DOI: 10.24996/ijs.2025.66.11.25

ISSN: 0067-2904

Applications of the $PG(5,q)$ in coding theory

Yahya ammar fawzy yahya[1]*, Nada yassen kasm yahya[2]

[1]Department of mathematics, College Computer sciences and mathematics, University of Mosul, City Nineveh, Country Iraq

[2]Department of mathematics, College of education for pure science, University of mosul, City Nineveh, Country Iraq

Received: 5/2/2024 Accepted: 29/9/2024 Published: 30/11/2025

Abstract

This paper's primary goal is to present the connection between coding theory and the projective space PG(5,q) of order q, where q={2,3,4,5,7,8,9,11} It was determined whether or not the $[n,k,d]$-code is perfect. Where n is the length of the code, k is the code dimension, and d is the minimum distance, were calculated with error correction e according to the incidence matrix. We have found that the $[n,k,d]$-code in PG(5,q), q= {2,3,4,5,7,8,9,11}, is perfect if e = 1.

Keywords: Finite projective space, incidence matrix, linear code,PG(5,q), perfect .

تطبيقات الفضاء الاسقاطي $PG(5,q)$في نظرية الترميز

يحيى عمار فوزي يحيى[1]*, ندى ياسين قاسم يحيى[2]

[1]قسم الرياضيات, كلية علوم الحاسوب و الرياضيات, جامعة الموصل, نينوى, العراق

[2]قسم الرياضيات, كلية التربية للعلوم الصرفة, جامعة الموصل, نينوى, العراق

الخلاصة

الهدف الأساسي لهذا البحث هو ان في الفضاء الاسقاطي عند ربطه في نظرية الترميز فان PG(5,q) على مجال جالوا من الرتبة q={2,3,4,5,7,8,9,11} لمعرفة اذا كان الرمز $q$code [n,k,d] مثالي ام لا حيث تم حساب معلمات طول الكود n وأبعاد الكود k والحد الأدنى للمسافة d مع تصحيح الخطأ e وفقًا لمصفوفة الوقوع , وجدنا ان $[n,k,d]$−code في PG(5,q), q= {2,3,4,5,7,8,9,11}, مثالية اذا كانت e=1

.

1. Introduction

In coding theory, many mathematicians have studied the application of a projective space over a Galois field and demonstrated more than a few theorems and definitions about the relations between finite projective geometry and coding theory. In 2018, Marcus and Natalia Silberstein [1] have been considered Coding and Subspace Designs. J.W.P. Hirschfeld [2, 3] showed theories and definitions of the relationships between finite projective geometry and coding theory. Sloane [4] then followed him by republishing a book that had been published in 1979 by the same scientist in 1981.Hill [8] classified the concepts and tools of coding theory. Al-Zangana [9, 10, 11, 12] described the link between the projective plane of order 19

*Email: yahya.23csp35@student.uomosul.edu.iq

and error- correcting codes . In 2023 Al-Zangan [13] described the geometry of the plane of order 27 and its application to Special NMDS Codes . We look at a five-dimensional projective space on a finite field of rank q, where q=2,3,4,5,7,8,9,11 and apply it to coding theory to find a general rule that shows whether the code is perfect or not without application. Where we found the points, lines, planes, and subspaces, then found the incidence matrix, and from each subspace, plane, or line in the space with the points located on it, then found the shortest distance, and thus we have found the symbol with which we will verify the condition of the perfect code.

Theorem 1.1: [14] Sphere packing or Hamming bound
A q-ary(n,M,2e+1) – code C satisfies
$$M\left\{\binom{n}{0}+\binom{n}{1}(q-1)+\cdots+\binom{n}{e}(q-1)^e\right\}\leq q^n$$

Definition 1.1 : (incidence matrix) The incidence matrix of a projective space gives the (0,1)-matrix which has a row for each line and column for each point, and (p,L)=1 Iff the point p lies on the line L . [15]

Corollary1.2 [15] A q-ary (n.M,2e+1) code C is perfect if and only if equality holds in Theorem 1.1

Definition 1.2 [16] A q-ary code C of length n is a subset of (F_q)

Definition 1.3 (Linear code) If Σ is a field and C ⊂ Σ^n is a subspace of Σ^n then C is said to be a linear code [17]
The minimum distance d of a non-trivial code C is given by
$d = min\{ d(x,y) | x \in C , y \in C , x \neq y\}$

1- The classification of PG(5, 3)

The polynomial $g(x) = x^6 + \varrho x^5 + x^4 + \varrho x^3 + \varrho x + \varrho$ is primitive over $F_3 = [0,1,\varrho]$, where ϱ =2 is a primitive element of $F3$, since ,$g(0) = \varrho$,$g(1) = 1$ and $g(\varrho) = 1$ The points and lines are generated as following:

$$P_i = [1,0,0,0]\begin{pmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&\varrho&1&0&1&1\end{pmatrix}^i, i = 1,2,\ldots\ldots,364, \qquad / \qquad l_i =$$

$$l_1\begin{pmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&\varrho&1&0&1&1\end{pmatrix}^{i^i}, i = 1,2,\ldots\ldots,364$$

$$PLAN_i = plane_1 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & \varrho & 1 & 0 & 1 & 1 \end{pmatrix}^{i^i}, i = 1,2, \dots \dots ,364 \qquad / \qquad SUBSPACE_i =$$

$$space_1 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & \varrho & 1 & 0 & 1 & 1 \end{pmatrix}^{i^i}, i = 1,2, \dots \dots ,364$$

$$pg(4,3)_i = pg(4,3)_1 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & \varrho & 1 & 0 & 1 & 1 \end{pmatrix}^{i^i}, i = 1,2, \dots \dots ,364$$

The points of PG(5, 3) are:

p1:(1,0,0,0,0,0)	p74:(0,ϱ,1,1,1,1)	p147:(1,0,1,1,0,0)	p220:(0,1,1,1,1,0)	p293:(1,1,0,1,1,0)
p2:(0,1,0,0,0,0)	p75:(ϱ,1,0,ϱ,1,1)	p148:(0,1,0,1,1,0)	p221:(0,0,1,1,1,1)	p294:(0,1,1,0,1,1)
p72:(0,1,ϱ,1,ϱ,1)	p145:(1,0,ϱ,1,0,1)	p218:(ϱ,0,1,ϱ,ϱ,1)	p291:(ϱ,1,0,0,1,1)	p364:(1,ϱ,0,ϱ,ϱ,1)
p73:(ϱ,1,1,1,1,0)	p146:(1,0,1,ϱ,ϱ,1)	p219:(1,1,1,1,0,0)	p292:(ϱ,ϱ,1,0,ϱ,1)	

The lines of (5,3) are:

L1	1	2	165	187	**L123**	123	124	287	309	**L245**	245	246	45	67
L2	2	3	166	188	**L124**	124	125	288	310	**L246**	246	247	46	68
L3	3	4	167	189	**L125**	125	126	289	311	**L247**	247	248	47	69
L120	120	121	284	306	**L242**	242	243	42	64	**L364**	364	1	164	186
L121	121	122	285	307	**L243**	243	244	43	65	**L245**	245	246	45	67
L122	122	123	286	308	**L244**	244	245	44	66	**L246**	246	247	46	68

The plane of PG(5, 3) are:

plane1	1	2	3	9	55	165	166	187	188	268	325	329	351
Plane2	2	3	4	10	56	166	167	188	189	269	326	330	352
plane3	3	4	5	11	57	167	168	189	190	270	327	331	353
plane362	362	363	364	6	52	162	163	184	185	265	322	326	348
plane363	363	364	1	7	53	163	164	185	186	266	323	327	349
plane364	364	1	2	8	54	164	165	186	187	267	324	328	350

The subspace of PG(5, 3) are:

space1

1 ,2 ,3 ,4 ,9 ,10 ,21 ,55 ,56 ,68 ,80 ,90 ,125 ,129 ,147 ,151 ,156 ,165 ,166 ,167 ,173 ,176 ,187 ,188 ,189 ,195 ,219 ,238 ,241 ,251 ,268 ,269 ,296 ,301 ,325 ,326 ,329 ,330 ,351 ,352

The PG(4, 3) of PG(5, 3) are:

$PG(4,3)1$

1 ,2 ,3 ,4 ,5 ,9 ,10 ,11 ,15 ,17 ,19 ,21 ,22 ,30 ,36 ,38 ,41 ,45 ,48 ,51 ,55 ,56 ,57 ,60 ,63 ,68 ,69 ,71 ,73 ,80 ,81 ,86 ,90 ,91 ,94 ,96 ,101 ,109 ,118 ,123 ,125 ,126 ,129 ,130 ,147 ,148 ,151 ,152 ,156 ,157 ,160 ,165 ,166 ,167 ,168 ,171 ,173 ,174 ,176 ,177 ,185 ,187 ,188 ,189 ,190 ,192

,195 ,196 ,207 ,213 ,219 ,220 ,225 ,228 ,232 ,238 ,239 ,241 ,242 ,244 ,251 ,252 ,254 ,262 ,266 ,268 ,269 ,270 ,276 ,278 ,283 ,285 ,289 ,293 ,296 ,297 ,299 ,301 ,302 ,306 ,311 ,313 ,315 ,317 ,320 ,322 ,325 ,326 ,327 ,329 ,330 ,331 ,333 ,337 ,340 ,342 ,351 ,352 ,353 ,359 ,362

If we use the line
in the following theorem, the parameters n, m, d are constructed.

Theorem 1,2 : The projective space of order 7 is a code with a parameter [n=364,M=3^{358},d=4]

Proof: the plane π_7 has an incidence matrix $A = (a_{ij})$ where $a_{ij} = \begin{cases} 1 & if\ p_i \in plane_i \\ 0 & if\ p_i \notin plane_i \end{cases}$ [18]

Then we have the insider Table(3,1).

L1

1 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0

$m_i = [\, 0,0,0,0,0,0, \ldots ,0,0,0,0,0]$
$v_i = [\, 1,1,1,1,1,1,1, \ldots ,1,1,1,1,1]$
$h_i = [\, \varrho, \varrho, \varrho, \varrho, \varrho, \varrho, \varrho, \ldots , \varrho, \varrho, \varrho, \varrho, \varrho]$

Combine vi with a binary system projective matrix we get a zero-one
Matrix $v_i = [\, 1,1,1,1,1,1,1,1, \ldots ,1,1,1,1,1]$
Let $a_i = v_i + L_i \quad 1 \leq i \leq 364$
Now the table of a_i

A1

ϱ ,ϱ ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1
,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1
,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1
,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1
,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,ϱ ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1
,1 ,ϱ ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1
,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1
,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1
,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1
,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1 ,1

Combine h_i with a binary system projective matrix we get a zero-one
Matrix $h_i = [\, \varrho, \varrho, \varrho, \varrho, \varrho, \varrho, \varrho, \ldots , \varrho, \varrho, \varrho, \varrho, 2]$
Let $b_i = h_i + L_i \quad 1 \leq i \leq 364$
Now the table of b_i

B1

0 ,0 ,ϱ
,ϱ ,ϱ
,ϱ ,ϱ
,ϱ ,ϱ
,ϱ ,ϱ ,ϱ ,ϱ ,ϱ ,ϱ ,ϱ ,ϱ ,ϱ ,ϱ ,ϱ ,ϱ ,ϱ ,ϱ ,ϱ ,ϱ ,0 ,ϱ
,ϱ ,0 ,ϱ

,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q
,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q
,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q
,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q ,Q

Find the shortest distance between two different code words in the matrices above [19], so that the shortest distance is 4 and the largest distance is 364 . $m_i , v_i , h_i , l_i , a_i , b_i$

d (m_i , L_i) = 4	d (a_i , L_i) = 364	d (v_i , m_i) = 364	d (b_i , m_i) = 364
d (v_i , L_i) = 360	d (b_i , L_i) = 364	d (h_i , m_i) = 364	d (a_i , v_i) = 360
d (h_i , L_i) = 364	d (h_i , v_i) = 364	d (a_i , m_i) = 364	d (b_i , v_i) = 364
d (a_i , h_i) = 364	d (b_i , h_i) = 4	d (b_i , a_i) = 364	

If we substitute the values of n = 364 , d = 4 , e = 1 in inequality of Theorem 3.1 we get $M = 3^{358}$. Hence C is a $(364, 3^{358}, 4) - cade.$

$M \left\{ \binom{n}{0} + \binom{n}{1}(q-1) + \cdots + \binom{n}{e} (q-1)^e \right\} \leq q^n$ [20]

As $n = q^3 + q^2 + q + 1$ And $M = q^{q^3+q^2+q+1-k}$ Then $M \left\{ \binom{n}{0} + \binom{n}{1}(q-1) + \cdots + \binom{n}{e} (q-1)^e \right\} = q^n$

Become $q^{q^3+q^2+q+1-k} \left\{ \binom{n}{0} + \binom{n}{1}(q-1) + \cdots + \binom{n}{e} (q-1)^e \right\} \leq q^{q^3+q^2+q+1}$

$q^{q^3+q^2+q+1} q^{-k} \left\{ \binom{n}{0} + \binom{n}{1}(q-1) + \cdots + \binom{n}{e} (q-1)^e \right\} \leq q^{q^3+q^2+q+1}$

$\left\{ \binom{n}{0} + \binom{n}{1}(q-1) + \cdots + \binom{n}{e} (q-1)^e \right\} \leq q^k$

$\left\{ \binom{364}{0} + \binom{364}{1}(3-1) \right\} = 3^6 \rightarrow 729 = 3^6$ but that is true

Then is perfect

Note: This result is if we use straight lines.

But if we use the plane in the following theorem the parameters n, m, d are constructed.

plane1

1 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,1 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
...

Find the shortest distance between two different code words in the matrices above, so that the shortest distance is 4 and the largest distance is 364 . $m_i , v_i , h_i , l_i , a_i , b_i$

d (m_i , L_i) = 13	d (a_i , L_i) = 364	d (v_i , m_i) = 364	d (b_i , m_i) = 364
d (v_i , L_i) = 351	d (b_i , L_i) = 364	d (h_i , m_i) = 364	d (a_i , v_i) = 351
d (h_i , L_i) = 364	d (h_i , v_i) = 364	d (a_i , m_i) = 364	d (b_i , v_i) = 364
d (a_i , h_i) = 364	d (b_i , h_i) = 13	d (b_i , a_i) = 364	

If we substitute the values of n = 364 , d = 13 , e = 6 in inequality of Theorem 3.1 we get $M = 3^{358}$. Hence C is a $(364, 3^{358}, 13) - cade.$

$\left\{ \binom{n}{0} + \binom{n}{1} (q-1) + \cdots + \binom{n}{e} (q-1)^e \right\} \leq q^k$

$\left\{ \binom{364}{0} + \binom{364}{1}(3-1) + \binom{364}{2} (3-1)^2 + \binom{364}{3} (3-1)^3 \right\} \geq q^k$ then is not perfect

Once again we will repeat the process and use subspaces.
In the following theorem, the parameters n, m, d are constructed.

subspace1

1 ,1 ,1 ,1 ,0 ,0 ,0 ,0 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0
,0 ,1 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,1 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0
,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,1 ,0 ,0 ,1 ,1 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0

Using the same steps that we used with the straight lines and planes, we will get the following table for the shortest distances.
Find the shortest distance between two different code words in the matrices above, so that the shortest distance is 40 and the largest distance is 364 . $m_i , v_i , h_i , l_i , a_i , b_i$

d (m_i , L_i) = 40	d (a_i , L_i) = 364	d (v_i , m_i) = 364	d (b_i , m_i) = 364
d (v_i , L_i) = 324	d (b_i , L_i) = 364	d (h_i , m_i) = 364	d (a_i , v_i) = 324
d (h_i , L_i) = 364	d (h_i , v_i) = 364	d (a_i , m_i) = 364	d (b_i , v_i) = 364
d (a_i , h_i) = 364	d (b_i , h_i) = 40	d (b_i , a_i) = 364	

If we substitute the values of n = 364 , d = 40 , e = 18 in inequality of Theorem 3.1 we get $M = 3^{358}$ Hence C is a $(364, 3^{358}, 40) - cade.$

$\left\{ \binom{n}{0} + \binom{n}{1}(q-1) + \cdots + \binom{n}{e}(q-1)^e \right\} \leq q^k$

$\left\{ \binom{364}{0} + \binom{364}{1}(3-1) + \binom{122}{2}(3-1)^2 + \binom{364}{3}(3-1)^3 + \binom{364}{4}(3-1)^4 + \binom{364}{5}(3-1)^5 + \binom{364}{6}(3-1)^6 + \cdots + \binom{364}{15}(3-1)^{15} + \binom{364}{16}(3-1)^{16} + \binom{364}{17}(3-1)^{17} + \binom{364}{18}(3-1)^{18} \right\} \geq q^k$ then is not perfect

Once again we will repeat the process and use $PG(4,3)$ in $PG(5,3)$
in the following theorem the parameters n, m, d are constructed.

***PG*(4, 3)1**

1 ,1 ,1 ,1 ,1 ,0 ,0 ,0 ,1 ,1 ,1 ,0 ,0 ,0 ,1 ,0 ,1 ,0 ,1 ,0 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,1
,0 ,0 ,1 ,0 ,0 ,0 ,1 ,0 ,0 ,1 ,0 ,0 ,1 ,0 ,0 ,0 ,1 ,1 ,1 ,0 ,0 ,1 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,1 ,1 ,0 ,1 ,0 ,1 ,0 ,0
,0 ,0 ,0 ,0 ,1 ,1 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,1 ,1 ,0 ,0 ,1 ,0 ,1 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,1 ,0 ,1 ,1 ,0 ,0 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,1 ,0
,0 ,1 ,1 ,0 ,0 ,0 ,1 ,1 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,1 ,1 ,1 ,1 ,0 ,0 ,1 ,0 ,1 ,1 ,0 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0
,1 ,1 ,1 ,1 ,0 ,1 ,0 ,0 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,1 ,1 ,0 ,0 ,0
,0 ,1 ,0 ,0 ,1 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,1 ,1 ,0 ,1 ,1 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,1 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0
,0 ,0 ,0 ,0 ,0 ,1 ,0 ,1 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,1 ,0 ,0 ,0 ,0 ,1 ,0 ,1 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,1 ,0 ,0 ,1 ,1
,0 ,1 ,0 ,1 ,1 ,0 ,0 ,0 ,1 ,0 ,0 ,0 ,0 ,1 ,0 ,1 ,0 ,1 ,0 ,1 ,0 ,0 ,1 ,0 ,1 ,0 ,0 ,1 ,1 ,1 ,0 ,1 ,1 ,1 ,0 ,1 ,0
,0 ,0 ,0 ,0 ,0 ,1 ,0 ,1 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1 ,1 ,1 ,0 ,0 ,0 ,0 ,0 ,1 ,0 ,0 ,1 ,0 ,0

Using the same steps that we used with the straight lines and planes, we will get the following table for the shortest distances.

Find the shortest distance between two different code words in the matrices above, so that the shortest distance is 40 and the largest distance is 364 . $m_i , v_i , h_i , l_i , a_i , b_i$

d (m_i , L_i) = 121	d (a_i , L_i) = 364	d (v_i , m_i) = 364	d (b_i , m_i) = 364
d (v_i , L_i) = 243	d (b_i , L_i) = 364	d (h_i , m_i) = 364	d (a_i , v_i) = 243
d (h_i , L_i) = 364	d (h_i , v_i) = 364	d (a_i , m_i) = 364	d (b_i , v_i) = 364
d (a_i , h_i) = 364	d (b_i , h_i) = 121	d (b_i , a_i) = 364	

If we substitute the values of n = 364 , d = 121 , e = 60 in inequality of Theorem 3.1 we get $M = 3^{358}$ Hence C is a $(364, 3^{358}, 121) - cade.$

$\{ \binom{n}{0} + \binom{n}{1}(q-1) + \cdots + \binom{n}{e} (q-1)^e \} \leq q^k$

$\{ \binom{364}{0} + \binom{364}{1}(3-1) + \binom{122}{2} (3-1)^2 + \binom{364}{3} (3-1)^3 + \binom{364}{4} (3-1)^4 +$
$\binom{364}{5} (3-1)^5 + \binom{364}{6} (3-1)^6 + \cdots + \binom{364}{59} (3-1)^{59} + \binom{364}{60} (3-1)^{60} \} \geq q^k$ then is not perfect

2- We use the same steps on $PG(5,2)$ To know in which case the perfect code condition can be met :

The polynomial $g(x) = x^6 + x^5 + x^4 + x + 1$ is primitive over, , since ,$g(0) = 1$ and $g(1) = 1$ The points and lines are generated as following:

$$P_i = [1,0,0,0] \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}^i , i = 1,2, \ldots \ldots ,63,$$

Table (9-1)The points of $PG(5,2)$ are:

p1:(1,0,0,0,0,0)	p14:(0,0,1,0,0,1)	p27:(1,0,1,1,0,1)	p40:(1,1,0,0,0,0)	p53:(1,1,1,1,1,0)
p5:(0,0,0,0,1,0)	p18:(0,0,1,0,1,0)	p31:(1,0,0,0,1,0)	p44:(0,0,0,0,1,1)	p57:(0,0,1,1,1,1)
p11:(1,0,1,1,1,1)	p24:(1,1,1,0,1,0)	p37:(0,1,1,1,0,0)	p50:(0,0,1,0,1,1)	p63:(1,0,0,1,1,1)
p12:(1,0,0,1,0,0)	p25:(0,1,1,1,0,1)	p38:(0,0,1,1,1,0)	p51:(1,1,0,1,1,0)	
p13:(0,1,0,0,1,0)	p26:(1,1,1,1,0,1)	p39:(0,0,0,1,1,1)	p52:(0,1,1,0,1,1)	

L1 = (1, 2, 40)
plane 1 = (1 ,2 ,3 ,16 ,36 ,40 ,41)
subspace 1 = (1 ,2 ,3 ,4 ,12 ,16 ,17 ,36 ,37 ,40 ,41 ,42 ,48 ,55 ,58)
$PG(4,2)1$ = (1 ,2 ,3 ,4 ,5 ,8 ,12 ,13 ,16 ,17 ,18 ,22 ,24 ,31 ,34 ,36 ,37 ,38 ,40 ,41 ,42 ,43 ,45 ,48 ,49 ,51 ,53 ,55 ,56 ,58 ,59)

3- We use the same steps on $PG(5,4)$ to know in which case the perfect code condition can be met :

The polynomial $g(x) = x^6 + tx^5 + t^2x^4 + x^3 + tx + t$ is primitive over $F_4 = [0,1,t,t^2]$, and ,$g(0) = t$,$g(1) = 1$, $g(t) = 1$, $g(t^2) = t$

$$P_i = [1,0,0,0,0] \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ t & t^2 & 1 & 0 & t & t \end{pmatrix}^i , i = 1,2, \ldots \ldots ,1365,$$

Table (10-1) The points of $PG(5,4)$ are:

p1:(1,0,0,0,0,0)	p274:(T2,T2,1,T,1,0)	p547:(T,T2,1,T,1,0)	p820:(1,T2,1,T,1,0)	p1093:(0,T2,1,T,1,0)
p2:(0,1,0,0,0,0)	p275:(0,T2,T2,1,T,1)	p548:(0,T,T2,1,T,1)	p821:(0,1,T2,1,T,1)	p1094:(0,0,T2,1,T,1)
p268:(1,T2,0,T2,0,1)	p541:(0,0,1,1,T,1)	p814:(T2,T,T2,1,1,0)	p1087:(T2,T,T,0,T2,1)	p1360:(T,1,T2,T,1,1)
p272:(0,T2,T,1,0,1)	p545:(0,T,T2,T,1,0)	p818:(0,T,0,0,T2,1)	p1091:(0,0,T2,T,T,1)	p1364:(0,1,1,T2,1,1)
p273:(1,T,1,1,T,1)	p546:(0,0,T,T2,T,1)	p819:(T,T2,T2,0,T,1)	p1092:(T2,1,T,1,0,0)	p1365:(T2,1,0,T,T,1)

L1 = (1 ,2 ,145 ,388 ,801)
plane 1 = (1 ,2 ,3 ,145 ,146 ,236 ,289 ,307 ,321 ,388 ,389 ,395 ,508 ,532 ,775 ,801 ,802 ,945 ,1184 ,1188 ,1319)
subspace 1 = (1 ,2 ,3 ,4 ,81 ,98 ,145 ,146 ,147 ,167 ,175 ,206 ,210 ,236 ,237 ,279 ,289 ,290 ,307 ,308 ,321 ,322 ,341 ,344 ,380 ,388 ,389 ,390 ,395 ,396 ,… ,1121 ,1151 ,1162 ,1172 ,1184 ,1185 ,1188 ,1189 ,1195 ,1308 ,1319 ,1320 ,1328 ,1332)
$Pg(4,4)1$ = (1 ,2 ,3 ,4 ,5 ,13 ,16 ,26 ,49 ,54 ,56 ,58 ,71 ,77 ,81 ,82 ,87 ,89 ,98 ,99 ,107 ,109 ,111 ,114 ,125 ,129 ,134 ,139 ,143 ,145 ,146 ,147 ,148 ,151 ,167 ,168 ,… ,1328 ,1329 ,1332 ,1333 ,1337 ,1339 ,1344 ,1352 ,1363)

4- We use the same steps on $PG(5,5)$ to know in which case the perfect code condition can be met :

The polynomial $g(x) = x^6 + \varrho x^5 + \varrho^3 x^4 + \varrho^2 x^3 + x + \varrho$ is primitive over $F_5 = [0,1,\varrho,\varrho^2,\varrho^3]$ where ϱ =2 is a primitive element of $F5$, since,$g(0) = \varrho$,$g(1) = 3$, $g(\varrho) = \varrho$, $g(\varrho^3) = 1$ and $g(\varrho^2) = \varrho^2$

$$P_i = [1,0,0,0,0] \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 & 4 & 3 \end{pmatrix}^i , i = 1,2, \dots \dots ,3906,$$

Table (11-1) The points of (5,5) are:

p1:(1,0,0,0,0,0)	p783:(ϱ2,ϱ,ϱ,ϱ3,1,1)	p1565:(ϱ2,ϱ,0,ϱ,ϱ,1)	p3129:(ϱ3,1,ϱ3,1,1,1)
p7:(1,ϱ2,ϱ,0,ϱ3,1)	p789:(ϱ3,ϱ3,0,ϱ3,ϱ2,1)	p1571:(ϱ3,0,ϱ,0,ϱ2,1)	p3135:(1,ϱ2,ϱ,ϱ3,1,0)
p8:(ϱ3,ϱ3,0,ϱ,ϱ2,1)	p790:(ϱ2,0,ϱ,0,1,1)	p1572:(ϱ2,0,ϱ3,1,ϱ,1)	p3136:(0,1,ϱ2,ϱ,ϱ3,1)
p777:(1,0,1,ϱ2,ϱ,1)	p1559:(1,0,ϱ3,1,1,0)	p2341:(1,1,1,1,ϱ,1)	p3905:(ϱ,ϱ2,1,ϱ3,ϱ3,1)
p782:(ϱ,ϱ3,1,ϱ3,ϱ2,1)	p1564:(ϱ,ϱ2,ϱ2,0,ϱ2,1)	p2346:(0,ϱ2,1,ϱ3,ϱ2,1)	p3132:(ϱ2,ϱ2,ϱ,1,0,0)

L1 = (1 ,2 ,1390 ,3542 ,3632 ,3888)
plane 1 = (1 ,2 ,3 ,56 ,134 ,905 ,1025 ,1115 ,1371 ,1390 ,1391 ,1938 ,1985 ,2126 ,2458 ,2617 ,2779 ,3177 ,3181 ,3267 ,3357 ,3523 ,3542 ,3543 ,3613 ,3626 ,3632 ,3633 ,3869 ,3888 ,3889)
subspace 1 = (1 ,2 ,3 ,4 ,13 ,37 ,56 ,57 ,100 ,115 ,134 ,135 ,229 ,234 ,262 ,425 ,526 ,540 ,572 ,602 ,630 ,638 ,660 ,664 ,750 ,799 ,840 ,886 ,905 ,906 ,1001 ,… ,3762 ,3765 ,3847 ,3850 ,3869 ,3870 ,3888 ,3889 ,3890)
$Pg(4,5)1$ = (1 ,2 ,3 ,4 ,5 ,13 ,14 ,16 ,18 ,30 ,37 ,38 ,42 ,56 ,57 ,58 ,60 ,63 ,65 ,81 ,90 ,96 ,100 ,101 ,105 ,111 ,115 ,116 ,122 ,126 ,134 ,135 ,… ,3860 ,3865 ,3869 ,3870 ,3871 ,3875 ,3877 ,3888 ,3889 ,3890 ,3891 ,3893 ,3900)

5- We use the same steps on PG(5,7) to know in which case the perfect code condition can be met :

The polynomial $g(x) = x^6 + ƀx^5 + ƀ^4x^4 + ƀ^2x^3 + ƀ^5x^2 + ƀ^4x + ƀ^3$ is primitive over $F_7 = [0,1,2,3,4,5,6]$, where ƀ =5 is a primitive element of $F7$, since ,$g(0) = ƀ^3$,$g(1) = ƀ$, $g(ƀ^4) = ƀ^4$, $g(ƀ^5) = ƀ^3$, $g(ƀ^2) = 1$, $g(ƀ) = ƀ^4$ and $g(ƀ^3) = 1$

$$P_i = [1,0,0,0,0]\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ ƀ^4 & ƀ & ƀ^5 & ƀ^2 & ƀ & 1 \end{pmatrix}^i , i = 1,2, \dots\dots ,19608 ,$$

Table (12-1) The points of $PG(5,7)$ are:

p1:(1,0,0,0,0,0)	p1221:(ƀ4,ƀ,0,ƀ3,ƀ4,1)	p8139:(ƀ2,ƀ3,ƀ5,0,ƀ5,1)	p19048:(ƀ2,ƀ3,ƀ,1,ƀ5,1)
p2:(0,1,0,0,0,0)	p1222:(ƀ5,0,ƀ,ƀ3,ƀ3,1)	p8140:(ƀ2,ƀ2,ƀ2,0,ƀ5,1)	p19049:(ƀ2,ƀ2,ƀ2,ƀ2,ƀ,1)
p551:(ƀ,ƀ5,1,0,0,1)	p1771:(ƀ5,ƀ5,0,ƀ,0,1)	p8689:(ƀ5,ƀ3,1,ƀ5,ƀ4,1)	p19598:(ƀ4,ƀ2,0,ƀ3,0,1)
p552:(ƀ4,ƀ5,ƀ3,ƀ,ƀ,1)	p1772:(ƀ4,1,ƀ3,ƀ2,ƀ5,1)	p8690:(ƀ5,ƀ,ƀ5,ƀ2,ƀ,1)	p19599:(ƀ4,0,0,ƀ2,ƀ2,1)
p561:(ƀ2,ƀ,ƀ4,ƀ2,ƀ,1)	p1781:(ƀ5,ƀ,ƀ,1,ƀ,1)	p8699:(0,ƀ5,ƀ5,ƀ2,ƀ4,1)	p19608:(ƀ4,ƀ2,ƀ5,ƀ4,ƀ3,1)

L1 = (1 ,2 ,4109 ,5123 ,5893 ,8224 ,17364 ,18571)

plane1 = (1 ,2 ,3 ,141 ,951 ,1864 ,2878 ,3071 ,3130 ,3648 ,4085 ,4109 ,4110 ,4305 ,4855 ,5123 ,5124 ,5893 ,5894 ,5979 ,6755 ,6954 ,7030 ,7186 ,… ,16856 ,17364 ,17365 ,17438 ,17533 ,17840 ,18571 ,18572 ,18859 ,18941 ,19182)

subspace1 = (1 ,2 ,3 ,4 ,141 ,142 ,187 ,400 ,506 ,633 ,642 ,697 ,725 ,785 ,826 ,842 ,885 ,947 ,951 ,952 ,1075 ,1285 ,1356 ,1403 ,1520 ,1548 ,1667 ,1711 ,1840 ,…,19182 ,19183 ,19227 ,19238 ,19408 ,19512 ,19521 ,19570)

$PG(4,7)1$ = (1 ,2 ,3 ,4 ,5 ,12 ,37 ,42 ,45 ,47 ,69 ,95 ,97 ,125 ,127 ,141 ,142 ,143 ,146 ,160 ,187 ,188 ,194 ,201 ,208 ,229 ,232 ,236 ,239 ,247 ,251 ,… ,19513 ,19517 ,19521 ,19522 ,19538 ,19570 ,19571 ,19576 ,19601)

6- We use the same steps on $PG(5,8)$ to know in which case the perfect code condition can be met :

The polynomial $g(x) = x^6 + s^6x^5 + s^5x^4 + x^3 + sx^2 + s^4x + s^3$ is primitive over $F_8 = [0,1,s,s^2,s^3,s^4,s^5,s^6]$, and ,$g(0) = s^3$,$g(1) = s^6$, $g(s) = s^3$, $g(s^2) = s$, $g(s^3) = s^3$, $g(s^4) = s^5$, $g(s^5) = s$ and $g(s^6) = 1$

$$P_i = [1,0,0,0,0]\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ s^6 & s^5 & 1 & s & s^4 & s^3 \end{pmatrix}^i , i = 1,2, \dots\dots ,37449,$$

Table (13-1) The points of $PG(5,8)$ are:

p1:(1,0,0,0,0,0)	p501:(s6,0,s3,s,s5,1)	p1001:(1,0,1,0,s2,1)	p33001:(s2,s3,s,s,s3,1)
p2:(0,1,0,0,0,0)	p502:(s4,s6,s5,s5,1,1)	p1002:(s,s6,s2,s5,s6,1)	p33002:(s4,s,s6,0,1,0)
p8:(s6,s2,s6,s2,1,1)	p508:(s6,s6,0,0,s5,1)	p1008:(s4,s4,s6,s3,s6,1)	p33008:(s5,s4,0,1,0,1)
p496:(s6,s2,1,1,s4,1)	p996:(0,s5,s4,s,0,1)	p1496:(s6,s4,s,s5,s3,1)	p33496:(s6,s,s5,s,s6,1)
p500:(s5,s,0,1,s,1)	p1000:(s5,s2,s,s2,s4,1)	p1500:(1,s3,1,s6,s5,1)	p33500:(s,s5,s3,1,s,1)

L1 = (1 ,2 ,8565 ,10578 ,10648 ,12028 ,15917 ,18370 ,24533)
plane1 = (1 ,2 ,3 ,2606 ,3000 ,3809 ,4079 ,5277 ,5441 ,5453 ,5560 ,6019 ,6673 ,8565 ,8566 ,10356 ,10578 ,10579 ,10648 ,10649 ,11616 ,… ,31833 ,33097 ,33177 ,33612 ,34182 ,34286 ,35110 ,35180 ,36560 ,36605 ,36739)
subspace1 = (1 ,2 ,3 ,4 ,62 ,132 ,187 ,281 ,411 ,476 ,487 ,588 ,798 ,860 ,868 ,1072 ,1142 ,1340 ,1512 ,1559 ,1617 ,1787 ,1874 ,1948 ,2047 ,2075 ,…,36765 ,37071 ,37100 ,37141 ,37144 ,37211 ,37246 ,37339 ,37409 ,37428)
$PG(4,8)$1 = (1 ,2 ,3 ,4 ,5 ,16 ,21 ,24 ,40 ,42 ,45 ,48 ,58 ,62 ,63 ,65 ,68 ,101 ,111 ,113 ,127 ,132 ,133 ,164 ,187 ,188 ,200 ,210 ,229 ,232 ,… ,37410 ,37416 ,37420 ,37422 ,37428 ,37429 ,37433 ,37436 ,37439 ,37443)

7- We use the same steps on $PG(5,9)$) to know in which case the perfect code condition can be met :

The polynomial $g(x) = x^6 + \eta x^5 + \eta^5 x^4 + x^3 + \eta^4 x^2 + \eta^7 x + \eta$ is primitive over $F_9 = [0,1,\eta,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6,\eta^7]$, and ,$g(0) = \eta$,$g(1) = \eta^5$, $g(\eta) = \eta^7$, $g(\eta^2) = \eta^5$, $g(\eta^3) = 1$, $g(\eta^4) = \eta$, $g(\eta^5) = \eta^5$, $g(\eta^6) = \eta$ and $g(\eta^7) = \eta^5$

$$P_i = [1,0,0,0,0] \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \eta^5 & \eta & \eta^4 & 1 & \eta^3 & \eta^5 \end{pmatrix}^i ,i = 1,2,\dots\dots,66430,$$

Table (14-1) The points of $PG(5,9)$ are:

p1: (1,0,0,0,0,0)	p7001: (η7,η2,η4,1,η,1)	p21501: (η3,1,1,η2,1,0)	p65931: (η2,η5,η6,η6,η6,1)
p2: (0,1,0,0,0,0)	p7002: (η7,1,η3,0,1,0)	p21502: (0,η3,1,1,η2,1)	p65932: (η6,η4,η7,η6,η2,1)
p498: (0,η5,η5,η6,0,1)	p7498: (η2,η2,η4,1,η,1)	p21998: (1,η3,1,1,η,1)	p66428: (η5,0,0,η2,η3,1)
p499: (1,η4,η,η6,η4,1)	p7499: (η7,η5,η3,0,1,0)	p21999: (η7,η4,η7,η6,1,0)	p66429: (η3,0,η2,η6,η2,1)
p500: (η7,η4,η2,η4,η3,1)	p7500: (0,η7,η5,η3,0,1)	p22000: (0,η7,η4,η7,η6,1)	p66430: (η5,1,η4,η7,η,1)

L1 = (1 ,2 ,11653 ,17240 ,17752 ,19268 ,20687 ,22015 ,41772 ,57374)
plane 1 = (1 ,2 ,3 ,1578 ,2596 ,5623 ,8183 ,8695 ,9584 ,10211 ,11286 ,11630 ,11653 ,11654 ,12958 ,13685 ,15437 ,16208 ,16861 ,17113 ,17240 ,17241 ,17752, … ,52019 ,52881 ,53424 ,54515 ,57374 ,57375 ,57439 ,59011 ,59523 ,61039 ,62458 ,63488 ,63786 ,65649 ,65679)
subspace1 = (1 ,2 ,3 ,4 ,10 ,22 ,319 ,335 ,472 ,527 ,1003 ,1016 ,1154 ,1207 ,1429 ,1528 ,1561 ,1578 ,1579 ,1633 ,1653 ,1702 ,1948 ,2169 ,2229 ,2422 ,2535 ,2573 ,…,66347 ,66365 ,66373 ,66376 ,66378 ,66381 ,66402 ,66407 ,66414 ,66422)

8- We use the same steps on $PG(5,11)$ to know in which case the perfect code condition can be met :

The polynomial $g(x) = x^6 + 2x^5 + x^4 + 10x^3 + 9x + 4$ is primitive over $F_{11} = [0,1,h,h^2,h^3,h^4,h^5,h^6,h^7,h^8,h^9]$ where h =7 is a primitive element of $F7$, since,$g(0) = h^6$,$g(1) = h^2$, $g(h^3) = h^6$, $g(h^4) = 2$, $g(h^6) = h$, $g(h^2) = h^7$, $g(h^7) = h^9$, $g(h) = h^6$, $g(h^9) = h^8$, $g(h^8) = h^5$ and $g(h^5) = h^7$

$$P_i = [1,0,0,0,0] \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ h^8 & h^5 & 1 & 0 & h^3 & h \end{pmatrix}^i ,i = 1,2,\dots\dots,177156,$$

Table (15-1) The points of $PG(5,11)$ are:

p1: (1,0,0,0,0,0)	p27501: (h4,h,h2,h7,h4,1)	p65001: (h2,h4,h3,h7,h9,1)	p107001: (h6,h5,h7,h7,h,1)
p2: (0,1,0,0,0,0)	p27502: (h3,h8,h4,h7,h4,1)	p65002: (h2,1,1,h7,h3,1)	p176658: (h2,h6,h4,h3,h6,1)
p494: (h3,0,1,h5,h4,1)	p27994: (h2,h6,1,h6,1,0)	p65494: (h,h5,h4,h3,h,1)	p176665: (h3,h7,h4,h6,1,0)
p499: (h8,h2,1,h7,1,1)	p27999: (1,h,0,h8,1,0)	p65499: (1,h6,h7,h6,h,1)	p177155: (h4,h5,h6,1,h3,1)
p500: (h9,1,h8,h,1,1)	p28000: (0,1,h,0,h8,1)	p65500: (h4,0,h8,h3,h3,1)	p177156: (1,h5,0,h8,h6,1)

L1 = (1 ,2 ,16642 ,21362 ,25244 ,28048 ,45855 ,74354 ,83293 ,123585 ,174817 ,176430 ,177157)
plane1 = (1 ,2 ,3 ,4218 ,7006 ,8844 ,12514 ,14302 ,15915 ,16642 ,16643 ,19022 ,20023 ,20635 ,20782 ,21362 ,21363 ,22904 ,24513 ,24517 ,25244 ,25245 ,25708 ,… ,176430 ,176431 ,177157)
subspace1 = (1 ,2 ,3 ,4 ,391 ,602 ,690 ,1470 ,1503 ,1622 ,1851 ,1878 ,1886 ,2088 ,2277 ,2359 ,2438 ,2523 ,2629 ,2783 ,2900 ,2917 ,2967 ,3107 ,3256 ,3331 ,…,176432 ,176515 ,176754 ,176805 ,176856 ,176875 ,176981 ,177089 ,177157)
$PG(4,11)1$ = (1 ,2 ,3 ,4 ,5 ,12 ,19 ,28 ,94 ,98 ,103 ,117 ,122 ,174 ,183 ,201 ,251 ,262 ,271 ,275 ,285 ,289 ,308 ,311 ,371 ,391 ,392 ,405 ,443 ,451 ,471 ,479 ,485 ,490 ,512 ,517,..., 177093 ,177111 ,177119 ,177129 ,177144 ,177157

When applying these steps to other $PG(n,q)$ We assumed that e=1

PG(5,2)	PG(5,3)	PG(5,4)	PG(5,5)
$\boldsymbol{(63, 2^{57}, 3)-cade.}$ Use e=1 And n=63 $\boldsymbol{\{\binom{n}{0}+\binom{n}{1}(q-1)+\cdots+\binom{n}{e}(q-1)^e\} = q^k = 2^6}$ 64 == 64 Is perfect	$(364, 3^{358}, 4)-cade$ Use e=1 And n=364 $\{\binom{n}{0}+\binom{n}{1}(q-1)+\cdots+\binom{n}{e}(q-1)^e\} = q^k = 3^6$ 729 == 729 Is perfect	$(1365, 4^{1359}, 5)-cade$ Use e=1 And n=1365 $\{\binom{n}{0}+\binom{n}{1}(q-1)+\cdots+\binom{n}{e}(q-1)^e\} = q^k = 4^6$ 4096 == 4096 Is perfect	$(3906, 5^{3900}, 6)-cade$ Use e=1 And n=3906 $\{\binom{n}{0}+\binom{n}{1}(q-1)+\cdots+\binom{n}{e}(q-1)^e\} = q^k = 5^6$ 15625 == 15625 Is perfect
PG(5,7)	PG(5,8)	PG(5,9)	PG(5,11)
$\boldsymbol{(19608, 7^{19602}, 8)-cade}$ Use the e=1 And n=19608 $\boldsymbol{\{\binom{n}{0}+\binom{n}{1}(q-1)+\cdots+\binom{n}{e}(q-1)^e\} = q^k = 7^6}$ 117649 == 117649 Is perfect	$(37449, 8^{37443}, 9)-cade$ Use the e=1 And n=37449 $\{\binom{n}{0}+\binom{n}{1}(q-1)+\cdots+\binom{n}{e}(q-1)^e\} = q^k = 8^6$ 262144 == 262144 Is perfect	$(66430, 9^{66424}, 10)-cade$ Use the e=1 And n=66430 $\{\binom{n}{0}+\binom{n}{1}(q-1)+\cdots+\binom{n}{e}(q-1)^e\} = q^k = 9^6$ 531441 == 531441 Is perfect	$(177156, 11^{177150}, 12)-cade$ Use the e=1 And n=177156 $\{\binom{n}{0}+\binom{n}{1}(q-1)+\cdots+\binom{n}{e}(q-1)^e\} = q^k = 11^6$ 1771561 == 1771561 Is perfect

Note: The operations in the previous sentence were calculated using algorithm on the Matlab 2019R program.

We note that when e is equal to one, the ideal equation is achieved, and since the value of e affects one side of the equation and the effect is positive, considering that the equation does not contain a negative value, and therefore, it is impossible to achieve equality when the value of e changes.
We can deduce the general formula in the case of e=1 that $n(q-1)+1 = q^k$

CONCLUSIONS

We have constructed the Projective linear codes with the parameters n, k and d depending on the order of Galois Field F_s . So that we have arrived at a general rule that states that any application of a projective space to the coding system has an error correction factor in it, equal to one, it satisfies the condition of the perfect code.
Our recommendations for future researchers complement these steps to find a general rule for good code

Acknowledgment:

The authors would like to thank the University of Mosul / College of Education for Pure Science and The college of Computer Science and Mathematics for providing the facilities that improved this work.

References

[1] M. Greferath , M. O. Pavčević , N. Silberstein and M. A. *Vázquez-Castro, Network Coding and Subspace Designs, Cham: Springer*, 2018.

[2] S. Ball and J. W. P. Hirschfeld, "Bounds on (n,r)(r)-arcs and their application to linear codes," *Finite Fields and Their Applications,* vol. 11, no. 3, pp. 326-336, 2005.

[3] J. W. P. Hirschfeld and J. A. Thas, "Open problems in finite projective spaces," *Finite Fields and Their Applications,* vol. 32, no. 1, pp. 44-81, 2015.

[4] N. Sloane, "Review of 'Projective Geometries over Finite Fields' (Hirschfeld, J. W. P.; 1979)," *Information Theory, IEEE Transactions on,* vol. 27, no. 3, pp. 385-385, 1981.

[5] Najm A. M. AL-Seraji and Raad I. K. AL-Humaidi, "Some application of coding theory in the projective plane of order three," *Iraqi Journal of Science,* vol. 59, no. 1, pp. 1947-1951, 2018.

[6] N. A. Makhrib Al-Seraji . B. A. Bakheet and Z. S. Jafar, "Study of orbits on the finite projective plane," *Journal of Interdisciplinary Mathematics,* vol. 23, no. 6, p. 1187–1195, 2020.

[7] Gyorgy Kiss and Tamas Szonyi, Finite Geometries, Boca Raton: Chapman and Hall/CRC, 2019.

[8] Raymond Hill, A First Course in Coding Theory, Oxford, New York: Oxford University Press, 1990.

[9] E. B. Al-Zangana and E. A. S. Shehab, "Certain Types of Linear Codes over the Finite Field of Order Twenty-Five," *Iraqi Journal of Science,* vol. 62, no. 1, pp. 4019-4031, 2021.

[10] E. B. Abdulkareem Al-Zangana, "Projective MDS Codes Over GF(27)," *Baghdad Science Journal,* vol. 18, no. 2(Suppl.), pp. 1125-1125, 2021.

[11] J. S. Radhi and E. B. Al-Zangana, "Complete (k,r)-Caps From Orbits In PG(3,11)," *Iraqi Journal of Science,* vol. 64, no. 1, pp. 347-353, 2023.

[12] J. S. Radhi and E. B. Al-Zangana, "Construction of Complete (k;r)-Arcs from Orbits in PG(3,11)," *Al-Mustansiriyah Journal of Science,* vol. 33, no. 3, pp. 48-53, 2022.

[13] S. H. Naji and E. B. Al-Zangana, "Special NMDS codes of dimension three over the finite field of order 27," *AIP Conference Proceedings,* vol. 2414, no. 1, p. 040068, 2023.

[14] M. Stynes and Hans-Görg Roos, "The midpoint upwind scheme," *Applied Numerical Mathematics,* vol. 23, no. 3, pp. 361-374, 1997.

[15] N. K. Yahya, "Applications geometry of space in PG(3, P)," *Journal of Interdisciplinary Mathematics,* vol. 25, no. 2, pp. 285-297, 2022.

[16] A. Neubauer , J. Freudenberger and V. Kuhn, Coding Theory: Algorithms, Architectures and

Applications, Chichester, England ; Hoboken, NJ: Wiley-Interscience, 2007.

[17] Venkatesan Guruswami, "*Introduction to Coding Theory," in Carnegie Mellon School of Computer Science, USA*, 2010.

[18] J. H. Van Lint, "*Introduction to Coding Theory (second edition)," in Introduction to Coding Theory, Berlin, Heidelberg, Springer*, 1992, pp. 69-76.

[19] J. Castiñeira Moreira and P. G. Farrell, *Essentials of Error-Control Coding, Weinheim: Wiley,* 2006.

[20] S. S. Adams, "Introduction to algebraic coding theory," *Franklin W. Olin College: NSF CCLI,* no. 1, pp. 1-67, 2008.