sigma}{q_1(\mathfrak{z})}+\frac{\tau}{q_1^2(\mathfrak{z})}\right)\mathfrak{z}q_1'(\mathfrak{z})\prec D_2(f,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,\alpha,\xi,p;\mathfrak{z})$$
$$\prec\rho+\delta q_2(\mathfrak{z})+\mu q_2^2(\mathfrak{z})+\frac{\eta}{q_2(\mathfrak{z})}+\left(\frac{\sigma}{q_2(\mathfrak{z})}+\frac{\tau}{q_2^2(\mathfrak{z})}\right)\mathfrak{z}q_2'(\mathfrak{z}),$$

then

$$q_1(\mathfrak{z})\prec\left(\frac{\Upsilon_\alpha^{\xi+p}f(\mathfrak{z})}{\Upsilon_\alpha^{\xi+p-1}f(\mathfrak{z})}\right)^\gamma\prec q_2(\mathfrak{z})$$

and q_1, q_2 are respectively the best subordinant and the best dominant.
By using Corollaries 2.4 and 3.4, we derive the next corollary:

Corollary 4.4. Consider q_1 and q_2 as convex univalent in $\mathfrak{D}$ with $q_1(0)=q_2(0)=1$, $\rho,\delta,\mu,\eta,\sigma,\tau,\gamma\in\mathbb{C}$ with $\gamma\neq 0$ and σ,τ are not simultaneously zero. Assume that q_2 fulfills condition (2.1) and q_1 fulfills condition (3.1). For $f\in\mathcal{A}_p$, let

$$\left(\frac{t_1\Upsilon_\alpha^{\xi+p}(\mathfrak{z})+t_2\Upsilon_\alpha^{\xi+p-1}(\mathfrak{z})}{(t_1+t_2)\mathfrak{z}^p}\right)^\gamma\in\mathcal{H}[1,1]\cap Q$$

and $D_4(f,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,t_1,t_2,\alpha,\xi,p;\mathfrak{z})$ as defined by Equation (2.13) be univalent in $\mathfrak{D}$. If

$$\rho+\delta q_1(\mathfrak{z})+\mu q_1^2(\mathfrak{z})+\frac{\eta}{q_1(\mathfrak{z})}+\left(\frac{\sigma}{q_1(\mathfrak{z})}+\frac{\tau}{q_1^2(\mathfrak{z})}\right)\mathfrak{z}q_1'(\mathfrak{z})$$
$$\prec D_4(f,\rho,\delta,\mu,\eta,\sigma,\tau,\gamma,t_1,t_2,\alpha,\xi,p;\mathfrak{z})$$
$$\prec\rho+\delta q_2(\mathfrak{z})+\mu q_2^2(\mathfrak{z})+\frac{\eta}{q_2(\mathfrak{z})}+\left(\frac{\sigma}{q_2(\mathfrak{z})}+\frac{\tau}{q_2^2(\mathfrak{z})}\right)\mathfrak{z}q_2'(\mathfrak{z}),$$

then

$$q_1(\mathfrak{z})\prec\left(\frac{t_1\Upsilon_\alpha^{\xi+p}(\mathfrak{z})+t_2\Upsilon_\alpha^{\xi+p-1}(\mathfrak{z})}{(t_1+t_2)\mathfrak{z}^p}\right)^\gamma\prec q_2(\mathfrak{z})$$

and q_1, q_2 are respectively the best subordinant and the best dominant.

References

[1] S. Altinkaya and S. Yalcin, "Poisson distribution series for certain subclasses of starlike function with negative coefficients", *Annals of Oradea University Mathematics Fascicola*, vol. 24, no. 2, pp. 5-8, 2017.

[2] S. M. El-Deeb, T. Bulboaca and J. Dziok, "Pascal distribution series connected with certain subclasses of univalent functions", *Kyungpook Math. J.*, vol. 59, no. 2, pp. 301-314, 2019.

[3] W. Nazeer, Q. Mehmood, S. M. Kang and A. U. Haq, "An application of Binomial distribution series on certain analytic function", *J. Comput. Anal. Appl.*, vol. 26, pp. 11-17, 2019.

[4] S. Porwal and M. Kumar, "A unified study on starlike and convex functions associated with Poisson distribution series", *Afr. Mat.*, vol. 27, pp. 10-21, 2016.

[5] S. Porwal, "An application of a Poisson distribution series on certain analytic function", *J. Complex Anal.*, vol. 2014, Art. ID: 984135, pp. 1-3, 2014.

[6] A. K. Wanas and J. A. Khuttar, "Applications of Borel distribution series on analytic functions", *Earthline J. Math. Sci.*, vol. 4, no. 2, pp. 71-82, 2020.

[7] S. Ruscheweyh, "New criteria for univalent functions", *Proc. Amer. Math. Soc.*, vol. 49, pp. 109-115, 1975.

[8] T. Bulboacă, "Classes of first order differential superordinations", *Demonstratio Math.*, vol. 35, no. 2, pp. 287-292, 2002.

[9] T. Bulboacă, "A class of superordination-preserving integral operators", *Indag. Math., New Ser.*, vol. 13, no. 3, pp. 301-311, 2002.

[10] S. S. Miller and P. T. Mocanu, "Subordinants of differential superordinations", *Complex Variables*, vol. 48, no. 10, pp. 815-826, 2003.

[11] T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, "Differential sandwich theorems for some subclasses of analytic functions", *Aust. J. Math. Anal. Appl.*, vol. 3, no. 1, pp. 1-11, 2006.
[12] S. P. Goyal, P. Goswami and H. Silverman, "Subordination and superordination results for a class of analytic multivalent functions", *Int. J. Math. Math. Sci.*, Art. ID: 561638, pp. 1-12, 2008.
[13] G. Murugusundaramoorthy and N. Magesh, "Differential sandwich theorems for analytic functions defined by Hadamard product", *Annales Univ. M. Curie-Sklodowska*, vol. 59, Sec. A, pp. 117-127, 2007.
[14] G. Murugusundaramoorthy and N. Magesh, "Differential subordinations and superordinations for analytic functions defined by convolution structure", *Studia Univ. Babes-Bolyai Math.*, vol. 54, no. 20, pp. 83-96, 2009.
[15] N. Magesh, G. Murugusundaramoorthy, T. Rosy and K. Muthunagai, "Subordination and superordination for analytic functions associated with convolution structure", *Int. J. Open Problems Complex Analysis*, vol. 2, no. 2, pp. 67-81, 2010.
[16] R. W. Ibrahim and M. Darus, "On a univalent class involving differential subordination with applications", *J. Math. Statistics*, vol. 7, no. 2, pp. 137-143, 2011.
[17] H. M. Anwar and H. F. Al-Janaby, "Sandwich Subordinations Imposed by New Generalized Koebe-Type Operator on Holomorphic Function Class", *Iraqi Journal of Science*, vol. 64, no. 10, pp. 5228-5240, 2023.
[18] W. G. Atshan and A. A. R. Ali, "On sandwich theorems results for certain univalent functions defined by generalized operators", *Iraqi Journal of Science*, vol. 62, no. 7, pp. 2376- 383, 2021.
[19] S. D. Theyab, W. G. Atshan, H. K. Abdullah, "On Some Sandwich Results of Univalent Functions Related by Differential Operator", *Iraqi Journal of Science*, vol. 63, no. 11, pp. 4928-4936, 2022.
[20] S. S. Miller and P. T. Mocanu, "Differential Subordinations: Theory and Applications", *Series on Monographs and Textbooks in Pure and Applied Mathematics*, Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.