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Using standard Bayesian estimation by simulation technique to estimate parameter of Exponential-Rayleigh models

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Abstract

In this paper, the point estimation method for parameters α and λ of the parameters of the Exponential-Rayleigh distribution have been estimated by using the simulation technique by using two Bayesian estimation methods; The first Bayesian method of estimation consists of standard Bayesian_estimation methods and the second Bayesian estimation method consists lindley approximation Bayes estimation method to estimate all the unknown parameters (α, λ) of Exponential-Rayleigh distribution. Comparisons between these two methods were made by employing the Mean squares error criterion. Applying a simulation technique with different sample sizes, these methods are being compared. The simulation procedure is used to generate some sample sizes and mean squares error measure, and when we compared between the above two methods, we find that lindley approximation estimation method has the lower Mean squares error.

Keywords: Bayesian method, Exponential-Rayleigh distribution, Standard Bayes estimation method, Lindlely_approximation Bayes estimation method, Simulation technique, Mean squares error.

استخدام أساليب تقدير البايزية القياسية وتقدير ليندلي عن طريق تقنية المحاكاة لتقدير معلمة لتوزيع رايلي – الأسي

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الخلاصة

في هذه الورقة، يتم استخدام طريقة تقدير النقطي للبارامترات α, λ بارامترات توزيع رايلي – الأسي التي تم تقديرها باستخدام تقنية محاكاة باستخدام طريقتين بايزيتين للتقدير ؛ تتكون الطريقة البايزية الأولى للتقدير من طرق التقدير البايزية القياسية وطريقة التقدير البايزية الثانية تتكون من طريقة التقريب ليندلي البايزية لتقدير جميع البارامترات غير المعروفة (α، λ) للتوزيع الأسي–رايلي. تمت المقارنات بين هاتين الطريقتين باستخدام معيار خطأ المربعات المتوسطة. بتطبيق تقنية محاكاة بأحجام عينات مختلفة، تتم مقارنة هذه الطرق. يتم استخدام إجراء المحاكاة لتوليد بعض أحجام العينات ومتوسط المربعات مقياس الخطأ، وعندما نقارن بين الطريقتين أعلاه نجد أن طريقة تقدير التقريب ليندلي تحتوي على خطأ المربعات الأقل متوسطة.

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1. Introduction

Bayesian estimation methods are usually based on the idea that the parameters to be estimated can be random variables and $(t_1, t_2, \ldots, t_n)$ be random variables. Bayesian estimation methods are usually based on the idea that the parameters$(\theta_1, \theta_2, \ldots, \theta_n)$ to be estimated can be random variables and $(t_1, t_2, \ldots, t_n)$ be random variables too [1]. In (2013), Tasnim H.K. Al-Baldawi [2] Compared to the Maximum Likelihood and some Bayes Estimators for Maxwell Distribution based on non-informative Priors. In (2019) Bastan and Mirmostafaee [3] used approximating Bayes Estimates by Means of the Tierney Kadane , Importance Sampling and Metropolis-Hastings within Gibbs Methods in the Poisson-Exponential Distribution. In (2018), Ghazal and Hasaballah [4] used Based on Unified Hybrid Censored Data from the Exponentiated Rayleigh Distribution by Bayesian Prediction. In (2021), Sana and Faizan [5] used lindley's approximation and prediction by Bayesian estimation of generalized exponential distribution based on lower record values. In (2023), Ruqaya and Iden and Manal [6] estimate of the fuzzy parameters for Exponential-Rayleigh Distribution by using a nonlinear Ranking function. In (2023), Maryam and Huda [7] used the Bayesian estimation method under three different loss functions of two parameters for the Exponential distribution. The probability density function of the Exponential-Rayleigh distribution is defined as follows [8]:

$$f(t;\alpha,\lambda) = (\alpha + \lambda t)e^{-\left(\alpha t + \frac{\lambda}{2}t^2\right)} \quad , \ t \geq 0 \ ; \ \alpha, \lambda > 0 \tag{1}$$

The cumulative distribution function (CDF):

$$F(t) = 1 - e^{-\left(\alpha t + \frac{\lambda}{2}t^2\right)} \quad , t \geq 0 \ ; \ \alpha, \lambda > 0 \tag{2}$$

And the survival function is:

$$S(t) = e^{-\left(\alpha t + \frac{\lambda}{2}t^2\right)} \quad , t \geq 0 \ ; \ \alpha, \lambda > 0 \tag{3}$$

The hazard function is:

$$h(t) = (\alpha + \lambda t) \quad , \ t \geq 0 \ ; \ \alpha, \ \lambda > 0 \tag{4}$$

The aim of this paper is to show how to derive and estimate the two parameters (scale) in Exponential-Rayleigh distribution by using two Bayesian estimation methods, standard Bayesian estimation methods and lindley approximation method. Therefore, finding and estimating survival function. Finally compare between these two methods to find the best method. The structure of this paper is as follows: In section two; study of Bayesian estimation method. Section three; derive Exponential-Rayleigh distribution by using the standard Bayes estimation methods. Section four; derive Exponential-Rayleigh distribution by using the lindley approximation method. Section five; the simulation technique is discussed. Section six; the main results are discussed. Finally, section seven presents the conclusions.

2. Bayesian Method

In the classical estimation methods assuming that the parameters α or λ of any distribution was be constant and fixed, but it is known to us, then these methods were as classical methods [9]. I am now describing another approach to estimating the parameters which are called Bayesian methods, These methods of estimation are typically predicated on the idea that the parameters to be estimated can be random variables as opposed to fixed values [10]. These Bayesian estimation methods are based on the previous information available on the anonymous parameter plus the information that comes from the sample observations [11]. A loss function is a measure of the amount of loss resulting from a decision to be made depends on while the decision to be made depends on θ [12] .The Bayesian estimation method has received a lot of interest recently for analyzing failure time data, which has mostly been proposed as an alternative to that of the traditional methods [13].The Bayesian estimate method employs both the available data and one's prior knowledge of the parameters. The non-informative prior in Bayesian estimation can be used when one's prior knowledge about the parameter is unavailable [14] .

3. Standard Bayes estimation method

In this section, using standard Bayesian methods to estimate the two unknown parameters of Exponential-Rayleigh distribution [15]. The standard Bayesian estimator method shows that unknown parameters(α,λ) are random variables in some real situations: Additional information on(α,λ) can often is available, suggesting that prior knowledge about the parameter (α,λ) exists [16]. Now applying this method to the Exponential-Rayleigh distribution to estimate the parameter λ, when the parameterα is known, assuming that the prior density function for the parameter as λ is [17]:

$$g_1(\lambda)=\begin{cases}\theta e^{-\theta\lambda} & \lambda>0\\ 0 & 0ther\ wise\end{cases} \tag{5}$$

The likelihood function for equation (1) is:

$$L(\lambda;t_{1,}t_2,\dots,t_n)=\prod_{i=1}^{n}(\alpha+\lambda t_i)e^{-\alpha\sum_{i=1}^{n}t_i-\frac{\lambda}{2}\sum_{i=1}^{n}t_i^{\,2}} \tag{6}$$

The posterior density function for the parameter λ is:

$$f(\lambda|t_{1,}t_2,\dots,t_n)=\frac{L(\lambda;t_{1,}t_2,\dots,t_n)\,g(\lambda)}{\int_0^\infty L(\lambda;t_{1,}t_2,\dots,t_n)\,g(\lambda)\,d\lambda}=\frac{\prod_{i=1}^{n}(\alpha+\lambda t_i)e^{-\frac{\lambda}{2}\sum_{i=1}^{n}t_i^{\,2}}e^{-\theta\lambda}}{\prod_{i=1}^{n}\frac{\alpha}{\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta}+\prod_{i=1}^{n}\frac{t_i}{\left(\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta\right)^2}} \tag{7}$$

Now, with the use of a squared error loss function: $l(t,\lambda)=\left(\hat{\lambda}-\lambda\right)^2$ (8)

Then the risk function is:

$R_i(\lambda)=E[l(t;\lambda)]=\int l(t;\lambda)f(\lambda|t_{1,}t_2,\dots,t_n)\,d\lambda$ $R_i(\lambda)=\int_0^\infty\left(\hat{\lambda}-\lambda\right)^2\frac{\prod_{i=1}^{n}(\alpha+\lambda t_i)e^{-\lambda\left(\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta\right)}}{\prod_{i=1}^{n}\frac{\alpha}{\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta}+\prod_{i=1}^{n}\frac{t_i}{\left(\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta\right)^2}}d\lambda$

$$R_i(\lambda)=\frac{1}{\prod_{i=1}^{n}\frac{\alpha}{\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta}+\prod_{i=1}^{n}\frac{t_i}{\left(\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta\right)^2}}\left[\hat{\lambda}^2\left[\prod_{i=1}^{n}\frac{\alpha}{\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta}+\prod_{i=1}^{n}\frac{t_i}{\left(\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta\right)^2}\right]\right.$$

$$-2\hat{\lambda}\left[\prod_{i=1}^{n}\frac{\alpha}{\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta}+\prod_{i=1}^{n}\frac{2t_i}{\left(\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta\right)^3}\right]+\frac{1}{2}\left[\prod_{i=1}^{n}\frac{\alpha}{\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta}+\right.$$

$$\left.\left.\prod_{i=1}^{n}\frac{t_i}{\left(\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta\right)^3}\right]\right]\quad R_i(\lambda)=\hat{\lambda}^2-2\hat{\lambda}\left[\frac{\prod_{i=1}^{n}\frac{\alpha}{\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta}+\prod_{i=1}^{n}\frac{2t_i}{\left(\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta\right)^3}}{\prod_{i=1}^{n}\frac{\alpha}{\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta}+\prod_{i=1}^{n}\frac{t_i}{\left(\frac{\sum_{i=1}^{n}t_i^{\,2}}{2}+\theta\right)^2}}\right]+\frac{1}{2}$$

(9)

The partial derivative with respect to estimator$\hat{\lambda}$ is:

$$\frac{\partial R_i(\lambda)}{\partial \lambda} = 2\hat{\lambda} - 2\left[\frac{\prod_{i=1}^{n}\frac{\alpha}{\frac{\sum_{i=1}^{n}t_i^2}{2}+\theta}+\prod_{i=1}^{n}\frac{2t_i}{\left(\frac{\sum_{i=1}^{n}t_i^2}{2}+\theta\right)^3}}{\prod_{i=1}^{n}\frac{\alpha}{\frac{\sum_{i=1}^{n}t_i^2}{2}+\theta}+\prod_{i=1}^{n}\frac{t_i}{\left(\frac{\sum_{i=1}^{n}t_i^2}{2}+\theta\right)^2}}\right] \tag{10}$$

The equation (10) for $\hat{\lambda}$ is equal to 0 as follows:

$$\hat{\lambda} = \frac{\prod_{i=1}^{n}\frac{\alpha}{\frac{\sum_{i=1}^{n}t_i^2}{2}+\theta}+\prod_{i=1}^{n}\frac{2t_i}{\left(\frac{\sum_{i=1}^{n}t_i^2}{2}+\theta\right)^3}}{\prod_{i=1}^{n}\frac{\alpha}{\frac{\sum_{i=1}^{n}t_i^2}{2}+\theta}+\prod_{i=1}^{n}\frac{t_i}{\left(\frac{\sum_{i=1}^{n}t_i^2}{2}+\theta\right)^2}} \tag{11}$$

To estimate the parameter α, when the parameter λ is known, assume that the prior density function for the parameter as λ is:

$$g_1(\alpha) = \begin{cases} \theta e^{-\theta\alpha} & \alpha > 0 \\ 0 & 0ther\ wise \end{cases} \tag{12}$$

The posterior density function for the parameter α is:

$$f(\lambda|t_1,t_2,\dots,t_n) = \frac{L(\lambda;t_1,t_2,\dots,t_n)\,g(\alpha)}{\int_0^\infty L(\lambda;t_1,t_2,\dots,t_n)\,g(\alpha)\,d\alpha} = \frac{\prod_{i=1}^{n}(\alpha+\lambda t_i)e^{-\alpha(\sum_{i=1}^{n}t_i+\theta)}}{\prod_{i=1}^{n}\frac{1}{\left(\sum_{i=1}^{n}t_i+\theta\right)^2}+\prod_{i=1}^{n}\frac{\lambda t_i}{\sum_{i=1}^{n}t_i+\theta}} \tag{13}$$

Now, with the use of a squared error loss function: $\mathrm{I}(t,\alpha) = (\hat{\alpha}-\alpha)^2$ (14)

Then the risk function is:

$$R_i(\alpha) = \int_0^\infty (\hat{\alpha}-\alpha)^2 \frac{\prod_{i=1}^{n}(\alpha+\lambda t_i)e^{-\alpha(\sum_{i=1}^{n}t_i+\theta)}}{\prod_{i=1}^{n}\frac{1}{\left(\sum_{i=1}^{n}t_i+\theta\right)^2}+\prod_{i=1}^{n}\frac{\lambda t_i}{\sum_{i=1}^{n}t_i+\theta}}\,d\alpha$$

$$R_i(\alpha) = \frac{1}{\prod_{i=1}^{n}\frac{1}{\left(\sum_{i=1}^{n}t_i+\theta\right)^2}+\prod_{i=1}^{n}\frac{\lambda t_i}{\sum_{i=1}^{n}t_i+\theta}}\left[\hat{\alpha}^2\left[\prod_{i=1}^{n}\frac{1}{\left(\sum_{i=1}^{n}t_i+\theta\right)^2}+\prod_{i=1}^{n}\frac{\lambda t_i}{\sum_{i=1}^{n}t_i+\theta}\right]\right.$$
$$\left. -2\hat{\alpha}\left[\prod_{i=1}^{n}\frac{2}{\left(\sum_{i=1}^{n}t_i+\theta\right)^3}+\prod_{i=1}^{n}\frac{\lambda t_i}{\left(\sum_{i=1}^{n}t_i+\theta\right)^2}\right]+\frac{1}{2}\left[\prod_{i=1}^{n}\frac{1}{\left(\sum_{i=1}^{n}t_i+\theta\right)^2}+\prod_{i=1}^{n}\frac{\lambda t_i}{\left(\sum_{i=1}^{n}t_i+\theta\right)}\right]\right]$$

$$R_i(\alpha) = \hat{\alpha}^2 - 2\hat{\alpha}\left[\frac{\prod_{i=1}^{n}\frac{2}{\left(\sum_{i=1}^{n}t_i+\theta\right)^3}+\prod_{i=1}^{n}\frac{\lambda t_i}{\left(\sum_{i=1}^{n}t_i+\theta\right)^2}}{\prod_{i=1}^{n}\frac{1}{\left(\sum_{i=1}^{n}t_i+\theta\right)^2}+\prod_{i=1}^{n}\frac{\lambda t_i}{\sum_{i=1}^{n}t_i+\theta}}\right]+\frac{1}{2} \tag{15}$$

The partial derivative with respect to estimator$\hat{\alpha}$ is:

$$\frac{\partial R_i(\alpha)}{\partial \alpha} = 2\hat{\alpha} - 2\left[\frac{\prod_{i=1}^{n}\frac{2}{\left(\sum_{i=1}^{n}t_i+\theta\right)^3}+\prod_{i=1}^{n}\frac{\lambda t_i}{\left(\sum_{i=1}^{n}t_i+\theta\right)^2}}{\prod_{i=1}^{n}\frac{1}{\left(\sum_{i=1}^{n}t_i+\theta\right)^2}+\prod_{i=1}^{n}\frac{\lambda x_i}{\sum_{i=1}^{n}t_i+\theta}}\right] \tag{16}$$

The equation (16) for $\hat{\alpha}$ is equal to 0 as follows:

$$\hat{\alpha} = \frac{\prod_{i=1}^{n}\frac{2}{\left(\sum_{i=1}^{n}t_i+\theta\right)^3}+\prod_{i=1}^{n}\frac{\lambda t_i}{\left(\sum_{i=1}^{n}t_i+\theta\right)^2}}{\prod_{i=1}^{n}\frac{1}{\left(\sum_{i=1}^{n}t_i+\theta\right)^2}+\prod_{i=1}^{n}\frac{\lambda t_i}{\sum_{i=1}^{n}t_i+\theta}} \tag{17}$$

4. Lindley approximation estimation method

This method was introduced by researcher lindley in 1980 [18]. Now using lindley approximation method to estimate the unknown two parameters for Exponential-Rayleigh distribution. Lindley approximated the ratio of the integrals which are the following form [19]:

$$\frac{\int w(\theta)e^{L(\theta)}d\theta}{\int v(\theta)e^{L(\theta)}d\theta} \tag{18}$$

Where $\theta = (\theta_1, \theta_2, \dots \theta_N)$are parameters, $w(\theta)$ and $v(\theta)$ are any arbitrary functions for parameters.
$L(\theta)$ is the logarithm of the likelihood function.
Suppose that $v(\theta)$is the prior density function of parameters θ and let $w(\theta) = u(\theta) \cdot v(\theta)$.
From integrals equation (1) getting the posterior expectation which is as follows:
$E[u(\theta)|t] = \frac{\int u(\theta) v(\theta) e^{L(\theta)+G(\theta)} d\theta}{\int v(\theta) e^{L(\theta)+G(\theta)} d\theta} = \frac{\int u(\theta) e^{L(\theta)+G(\theta)} d\theta}{\int e^{L(\theta)+G(\theta)} d\theta}$. Where $G(\theta) = \log_e[v(\theta)]$
The likelihood function of exponential-Rayleigh distribution is:

$$L(\lambda; t_{1,} t_2, \dots, t_n) = \prod_{i=1}^{n} (\alpha + \lambda t_i) e^{-\left(\alpha t_i + \frac{\lambda}{2} t_i^{2}\right)} \tag{19}$$

The natural algorithm of likelihood function is:

$$\ln L(\lambda; t_{1,} t_2, \dots, t_n) = \sum_{i=1}^{n} (\alpha + \lambda t_i) - \alpha \sum_{i=1}^{n} t_i - \frac{\lambda}{2} \sum_{i=1}^{n} t_i^{2} \tag{20}$$

To apply this method, assuming the prior density function $g_1(\alpha)$ and $g_2(\lambda)$ for the parameters α and λ respectively, to get the posterior distribution. Assuming the prior density function for parameters α and λ is:

$$g_1(\alpha) = \begin{cases} \theta e^{-\theta\alpha} & \alpha > 0 \\ 0 & 0ther\ wise \end{cases}$$

$$g_2(\lambda) = \begin{cases} \theta e^{-\theta\lambda} & \lambda > 0 \\ 0 & 0ther\ wise \end{cases}$$

The joint prior density functions for parameters α and λ is:

$$g(\alpha, \lambda) = g_1(\alpha) \cdot g_2(\lambda) = \theta e^{-\theta\alpha} \cdot \theta e^{-\theta\lambda} \tag{21}$$

The natural algorithm of joint prior density functions for parameters α and λ is:

$$\ln g(\alpha, \lambda) = 2 \ln \theta - \theta\alpha - \theta\lambda \tag{22}$$

By using the reverse Bayes rule in the integration of the prior density function with the likelihood function we obtain the function of the posterior distribution of the parameters α and λ as follows [20]:

$$H(\alpha, \lambda; t_{1,} t_2, \dots, t_n) = \prod_{i=1}^{n} (\alpha + \lambda t_i) e^{-\alpha \sum_{i=1}^{n} t_i - \frac{\lambda}{2} \sum_{i=1}^{n} t_i^{2}} \theta e^{-\theta\alpha} \cdot \theta e^{-\theta\lambda} \tag{23}$$

The squared error loss function is given by following:

$$Loss(\hat{\alpha} - \alpha) = (\hat{\alpha} - \alpha)^2$$

$$Loss(\hat{\lambda} - \lambda) = (\hat{\lambda} - \lambda)^2$$

The Bayes estimators for α and λ for exponential-Rayleigh distribution under squared error loss function is the posterior which given as follows:

$$\hat{\phi}_{Bayes} = E[\alpha, \lambda] = \frac{\int\int \phi(\alpha,\lambda) H(\alpha,\lambda; t_{1,} t_2, \dots, t_n) d\alpha\, d\lambda}{\int\int H(\alpha,\lambda; t_{1,} t_2, \dots, t_n) d\alpha\, d\lambda} \tag{24}$$

$$\hat{\phi}_{Bayes} = E[\alpha, \lambda] = \frac{\int\int \phi(\alpha,\lambda) \prod_{i=1}^{n} (\alpha+\lambda t_i) e^{-\alpha \sum_{i=1}^{n} t_i - \frac{\lambda}{2} \sum_{i=1}^{n} t_i^{2}} \theta e^{-\theta\alpha} \cdot \theta e^{-\theta\lambda} d\alpha\, d\lambda}{\int\int \prod_{i=1}^{n} (\alpha+\lambda t_i) e^{-\alpha \sum_{i=1}^{n} t_i - \frac{\lambda}{2} \sum_{i=1}^{n} t_i^{2}} \theta e^{-\theta\alpha} \cdot \theta e^{-\theta\lambda} d\alpha\, d\lambda} \tag{25}$$

The posterior density function is difficult to solve, the ratio of these integrals does not seem to take a theoretical formula for the difficulty of calculating these integrals. Then utilizing the lindley approximation to solve the posterior distribution as follows:

$$\mathrm{I}(x) = E(u; \alpha, \lambda) = \frac{\int_0^\infty \int_0^\infty u(\alpha,\lambda) e^{L(\alpha,\lambda;t_i) + G(\alpha,\lambda)} d\alpha d\lambda}{\int_0^\infty \int_0^\infty e^{L(\alpha,\lambda;t_i) + G(\alpha,\lambda)}\, d\alpha d\lambda} \tag{26}$$

Where: $u(\alpha, \lambda)$ is function for α andλ. $L(\alpha, \lambda; t_i)$ is natural logarithm of likeihood function.
$G(\alpha, \lambda)$is natural logorithm of joint prior density function of α and λ.
Then, we can calculate $\mathrm{I}(x) = E(u; \alpha, \lambda)$ as follows:

$$\mathrm{I}(x) = u(\hat{\alpha}, \hat{\lambda}) + \frac{1}{2}\left[\left(\hat{u}_{\lambda\lambda} + 2\hat{u}_\lambda \hat{P}_\lambda\right)\hat{\sigma}_{\lambda\lambda} + \left(\hat{u}_{\alpha\lambda} + 2\hat{u}_\alpha \hat{P}_\lambda\right)\hat{\sigma}_{\alpha\lambda} + \left(\hat{u}_{\lambda\alpha} + 2\hat{u}_\lambda \hat{P}_\alpha\right)\hat{\sigma}_{\lambda\alpha} + \left(\hat{u}_{\alpha\alpha} + 2\hat{u}_\alpha \hat{P}_\alpha\right)\hat{\sigma}_{\alpha\alpha}\right]$$

$$+ \frac{1}{2}\left[(\hat{u}_\lambda \hat{\sigma}_{\lambda\lambda} + \hat{u}_\alpha \hat{\sigma}_{\lambda\alpha})\left(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha} + \hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda} + \hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}\right)\right.$$

$$+(\hat{u}_\lambda\hat{\sigma}_{\alpha\lambda}+\hat{u}_\alpha\hat{\sigma}_{\alpha\alpha})\big(\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha}+\hat{L}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\big)\big]$$

Then applying the lindley approach as loss function we get:

$$u(\hat{\alpha},\hat{\lambda})=\alpha,\hat{u}_\alpha=\frac{\partial u(\hat{\alpha},\hat{\lambda})}{\partial\alpha}=1,\hat{u}_{\alpha\alpha}=0,\hat{u}_\lambda=0,\hat{u}_{\lambda\lambda}=0,\hat{u}_{\lambda\alpha}=0,\hat{u}_{\alpha\lambda}=0$$

$$\mathrm{I}(x)=\hat{\alpha}+\frac{1}{2}\big[\big(2\hat{P}_\lambda\big)\hat{\sigma}_{\alpha\lambda}+\big(2\hat{P}_\alpha\big)\hat{\sigma}_{\alpha\alpha}\big]+\frac{1}{2}\big[(\hat{\sigma}_{\lambda\alpha})\big(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha}+\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda}+\hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}\big)$$
$$+(\hat{\sigma}_{\alpha\alpha})\big(\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha}+\hat{L}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\big)\big] \qquad (27)$$

Now derivate $P=\ln g(\alpha,\lambda)$ with respect to α and λ in equation (27) we get:

$\hat{\sigma}_{\lambda\lambda}=-\frac{1}{\hat{L}_{\lambda\lambda}}=\frac{1}{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^2}}$, $\hat{\sigma}_{\lambda\lambda}=-\frac{1}{\hat{L}_{\lambda\lambda}}=\frac{1}{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^2}}$, $\hat{\sigma}_{\alpha\lambda}=-\frac{1}{\hat{L}_{\alpha\lambda}}=\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}$, $\hat{\sigma}_{\lambda\alpha}=-\frac{1}{\hat{L}_{\lambda\alpha}}=\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}$, $\hat{P}_\lambda=\frac{\partial\ln g(\alpha,\lambda)}{\partial\lambda}=-\theta$, $\hat{P}_\alpha=\frac{\partial\ln g(\alpha,\lambda)}{\partial\alpha}=-\theta$, $\hat{L}_\alpha=\frac{\partial\ln L}{\partial\alpha}=\sum_{i=1}^{n}\frac{1}{\alpha+\lambda t_i}-\sum_{i=1}^{n}t_i$, $\hat{L}_{\lambda\lambda}=\frac{\partial^2\ln L}{\partial\lambda^2}=-\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^2}$, $\hat{L}_{\alpha\alpha}=\frac{\partial^2\ln L}{\partial\alpha^2}=-\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}$, $\hat{L}_{\lambda\alpha}=\frac{\partial^2\ln L}{\partial\lambda\partial\alpha}=-\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}$

$\hat{L}_{\alpha\lambda}=\frac{\partial^2\ln L}{\partial\alpha\partial\lambda}=-\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}$, $\hat{L}_{\lambda\lambda\alpha}=\frac{\partial^3\ln L}{\partial\lambda^2\partial\alpha}=2\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^3}$, $\hat{L}_{\lambda\lambda\lambda}=\frac{\partial^3\ln L}{\partial\lambda^3}=2\sum_{i=1}^{n}\frac{t_i^3}{(\alpha+\lambda t_i)^3}$ $\hat{L}_{\lambda\alpha\lambda}=\frac{\partial^3\ln L}{\partial\lambda\partial\alpha\partial\lambda}=2\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^3}$, $\hat{L}_{\alpha\alpha\lambda}=\frac{\partial^3\ln L}{\partial\alpha\partial\alpha\partial\lambda}=2\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}$, $\hat{L}_{\alpha\lambda\lambda}=\frac{\partial^3\ln L}{\partial\alpha\partial\lambda\partial\lambda}=2\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^3}$

$\hat{L}_{\lambda\alpha\alpha}=\frac{\partial^3\ln L}{\partial\lambda\partial\alpha\partial\alpha}=2\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}$, $\hat{L}_{\alpha\alpha\alpha}=\frac{\partial^3\ln L}{\partial\alpha^3}=2\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^3}$, $\hat{L}_{\alpha\lambda\alpha}=\frac{\partial^3\ln L}{\partial\alpha\partial\lambda\partial\alpha}=2\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}$

To estimate the parameter α at case of squared error loss function, assuming that λ is information.

$$\hat{\alpha}_{LAE}=\hat{\alpha}+\left[(-\theta)\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^2}}\right)+(-\theta)\left(\frac{1}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\right)\right]+$$
$$\frac{1}{2}\left[\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\left(2\sum_{i=1}^{n}\frac{t_i^3}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^2}}\right)\right.\right.$$
$$+2\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right)+2\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right)+$$
$$\left.2\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\right)\right)$$
$$+\frac{1}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\left(2\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^2}}\right)+2\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right)\right.$$
$$\left.\left.+2\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right)+2\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\right)\right)\right]$$

$$\hat{\alpha}_{LAE} = \hat{\alpha} + \left[- \frac{\theta}{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^2}} - \frac{\theta}{\sum_{i=1}^{n} \frac{1}{(\alpha+\lambda t_i)^2}} \right] + \frac{1}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} \left(\frac{\sum_{i=1}^{n} \frac{{t_i}^3}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^2}} + \frac{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} + \frac{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^2}} + \frac{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n} \frac{1}{(\alpha+\lambda t_i)^2}} \right) + \frac{1}{\sum_{i=1}^{n} \frac{1}{(\alpha+\lambda t_i)^2}} \left(\frac{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^2}} + \frac{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} + \frac{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} + \frac{\sum_{i=1}^{n} \frac{1}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n} \frac{1}{(\alpha+\lambda t_i)^2}} \right)$$

(28)

To estimate the parameter λ at case of squared error loss function, assuming that α is information.

We choose: $(\hat{\alpha}, \hat{\lambda}) = \lambda$, $\hat{u}_\lambda = \frac{\partial u(\hat{\alpha}, \hat{\lambda})}{\partial \lambda} = 1$, $\hat{u}_\alpha = 0, \hat{u}_{\alpha\alpha} = 0, \hat{u}_{\lambda\lambda} = 0, \hat{u}_{\lambda\alpha} = 0, \hat{u}_{\alpha\lambda} = 0$

$$\mathrm{I}(x) = \hat{\lambda} + \frac{1}{2}\left[(2\hat{u}_\lambda \hat{P}_\lambda)\hat{\sigma}_{\lambda\lambda} + (2\hat{u}_\lambda \hat{P}_\alpha)\hat{\sigma}_{\lambda\alpha} \right] + \frac{1}{2}\left[(\hat{\sigma}_{\lambda\lambda})(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha} + \hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda} + \hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}) + (\hat{\sigma}_{\alpha\lambda})(\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha} + \hat{L}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda} + \hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}) \right]$$

$$\hat{\lambda}_{LAE} = \hat{\lambda} + \left[(-\theta)\left(\frac{1}{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^2}} \right) + (-\theta)\left(\frac{1}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} \right) \right] + \frac{1}{2}\left[\frac{1}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} \left(2\sum_{i=1}^{n} \frac{{t_i}^3}{(\alpha+\lambda t_i)^3} \left(\frac{1}{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^2}} \right) + 2\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^3} \left(\frac{1}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} \right) + 2\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^3} \left(\frac{1}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} \right) + 2\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^3} \left(\frac{1}{\sum_{i=1}^{n} \frac{1}{(\alpha+\lambda t_i)^2}} \right) \right) + \frac{1}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} \left(2\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^3} \left(\frac{1}{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^2}} \right) + 2\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^3} \left(\frac{1}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} \right) + 2\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^3} \left(\frac{1}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} \right) + 2\sum_{i=1}^{n} \frac{1}{(\alpha+\lambda t_i)^3} \left(\frac{1}{\sum_{i=1}^{n} \frac{1}{(\alpha+\lambda t_i)^2}} \right) \right) \right]$$

$$\hat{\lambda}_{LAE} = \hat{\lambda} + \left[- \frac{\theta}{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^2}} - \frac{\theta}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} \right] + \frac{1}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} \left(\frac{\sum_{i=1}^{n} \frac{{t_i}^3}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^2}} + \frac{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} + \frac{\sum_{i=1}^{n} \frac{{t_i}^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}} \right.$$

$$+\frac{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}+\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\left(\frac{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^2}}+\frac{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}+\frac{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\frac{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\right) \qquad (29)$$

5. Simulation technique

The simulation procedure is now generally used in many branches of statistics. They can be used to evaluate the behavior of models as well as for some random variables. A simulation is defined as a numerical scientific method that uses logical mathematical methods to describe the behavior of a certain approved system. Different techniques have been conducted.

- We select $n = 10,20,30,50$
- We select $\alpha = 0.25,0.5,0.75$
- We select $\lambda = 0.5,1,1.5$, These values were selected at random.
- N=represent the Replicate of Experiment then N=1000

- $F(t) = 1 - S(t) = 1 - e^{-\left(\alpha t+\frac{\lambda}{2}t^2\right)}$

$u = 1 - e^{-\left(\alpha t+\frac{\lambda}{2}t^2\right)}$

$\ln(1-u) = -\left(\alpha t + \frac{\lambda}{2}t^2\right)$

$\ln(1-u) + \alpha t + \frac{\lambda}{2}t^2 = 0\ ,\quad a = \frac{\lambda}{2}\ ,\ b = \alpha\ ,\quad c = \ln(1-u)$

$t_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} \Rightarrow t = \frac{-\alpha\pm\sqrt{\alpha^2-2\lambda\ln(1-u)}}{\lambda}$

- We find Absolut error value as follows:

$\phi_\alpha = Min|\hat{\alpha}_i - \alpha|\ ,\ \phi_\lambda = Min\left|\hat{\lambda}_i - \lambda\right|$

- We find mean squares errors for parameter(α) and parameter(λ).

$MSE[S(t)] = \sum_{I=1}^{N}\frac{[\hat{S}(t_i)-S(t_i)]^2}{N}\ , MSE[\alpha] = \sum_{I=1}^{N}\frac{[\hat{\alpha}_i-\alpha]^2}{N}, MSE[\lambda] = \sum_{I=1}^{N}\frac{\left[\hat{\lambda}_i-\lambda\right]^2}{N}$

6. Numerical Results

The numerical results for the simulation approach are exhibited in the following tables after the simulation procedure has been used to determine the parameter estimation, reliability function, and mean square error technique. With a 1000 iteration rate, the MATLAB software employed the Monte Carlo methodology for the simulation and the Newton-Raphson method for the numerical solution. As shown in table (1, 2, 3, 4, 5)

Table1: represent the estimate for parameter(α) of all Bayesian method with Absolut error values

α	λ	n	$\hat{\alpha}_{SBA}$	$\hat{\alpha}_{LABE}$	$\emptyset$	Best
0.25	0.5	10	0.423445	0.232955	0.017045	2
0.25	0.5	20	0.25293	0.24949	0.00051	2
0.25	0.5	30	0.24972	0.220153	0.00028	1
0.25	0.5	50	0.249888	0.250018	1.77E-05	2
0.25	1	10	0.383786	0.259541	0.009541	2
0.25	1	20	0.343459	0.248875	0.001125	2
0.25	1	30	0.240483	0.259211	0.009517	1
0.25	1	50	0.23791	0.25004	3.95E-05	2
0.25	1.5	10	0.289665	0.267571	0.017571	2
0.25	1.5	20	0.262254	0.247656	0.002344	2
0.25	1.5	30	0.247634	0.249608	0.000392	2
0.25	1.5	50	0.25869	0.250008	7.72E-06	2
0.5	0.5	10	0.786435	0.533735	0.033735	2
0.5	0.5	20	0.760926	0.50599	0.00599	2
0.5	0.5	30	0.540539	0.49971	0.00029	2
0.5	0.5	50	0.599983	0.499999	8.39E-07	2
0.5	1	10	0.480737	0.484835	0.015165	2
0.5	1	20	0.749434	0.503185	0.003185	2
0.5	1	30	0.496147	0.499308	0.000692	2
0.5	1	50	0.454461	0.500012	1.22E-05	2
0.5	1.5	10	0.621616	0.534858	0.034858	2
0.5	1.5	20	0.585324	0.495619	0.004381	2
0.5	1.5	30	0.501684	0.500308	0.000308	2
0.5	1.5	50	0.622741	0.499961	3.91E-05	2
0.75	0.5	10	0.396121	0.696886	0.053114	2
0.75	0.5	20	0.741182	0.749098	0.000902	2
0.75	0.5	30	0.749422	0.750103	0.000103	2
0.75	0.5	50	1.027174	0.745595	0.004405	2
0.75	1	10	1.095415	0.72881	0.02119	2
0.75	1	20	0.805887	0.750642	0.000642	2
0.75	1	30	0.919627	0.749779	0.000221	2
0.75	1	50	1.029006	0.750033	3.27E-05	2
0.75	1.5	10	0.731602	0.736458	0.013542	2
0.75	1.5	20	0.943859	0.750101	0.000101	2
0.75	1.5	30	0.741523	0.750304	0.000304	2
0.75	1.5	50	0.731243	0.749938	6.23E-05	2

Noting from the table 1 that Absolut error values of parameter (α)for lindley approximation estimate values method are the best for the Standard Bayesian Estimation method.

Table 2: represent the estimated for parameter(λ) of all Bayesian method with Absolut error values

α	λ	n	$\hat{\lambda}_{SBE}$	$\hat{\lambda}_{LABE}$	$\emptyset$	Best
0.25	0.5	10	0.502345	0.509598	0.002345	1
0.25	0.5	20	0.446861	0.499007	0.000993	2
0.25	0.5	30	0.433642	0.500283	0.000283	2
0.25	0.5	50	0.469063	0.500042	4.2E-05	2
0.25	1	10	0.921151	1.026021	0.026021	2
0.25	1	20	1.03179	0.998455	0.001545	2
0.25	1	30	0.923987	0.999999	5.62E-07	2
0.25	1	50	1.116258	1.000025	2.54E-05	2
0.25	1.5	10	0.937313	1.484813	0.015187	2
0.25	1.5	20	1.672856	1.499029	0.000971	2
0.25	1.5	30	1.395744	1.500487	0.000487	2
0.25	1.5	50	1.467978	1.499997	2.5E-06	2
0.5	0.5	10	0.630772	0.48869	0.01131	2
0.5	0.5	20	0.48766	0.493391	0.006609	2
0.5	0.5	30	0.423941	0.49978	0.00022	2
0.5	0.5	50	0.508803	0.499963	3.67E-05	2
0.5	1	10	0.992885	1.016357	0.007115	1
0.5	1	20	1.000974	1.004066	0.000974	1
0.5	1	30	1.066751	1.000015	1.47E-05	2
0.5	1	50	1.11038	0.999995	5.04E-06	2
0.5	1.5	10	1.480908	1.543717	0.019092	1
0.5	1.5	20	1.540375	1.495319	0.004681	2
0.5	1.5	30	1.219139	1.500462	0.000462	2
0.5	1.5	50	1.54325	1.500013	1.34E-05	2
0.75	0.5	10	0.338967	0.52599	0.02599	2
0.75	0.5	20	0.55259	0.50068	0.00068	2
0.75	0.5	30	0.489326	0.499906	9.44E-05	2
0.75	0.5	50	0.499598	0.499992	8.31E-06	2
0.75	1	10	1.45723	1.0467	0.0467	2
0.75	1	20	0.780371	0.995807	0.004193	2
0.75	1	30	1.196467	1.000245	0.000245	2
0.75	1	50	0.99088	0.999999	1.16E-06	2
0.75	1.5	10	1.321738	1.545809	0.045809	2
0.75	1.5	20	1.637508	1.498723	0.001277	2
0.75	1.5	30	1.440195	1.500127	0.000127	2
0.75	1.5	50	1.298431	1.499983	1.71E-05	2

Noting from the table 2 that Absolut error values of parameter $(\lambda)$for lindley approximation estimate values method are the best for the Standard Bayesian Estimation method.

Table 3: represent the MSE for parameter(α) of all Bayesian method

α	λ	n	$MSE_{\hat{\alpha}_{SBE}}$	$MSE_{\hat{\alpha}_{LAB}}$	min	Best
0.25	0.5	10	0.052499	0.061014	0.052499	1
0.25	0.5	20	8.95E-06	4.11E-06	4.11E-06	2
0.25	0.5	30	9.37E-08	2.53E-07	9.37E-08	1
0.25	0.5	50	1.6E-08	7.47E-10	7.47E-10	2
0.25	1	10	0.049141	0.002126	0.002126	2
0.25	1	20	0.017511	8.36E-06	8.36E-06	2
0.25	1	30	0.004913	2.22E-07	2.22E-07	2
0.25	1	50	0.000248	2.49E-09	2.49E-09	2
0.25	1.5	10	0.001836	0.000907	0.000907	2
0.25	1.5	20	0.000696	1.24E-05	1.24E-05	2
0.25	1.5	30	7.45E-06	1.7E-07	1.7E-07	2
0.25	1.5	50	0.000195	8.12E-11	8.12E-11	2
0.5	0.5	10	0.181572	0.002175	0.002175	2
0.5	0.5	20	0.136121	4.75E-05	4.75E-05	2
0.5	0.5	30	2.92E-07	2.09E-07	1.4E-08	2
0.5	0.5	50	0.019992	7.9E-11	7.9E-11	2
0.5	1	10	0.003393	0.000705	0.000705	2
0.5	1	20	0.077531	1.62E-05	1.62E-05	2
0.5	1	30	1.5E-05	5.35E-07	5.35E-07	2
0.5	1	50	0.003881	4.62E-10	4.62E-10	2
0.5	1.5	10	0.023868	0.001242	0.001242	2
0.5	1.5	20	0.015609	3.03E-05	3.03E-05	2
0.5	1.5	30	0.000345	1.59E-07	1.59E-07	2
0.5	1.5	50	0.015199	4.42E-09	4.42E-09	2
0.75	0.5	10	0.254898	0.002838	0.002838	2
0.75	0.5	20	0.000202	2.4E-06	2.4E-06	2
0.75	0.5	30	3.69E-07	1.28E-07	1.28E-07	2
0.75	0.5	50	0.078889	7.23E-11	7.23E-11	2
0.75	1	10	0.133691	0.000452	0.000452	2
0.75	1	20	0.038778	1.31E-05	1.31E-05	2
0.75	1	30	0.061803	7.36E-08	7.36E-08	2
0.75	1	50	0.08462	1.18E-09	1.18E-09	2
0.75	1.5	10	0.194895	0.000532	0.000532	2
0.75	1.5	20	0.041827	1.74E-08	1.74E-08	2
0.75	1.5	30	0.000111	3.63E-07	3.63E-07	2
0.75	1.5	50	0.000386	4.55E-09	4.55E-09	2

Noting from the table 3 that mean square error (α) for lindley approximation estimate values method are the best for the Standard Bayesian Estimation method.

Table 4: represent the MSE for parameter(λ) of all Bayesian method:

α	λ	n	$MSE_{\hat{\lambda}_{SBE}}$	$MSE_{\hat{\lambda}_{LAB}}$	min	Best
0.25	0.5	10	0.031327	0.000858	0.000858	2
0.25	0.5	20	0.012774	1.5E-06	1.5E-06	2
0.25	0.5	30	0.006241	1.69E-07	1.69E-07	2
0.25	0.5	50	0.000981	2.85E-09	2.85E-09	2
0.25	1	10	0.147634	0.002652	0.002652	2
0.25	1	20	0.009541	5.36E-06	5.36E-06	2
0.25	1	30	0.005785	1.7E-07	1.7E-07	2
0.25	1	50	0.025103	1.43E-09	1.43E-09	2
0.25	1.5	10	0.592722	0.000791	0.000791	2
0.25	1.5	20	0.029993	1.1E-06	1.1E-06	2
0.25	1.5	30	0.239804	4.33E-07	4.33E-07	2
0.25	1.5	50	0.010338	3.21E-10	3.21E-10	2
0.5	0.5	10	0.0183	0.005259	0.005259	2
0.5	0.5	20	0.00055	4.92E-05	4.92E-05	2
0.5	0.5	30	0.012369	8.87E-08	8.87E-08	2
0.5	0.5	50	0.00043	1.48E-09	1.48E-09	2
0.5	1	10	0.004071	0.004654	0.002195	1
0.5	1	20	0.031053	3.26E-05	3.26E-05	2
0.5	1	30	0.012976	3.42E-07	3.42E-07	2
0.5	1	50	0.014294	8.28E-11	8.28E-11	2
0.5	1.5	10	0.069081	0.004483	0.004483	2
0.5	1.5	20	0.013157	3.71E-05	3.71E-05	2
0.5	1.5	30	0.116063	4.08E-07	4.08E-07	2
0.5	1.5	50	0.008977	2.63E-10	2.63E-10	2
0.75	0.5	10	0.043875	0.003688	0.003688	2
0.75	0.5	20	0.003222	2.55E-05	2.55E-05	2
0.75	0.5	30	0.000116	8.31E-08	8.31E-08	2
0.75	0.5	50	0.000295	8.9E-11	8.9E-11	2
0.75	1	10	0.262029	0.002613	0.002613	2
0.75	1	20	0.078116	2.49E-05	2.49E-05	2
0.75	1	30	0.038614	6.02E-08	6.02E-08	2
0.75	1	50	0.006711	5.13E-11	5.13E-11	2
0.75	1.5	10	0.038133	0.002863	0.002863	2
0.75	1.5	20	0.141032	2.1E-06	2.1E-06	2
0.75	1.5	30	0.010576	1.06E-07	1.06E-07	2
0.75	1.5	50	0.072074	4.03E-10	4.03E-10	2

Noting from the table 4 that mean square error of parameter $(\lambda)$for lindley approximation estimate values method are the best for the Standard Bayesian Estimation method.

Table 5: represent the MSE for survival function methods of all Bayesian method

α	λ	n	$S_{MSE_{SBE}}$	$S_{MSE_{LAB}}$	min	index
0.25	0.5	10	0.003719622	1.79E-05	1.78826E-05	2
0.25	0.5	20	4.48456E-05	6.69E-08	6.68907E-08	2
0.25	0.5	30	0.000441299	0.000186	0.00018582	2
0.25	0.5	50	5.93381E-05	2.42E-10	2.42355E-10	2
0.25	1	10	0.001083252	3.46E-05	3.45513E-05	2
0.25	1	20	0.00118467	3.87E-07	3.87255E-07	2
0.25	1	30	0.000175551	6.94E-06	6.9434E-06	2
0.25	1	50	0.000249509	2.22E-10	2.21642E-10	2
0.25	1.5	10	0.004894101	8.16E-06	8.15897E-06	2
0.25	1.5	20	0.000370517	3.59E-07	3.59237E-07	2
0.25	1.5	30	9.09589E-05	1.46E-09	1.46397E-09	2
0.25	1.5	50	2.47795E-06	2.01E-12	2.00982E-12	2
0.5	0.5	10	0.005271193	5.21E-05	5.21315E-05	2
0.5	0.5	20	0.003658881	9.98E-07	9.97916E-07	2
0.5	0.5	30	3.67164E-05	1.57E-08	1.57214E-08	2
0.5	0.5	50	0.000715931	2E-11	2.00342E-11	2
0.5	1	10	4.62543E-05	4.54E-06	4.54484E-06	2
0.5	1	20	0.003464237	1.58E-06	1.57777E-06	2
0.5	1	30	3.90295E-05	2.07E-08	2.06807E-08	2
0.5	1	50	1.10884E-05	3.8E-12	3.79964E-12	2
0.5	1.5	10	0.000498358	0.000107	0.000107102	2
0.5	1.5	20	0.000381746	1.55E-06	1.55265E-06	2
0.5	1.5	30	0.000624501	8.5E-09	8.49529E-09	2
0.5	1.5	50	0.000875097	5.85E-11	5.84854E-11	2
0.75	0.5	10	0.011401826	5.55E-05	5.55124E-05	2
0.75	0.5	20	7.658E-06	1.69E-08	1.69141E-08	2
0.75	0.5	30	1.89722E-06	2.77E-10	2.77344E-10	2
0.75	0.5	50	0.003646197	1.12E-06	1.12329E-06	2
0.75	1	10	0.006762336	1.16E-06	1.16403E-06	2
0.75	1	20	4.38976E-05	2.29E-08	2.29383E-08	2
0.75	1	30	0.001050262	3.09E-10	3.09264E-10	2
0.75	1	50	0.002992406	4.93E-11	4.92781E-11	2
0.75	1.5	10	0.000207977	1.98E-06	1.98306E-06	2
0.75	1.5	20	0.001114784	3.31E-09	3.31358E-09	2
0.75	1.5	30	1.63814E-05	2.99E-09	2.98614E-09	2
0.75	1.5	50	0.000217347	9.37E-11	9.37347E-11	2

Noting from the table 4 that mean square error of survival function for lindley approximation estimate values method are the best for the Standard Bayesian Estimation method.

7. Conclusion

In this paper, we have considered the Bayesian estimation of the unknown parameters of the two-parameter Exponential-Rayleigh distribution. The simulation procedure using Monte-Carlo technique and through previous tables show us that the estimated values and two distribution parameters are influenced by both (sample size, real value for estimator and estimation method), We find that the lindley approximation estimation method is the best for the parameters (α and λ). Bayes estimators were obtained using lindley approximation and Standard Bayesian Estimation method while MLE were obtained using Newton-Raphson method, simulation study was conducted to examine and compare the lindley approximation estimation method with Standard Bayesian Estimation method for different sample sizes with different values showing that the lindley approximation estimation method is the best when we use mean squares error for the parameter (α and λ). Noting that, the lindley approximation estimation method is the best when we use mean square error for survival function.

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