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On coefficients estimate for subclasses of bi-univalent functions defined by Ruscheweyh derivative operator

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Abstract
This paper presents two subclasses of analytic and bi-univalent functions associated with the Ruscheweyh derivative operator to investigate the bounds for $|c_2|$ and $|c_3|$, where c_2 and c_3 are the initial Tayler-Maclaurin coefficients. The current results would generalize and improve some corresponding recent works. Additionally, in certain cases, our estimates correct some of the existing coefficient bounds.

Keywords: bi-univalent; Ruscheweyh derivative, Coefficients bounds, Analytic function, Starlike function, Convex function.

حول تخمين المعاملات للفئات الفرعية للدوال الثنائية التكافؤ المعرفة بمؤثر راشوية التفاضلية

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الخلاصة:
ان هذا العمل يرتبط بين فاتين محددتين من الدوال التحليلية وثنائية التكافؤ بمشتقة راشوية . الهدف من ذلك هو فحص قيم القيود على المعاملات $|c_2|$ و $|c_3|$, حيث تمثل c_2 و c_3 المعاملات الاولية لسلسله تايلور_ماكلورين . تجدر الاشارة الى ان النتائج الحالية ستوسع وتعزز العديد من الدراسات الحالية في سياق واسع. في بعض الحالات, تقوم تقديراتنا بتصحيح بعض حدود المعاملات الموجودة بالفعل.

1. Introduction

Let $\mathcal{A}$ denote the class of the function $\mathcal{F}$ that may be expressed in the manner

$$\mathcal{F}(\xi) = \xi + \sum_{j=2}^{\infty} c_j \xi^j, \qquad (1)$$

which analytic in the open unit disc
$U = \{\xi : \xi \in \mathbb{C} \text{ and } |\xi| < 1\}$, together with a normalization given by $\mathcal{F}(0) = \mathcal{F}'(0) - 1 = 0$.

The convolution $\mathcal{F}(\xi) * l(\xi)$ of $\mathcal{F}(\xi)$ and $l(\xi)$ is defined by

$$(\mathcal{F} * l)(\xi) = \xi + \sum_{j=2}^{\infty} c_j d_j \xi^j = (l * \mathcal{F})(\xi) \ (\xi \in U),$$

where the function $l(\xi) = \xi + \sum_{j=2}^{\infty} d_j \xi^j$ is also analytic in U.

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The function $\mathcal{F}$, belonging to the class $\mathcal{A}$ and Equation (1), is associated with the Ruscheweyh
derivative R^{β}. This operator maps elements form class $\mathcal{A}$ to $\mathcal{A}$, [1]. This operator definition described below

$$R^{\beta}\mathcal{F}(\xi)=\frac{\xi\left(\xi^{\beta}f(\xi)\right)^{(\beta)}}{\beta!}=$$
$$\frac{\xi}{(1-\xi)^{\beta+1}}*\mathcal{F}(\xi)=\xi+\sum_{j=2}^{\infty}\frac{\Gamma(\beta+j)}{\Gamma(j)\Gamma(\beta+1)}c_j\xi^j \quad (\beta\in N_0,\ \xi\in U).$$

A function $\mathcal{F}$ is considered univalent in U if it is injective, meaning that it maps distinct elements in U to distinct elements in its range. The subclass S of $\mathcal{A}$ that contains functions are univalent in U. One of the most important examples of a function in S is the Koebe function

$$k(\xi)=\frac{1}{(1-\xi)^2}+\sum_{j=1}^{\infty}j\xi^j \qquad (\xi\in U).$$

This function is very important for figuring out a of problems in the theory of univalent functions. It moves the unit disc U onto the complex plane in that conformal, the exception of a slice through the opposite real axis from to -1/4.

The class $S^*(\gamma)$ of starlike functions of order γ $(0\le\gamma<1)$ in U and the class $K(\gamma)$ of convex functions of order γ $(0\le\gamma<1)$ in U are two of the most important and well-investigated subclasses of the analytic and univalent function class S.

According to the Koebe one quarter Theorem [2] the image of the open unit disk U under any of the univalent
The function encompasses a disc with a radius 4. Therefore, for all functions $\mathcal{F}$ belonging to the class S there exists an inverse function $\mathcal{F}^{-1}$that is

$$\mathcal{F}^{-1}(\mathcal{F}(\xi))=\xi \quad (\xi\in U) \text{ and } \mathcal{F}^{-1}\big(\mathcal{F}(w)\big)=w(|w|<r_0(\mathcal{F}),r_0(\mathcal{F})\ge 1/4).$$

The inverse function $g=\mathcal{F}^{-1}$ has the form

$$g(w)=\mathcal{F}^{-1}(w)=w-c_2w^2+(2c_2^2-c_3)w^3-(5c_2^3-5c_2c_3+c_4)w^4+\cdots. \quad (2)$$

A function $\mathcal{F}$ is considered bi- univalent if either F and its invers $\mathcal{F}^{-1}$ are univalent in the domain U. The symbol Σrepresents the of all bi- univalent functions in the domain U, as defined by Equation (1).
Liewin [3] constructed a study on the class Σ of bi-univalent functions and found that the absolute value of c_2 is lest then 1.51 for the functions in the class Σ. Berannan and Clunei [4] subsequently presented the hypothesis that the absolute value of c_2 is less than or equal to the square root of 2. Later, Natenyahu [5] proved that the maximum value of $|c_2|$ is 4/3 for $\mathcal{F}\in\Sigma$ Srivasteva et al. [6], did groundbreaking research that has effectively rekindled the study of analytic and bi- univalent function in the last few decades. If you want to look at interesting of function in class Σ, you shroud read their work.
Based on the findings of Srivasteva et al. [6], here are some examples of functions in the class Σ

$$\frac{1}{1-\xi}=\sum_{j=1}^{\infty}\xi^j \; -log(1-\xi)=\sum_{j=1}^{\infty}\frac{\xi^j}{j} \text{ and } \frac{1}{2}log\left(\frac{1+\xi}{1-\xi}\right)=\sum_{j=1}^{\infty}(-1)^{j-1}\frac{\xi^{2j+1}}{2j+1}.$$

The fact that the class Σ cannot be complete is apparent. However, the Koebe function is not a member of the Σ. Braennan and Taha [7] suggested making new subclasses inside the bi- univalent class K. These subclasses would be like the will know well-known $S^*{}_\Sigma$ and $K(\gamma)$ subclasses, starlike convex functions of order γ (0 ≤ γ< 1), respectively. Thus, for 0 ≤ γ < 1, a function $\mathcal{F} \in \Sigma$ falls into the class $S^*{}_\Sigma(\gamma)$ of bi-starlike functions of order γ are defined as functions F for which $\mathcal{F}$ and $\mathcal{F}^{-1}$ are starlike functions of order γ. Or into the class $k_\Sigma(\gamma)$ of bi-univalent functions of order γ, if both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are convex functions of order γ. Furthermore, A function $\mathcal{F} \in \mathcal{A}$ is classified as a strongly bi-starlike functions of order γ (0 < γ ≤ 1), denoted by $S^*{}_\Sigma[\gamma]$, K. Bilal [8] and A. Delph and K.A. Jassim [9], if it satisfies each of the following conditions:

$$\left|\arg\left(\frac{\xi\mathcal{F}'(\xi)}{\mathcal{F}'(\xi)}\right)\right| < \frac{\gamma\pi}{2} \quad \text{and} \quad \left|\arg\left(\frac{wg'(w)}{g'(w)}\right)\right| < \frac{\gamma\pi}{2},$$

where, g is the univalent extension of $\mathcal{F}^{-1}$ to U.

There have been numerous recent works dedicated to studying the class Σ of bi-univalent functions and tried to fine the highest values the Taylor-Maclaurin coefficients$|c_2|$ and $|c_3|$. Notably, the pioneering work by Srivasteva et al. [6] has made great progress of several subclasses inside the bi-univalent functions class Σ. Additionally, they have identified on the magnitudes of $|c_2|$ and $|c_3|$. The scientific literature has made a significant amount of additional work available, expanding on the pioneering research by Srivasteva et al. [10] and focusing on coefficient problems for different subclasses of the analytic and bi-univalent function class Σ . D. Ali [11], F. Ghanin and Hiba [12,13], provide of this type of study. The unsolved problem, however, is to find the generally applicable coefficient estimate bounds on $|c_j|$ (j∈ {4, 5, 6, . . .}) for a function $\mathcal{F}'$that is defined by Equation (1) and is the class. Indeed, when the coefficient exceed three, there is no inherent approach to determining an upper limit. Several studies have employed fiber polynomial approaches to calculate upper limits for coefficient with greater order Wanas and A. H. Majeed [14] and other [15-16].

The determination of estimates for the Tayler –Maclaurin variables in an algebraic equation are those multiply value. An essential issue in geometric function theory is the analysis of an, since it yields valuable insight into the geometric characteristics of these functions. For instance, the bounds for the second and third coefficients a_2 anda_3 of functions $\mathcal{F} \in \Sigma$ yield growth and distortion bounds, as well as covering theorems. Motivated by the aforementioned works and making use of Ruscheweyh derivative operator, we examine two specific categories of analytic and bi-univalent functions during our investigation. The techniques employed by Srivasteva et al. [6], Fraisn and Aouf [17]. The obtained results improve some recent works and rectify remarkable mistakes in existing coefficient estimates.

Lemma 1.1. [2] If $h \in P$, then $|h_k| \le 2$ for each $k \in N$, where P is the subclass of functions $h(\xi)$ of the form

$$h(\xi) = 1 + h_1\,\xi + h_2\xi^2 + h_3\xi^3 + \cdots. \tag{3}$$

which is analytic in U and the actual component , $\Re(h(\xi))$, is affirmative.

2. Bounds for the coefficient functions in the class $\mathcal{T}_\Sigma(\eta, \varpi, \beta;\ \sigma)$

Consider an element $h \in P$defined by Equation (3), and let $K(z)$ be a complex-valued function such that $K(\xi) = [h(\xi)]^\sigma$, $\sigma \in (0,1]$.

$$|\arg(K(\xi)) = \sigma|\arg(\mathrm{h}(\xi))| < \frac{\sigma\pi}{2}.$$

Therefore, if $|\arg(K(\xi))| < \frac{\sigma\pi}{2}$, then it can be said that there exists $h \in P$ such that k(ξ) can be written in terms of h and α as follows

Definition 2.1. A function $\mathcal{F} \in \Sigma$ given by Equation (1) is called in the class

$$\mathcal{T}_{\Sigma}(\eta, \varpi, \beta; \sigma)(\xi, w \in U, \eta > 0, \varpi \in \mathbb{C}\{0\}, \beta \in N_0, 0 < \sigma \leq 1)$$

if it is satisfied each the conditions

$$\left|\arg\left(1 + \frac{1}{\varpi}\left[(1-\eta)\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)'}{R^{\beta}\mathcal{F}(\xi)} + \eta\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)''}{\left(R^{\beta}\mathcal{F}(\xi)\right)'}\left(\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)' + \xi\left(R^{\beta}\mathcal{F}(\xi)\right)''}{\left(R^{\beta}\mathcal{F}(\xi)\right)'}\right) - 1\right]\right)\right| < \frac{\sigma\pi}{2} \qquad (4)$$

and

$$\left|\arg\left(1 + \frac{1}{\varpi}\left[(1-\eta)\frac{w\left(R^{\beta}g(w)\right)'}{R^{\beta}g(w)} + \eta\frac{w\left(R^{\beta}g(w)\right)''}{\left(R^{\beta}g(w)\right)'}\left(\frac{w\left(R^{\beta}g(w)\right)' + w\left(R^{\beta}g(w)\right)''}{\left(R^{\beta}g(w)\right)'}\right) - 1\right]\right)\right| < \frac{\sigma\pi}{2} \qquad (5)$$

where the function $g = \mathcal{F}^{-1}$is defined by Equation (2).

Theorem 2.2. If $\mathcal{F} \in \mathcal{T}_{\Sigma}(\eta, \varpi, \beta; \sigma)$ be given by $\mathcal{F}(\xi) = \xi + \sum_{j=2}^{\infty} c_j \xi^j$. Then

$$|c_2| \leq \frac{2\sigma|\varpi|}{\sqrt{|2\sigma\varpi[(\beta+2)(\beta+1)(1+2\eta) - 2(\beta+1)^2(1-\eta)] - (\sigma-1)(\beta+1)^2(\eta+1)^2|}} \qquad (6)$$

and

$$|c_3| \leq \frac{2\sigma|\varpi|}{(\beta+2)(\beta+1)(1+2\eta)} + \frac{4\sigma^2|\varpi|^2}{(\beta+1)^2(\eta+1)^2} \qquad (7)$$

Proof. It follows from Equations (4) and (5) that

$$1 + \frac{1}{\varpi}\left[(1-\eta)\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)'}{R^{\beta}\mathcal{F}(\xi)} + \eta\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)''}{\left(R^{\beta}\mathcal{F}(\xi)\right)'}\left(\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)' + \xi\left(R^{\beta}\mathcal{F}(\xi)\right)''}{\left(R^{\beta}\mathcal{F}(\xi)\right)'}\right) - 1\right] \qquad (8)$$

and

$$1 + \frac{1}{\varpi}\left[(1-\eta)\frac{w\left(R^{\beta}g(w)\right)'}{R^{\beta}g(w)} + \eta\frac{w\left(R^{\beta}g(w)\right)''}{\left(R^{\beta}g(w)\right)'}\left(\frac{w\left(R^{\beta}g(w)\right)' + w\left(R^{\beta}g(w)\right)''}{\left(R^{\beta}g(w)\right)'}\right) - 1\right] \qquad (9)$$

where $p, q \in P$ have the following representations

$$p(\xi) = 1 + p_1\xi + p_2\xi^2 + p_3\xi^3 + \cdots, \qquad (10)$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots. \qquad (11)$$

Clearly, we have

$$[p(\xi)]^{\sigma} = 1 + \sigma p_1\xi + \left(\frac{1}{2}\sigma(\sigma-1)p_2{}^2\right)\xi^2 + \left(\frac{1}{6}\sigma(\sigma-1)(\sigma-1)p_1{}^3 + \sigma(1-\alpha)p_1p_2\right)\xi^2 + \cdots, \qquad (12)$$

and

$$[q(w)]^{\sigma} = 1 + \sigma q_1 w + \left(\frac{1}{2}\sigma(\sigma-1)q_2{}^2\right)w^2 + \left(\frac{1}{6}\sigma(\sigma-1)(\sigma-1)q_1{}^3 + \sigma(1-\sigma)q_1q_2\right)w^2 + \cdots. \qquad (13)$$

We also find that

$$1 + \frac{1}{\varpi}\left[(1-\eta)\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)'}{R^{\beta}\mathcal{F}(\xi)} + \eta\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)''}{\left(R^{\beta}\mathcal{F}(\xi)\right)'}\left(\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)' + \xi\left(R^{\beta}\mathcal{F}(\xi)\right)''}{\left(R^{\beta}\mathcal{F}(\xi)\right)'}\right) - 1\right] = 1 + \frac{(\beta+1)(1+\eta)}{\varpi}c_2\xi + \left(\frac{(\beta+2)(\beta+1)(1+2\eta)}{\varpi}c_3 - \frac{2(\beta+1)^2(1-\eta)}{\varpi}c_2{}^2\right)\xi^2 + \cdots, \qquad (14)$$

and

$$1+\frac{1}{\varpi}\left[(1-\eta)\frac{w\left(R^{\beta}g(w)\right)'}{R^{\beta}g(w)}+\eta\frac{w\left(R^{\beta}g(w)\right)''}{\left(R^{\beta}g(w)\right)'}\left(\frac{w\left(R^{\beta}g(w)\right)'+w\left(R^{\beta}g(w)\right)''}{\left(R^{\beta}g(w)\right)'}\right)-1\right]=1-\frac{(\beta+1)(1+\eta)}{\varpi}\mathrm{c}_2w+\left(\frac{2(\beta+2)(\beta+1)(1+2\eta)-2(\beta+1)^2(1-\eta)}{\varpi}\mathrm{c}_2{}^2-\frac{(\beta+2)(1+2\eta)}{\varpi}\mathrm{c}_3\right)w^2+\cdots. \quad (15)$$

Now, by using Equations (12), (13), (14) and (15), together with comparing the coefficients of Equations (8) and (9), we get

$$\frac{(\beta+1)(1+\eta)}{\varpi}\mathrm{c}_2=\sigma p_1, \quad (16)$$

$$\frac{(\beta+2)(\beta+1)(1+2\eta)}{\varpi}\mathrm{c}_3-\frac{2(\beta+1)^2(1-\eta)}{\varpi}\mathrm{c}_2{}^2=\frac{1}{2}\sigma(\sigma-1)p_1{}^2+\sigma p_2, \quad (17)$$

$$-\frac{(\beta+1)(1+\eta)}{\varpi}\mathrm{c}_2=\sigma q_1, \quad (18)$$

and

$$\frac{2(\beta+2)(\beta+1)(1+2\eta)-2(\beta+1)^2(1-\eta)}{\varpi}\mathrm{c}_2{}^2-\frac{(\beta+2)(\beta+1)(1+2\eta)}{\varpi}\mathrm{c}_3=\frac{1}{2}\sigma(\sigma-1)q_1{}^2+\sigma q_2. \quad (19)$$

In view of Equations (16) and (18), we conclude that

$$p_1=-q_1, \quad (20)$$

$$\frac{2(\beta+1)^2(1+\eta)^2}{\varpi^2}\mathrm{c}_2{}^2=\mathrm{c}^2(p_1{}^2+q_1{}^2). \quad (21)$$

Adding Equation (17) to Equation (19), we obtain

$$\frac{2(\beta+2)(\beta+1)(1+2\eta)-4(\beta+1)^2(1-\eta)}{\varpi}\mathrm{c}_2{}^2=\frac{1}{2}\sigma(\sigma-1)(p_1{}^2+q_1{}^2)+\sigma(p_2+q_2). \quad (22)$$

Substituting the value of $p_1{}^2+q_1{}^2$ form Equation (21) into Equation (22) and further computations imply that

$$\mathrm{c}_2{}^2=\frac{\sigma^2\varpi^2(p_2+q_2)}{2\sigma\varpi[(\beta+2)(\beta+1)(1+2\eta)-2(\beta+1)^2(1-\eta)]-(\sigma-1)(\beta+1)^2(1+\eta)^2}. \quad (23)$$

Applying Lemma 1.1 for the coefficients p_2 and q_2 on Equation (23) imply that

$$|\mathrm{c}_2|\leq\frac{2\sigma|\varpi|}{\sqrt{|2\sigma\varpi[(\beta+2)(\beta+1)(1+2\eta)-2(\beta+1)^2(1-\eta)]-(\sigma-1)(\beta+1)^2(1+\eta)^2|}}.$$

Next, in order to derive the bound on $|\mathrm{c}_3|$, by subtracting Equation (19) from Equation (17), we obtain

$$\frac{2(\beta+2)(\beta+1)(1+2\eta)}{\varpi}(\mathrm{c}_3-\mathrm{c}_2{}^2)=\frac{1}{2}\sigma(\sigma-1)(p_1{}^2+q_1{}^2)+\sigma(p_2+q_2). \quad (24)$$

Now, substituting the value of $\mathrm{c}_2{}^2$from Equation (21) into Equation (24) and using (20), we conclude that

$$\mathrm{c}_3=\frac{\sigma\varpi(p_2-q_2)}{2(\beta+2)(\beta+1)(1+2\eta)}+\frac{\sigma^2\varpi^2(p_2+q_2)}{2(\beta+1)^2(1+\eta)^2}, \quad (25)$$

Finally, by using Lemma 1.1 again over the coefficients p_1, p_2, q_1 as well as q_2 on Equation (25), we may conclude that

$$|\mathrm{c}_3|\leq\frac{2\sigma|\varpi|}{(\beta+2)(\beta+1)(1+2\eta)}+\frac{4\sigma^2|\varpi|^2}{(\beta+1)^2(1+\eta)^2}.$$

This completes the proof.

3. Bounds for the coefficient functions in the class $\mathcal{T}^*{}_{\Sigma}(\eta,\varpi,\beta;\ \sigma,\tau)$

Let h∈ P be defined by Equation (3), and let $L(\xi)$ be an arbitrary complex value function$L(\xi)=\tau+(1-\tau)h(\xi)$, $0\leq\tau<1$, then

$$\Re\{L(\xi)\}=\tau+(1-\tau)\Re\{h(\xi)\}>\tau.$$

Therefore, if $\Re\{L(\xi)\} > \tau$, it can be said that there exists $h \in P$ such that $L(\xi)$ can be written in terms of h and τ as follows

$$L(\xi) = \tau + (1 - \tau)h(\xi).$$

Definition 3.1. A function $\mathcal{F} \in \Sigma$ given by Equation (1) is called in the class

$$\mathcal{T}^*_{\Sigma}(\eta, \varpi, \beta;\ \sigma, \tau)(\xi, w \in U, \eta > 0, \varpi \in \mathbb{C}\backslash\{0\}, \beta \in N_0, 0 \leq \tau < 1),$$

if it meets the following requirements

$$\Re\left\{1 + \frac{1}{\varpi}\left[(1-\eta)\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)'}{R^{\beta}\mathcal{F}(\xi)} + \eta\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)''}{\left(R^{\beta}\mathcal{F}(\xi)\right)'}\left(\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)' + \xi\left(R^{\beta}\mathcal{F}(\xi)\right)''}{\left(R^{\beta}\mathcal{F}(\xi)\right)'}\right) - 1\right]\right\} > \tau, \quad (26)$$

and

$$\Re\left\{1 + \frac{1}{\varpi}\left[(1-\eta)\frac{w\left(R^{\beta}g(w)\right)'}{R^{\beta}g(w)} + \eta\frac{w\left(R^{\beta}g(w)\right)''}{\left(R^{\beta}g(w)\right)'}\left(\frac{w\left(R^{\beta}g(w)\right)' + w\left(R^{\beta}g(w)\right)''}{\left(R^{\beta}g(w)\right)'}\right) - 1\right]\right\} > \tau,$$

(27)

the function $g = f^{-1}$is defined by Equation (2).

Theorem 3.2. Let $\mathcal{F} \in \mathcal{T}^*_{\Sigma}(\eta, \varpi, \beta;\ \alpha, \tau)$be given by Equation (1). Then

$$|c_2| \leq \begin{cases} \left(\frac{2|\varpi|(1-\tau)}{|(\beta+2)(\beta+1)(1+2\eta)-2(\beta+1)^2(1-\eta)|}\right)^{\frac{1}{2}} & 0 \leq \tau \leq 1 - \frac{(\beta+1)^2(1+\eta)^2}{2|\varpi||(\beta+2)(\beta+1)(1+2\eta)-2(\beta+1)^2(1-\eta)|}; \\ \frac{2|\varpi|(1-\tau)}{(\beta+1)(1+\eta)} & 1 - \frac{(\beta+1)^2(1+\eta)^2}{2|\varpi||(\beta+2)(\beta+1)(1+2\eta)-2(\beta+1)^2(1-\eta)|} \leq \tau < 1; \end{cases}$$

and

$$|c_3| \leq \begin{cases} \frac{2|\varpi|(1-\tau)}{(\ell\beta+2)(\beta+1)(1+2\eta)} + \frac{4|\varpi|(1-\tau)}{|(\beta+2)(\beta+1)(1+2\eta)-2(\beta+1)^2(1-\eta)|} \\ \qquad 0 \leq \tau \leq 1 - \frac{(\beta+1)^2(1+\eta)^2}{2|\varpi||(\beta+2)(\beta+1)(1+2\eta)-2(\beta+1)^2(1-\eta)|}; \\ \frac{2|\varpi|(1-\tau)}{(\beta+2)(\beta+1)(1+2\eta)} + \frac{4|\varpi|^2(1-\tau)^2}{(\beta+1)^2(1+\eta)^2} \quad 1 - \frac{(\beta+1)^2(1+\eta)^2}{2|\varpi||(\beta+2)(\beta+1)(1+2\eta)-2(\beta+1)^2(1-\eta)|} \leq \tau < 1. \end{cases}$$

Proof. It follows from Equations (26) and (27) that

$$1 + \frac{1}{\varpi}\left[(1-\eta)\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)'}{R^{\beta}\mathcal{F}(\xi)} + \eta\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)''}{\left(R^{\beta}\mathcal{F}(\xi)\right)'}\left(\frac{\xi\left(R^{\beta}\mathcal{F}(\xi)\right)' + \xi\left(R^{\beta}\mathcal{F}(\xi)\right)''}{\left(R^{\beta}\mathcal{F}(\xi)\right)'}\right) - 1\right] = \tau + (1-\tau)p(\xi), \quad (28)$$

and

$$1 + \frac{1}{\varpi}\left[(1-\eta)\frac{w\left(R^{\beta}g(w)\right)'}{R^{\beta}g(w)} + \eta\frac{w\left(R^{\beta}g(w)\right)''}{\left(R^{\beta}g(w)\right)'}\left(\frac{w\left(R^{\beta}g(w)\right)' + w\left(R^{\beta}g(w)\right)''}{\left(R^{\beta}g(w)\right)'}\right) - 1\right] = \tau + (1-\tau)q(w), \quad (29)$$

where p, q ∈ P have the representations of Equations (10) and (11), respectively. Clearly, we have

$$\tau + (1-\tau)p(\xi) = 1 + (1-\tau)p_1\xi + (1-\tau)p_2\xi^2 + \cdots, \quad (30)$$

and

$$\tau + (1-\tau)q(w) = 1 + (1-\tau)q_1 w + (1-\tau)q_2 w^2 + \cdots. \quad (31)$$

Now, by using Equations (30), (31), (14) and (15), together with comparing the coefficients of Equations (28) and (29), yields

$$\frac{(\beta+1)(1+\eta)}{\varpi}\mathfrak{c}_2=(1-\tau)q_1, \tag{32}$$

$$\frac{(\beta+2)(\beta+1)(1+2\eta)}{\varpi}\mathfrak{c}_3-\frac{2(\beta+1)^2(1-\eta)}{\varpi}{\mathfrak{c}_2}^2=(1-\tau)p_2, \tag{33}$$

$$-\frac{(\beta+1)(1+\eta)}{\varpi}\mathfrak{c}_2=(1-\tau)q_1, \tag{34}$$

and

$$\frac{2(\beta+2)(\beta+1)(1+2\eta)-2(\beta+1)^2(1-\eta)}{\varpi}{\mathfrak{c}_2}^2-\frac{(\beta+2)(\beta+1)(1+2\eta)}{\varpi}\mathfrak{c}_3=(1-\tau)q_2 \tag{35}$$

From Equations (32) and (34), we get

$$p_1=-q_1, \tag{36}$$

$$\frac{2(\beta+1)^2(1+\eta)^2}{\varpi^2}{\mathfrak{c}_2}^2=(1-\tau)^2({p_1}^2+{q_1}^2). \tag{37}$$

Adding Equations (33) to Equation (35), we obtain

$$\frac{2(\beta+2)(\beta+1)(1+2\eta)-4(\beta+1)^2(1-\eta)}{\varpi}{\mathfrak{c}_2}^2=(1-\tau)(p_2+q_2) \tag{38}$$

From Equation (37) and Equation (38), we find

$${\mathfrak{c}_2}^2=\frac{\varpi^2(1-\tau)^2({p_1}^2+{q_1}^2)}{2(\beta+1)^2(1+\eta)^2}, \tag{39}$$

and

$${\mathfrak{c}_2}^2=\frac{\varpi(1-\tau)(p_2+q_2)}{2(\beta+2)(\beta+1)(1+2\eta)-4(\beta+1)^2(1-\eta)}, \tag{40}$$

respectively.

The Equations (39) and (40) together with applying Lemma 1.1 for the coefficientsp_1, q_1, p_2 and q_2, we find that

$$|\mathfrak{c}_2|\le\frac{2|\varpi|(1-\tau)}{(\beta+1)(1+\eta)},$$

and

$$|\mathfrak{c}_2|\le\sqrt{\frac{2|\varpi|(1-\tau)}{|(\beta+2)(\beta+1)(1+2\eta)-2(\beta+11)^2(1-\eta)|}},$$

respectively.

To calculate the projection for the $|\mathfrak{c}_3|$, we subtract Equation (35) from Equation (33),

$$\frac{(\beta+2)(\beta+1)(1+2\eta)}{\omega}(\mathfrak{c}_3-{\mathfrak{c}_2}^2)=(1-\tau)(p_2-q_2)$$

or, equivalently,

$$\mathfrak{c}_3={\mathfrak{c}_2}^2+\frac{\varpi(1-\tau)(p_2-q_2)}{(\beta+2)(\beta+1)(1+2\eta)}. \tag{41}$$

Substituting the value of ${\mathfrak{c}_2}^2$ from Equation (39) and Equation (40) into Equation (41), imply that

$$\mathfrak{c}_3=\frac{\varpi(1-\tau)(p_2-q_2)}{(\beta+2)(\beta+1)(1+2\eta)}+\frac{\varpi^2(1-\tau)^2({p_1}^2+{q_1}^2)}{2(\beta+1)^2(1+\eta)^2}, \tag{42}$$

$$\mathfrak{c}_3=\frac{\varpi(1-\tau)(p_2-q_2)}{(\beta+2)(\beta+1)(1+2\eta)}+\frac{\varpi(1-\tau)(p_2+q_2)}{2(\beta+2)(\beta+1)(1+2\eta)-4(\beta+1)^2(1-\eta)}, \tag{43}$$

respectively.

Finally, applying Lemma 1.1 once again for the coefficients p_1, q_1, p_2 and q_2 on Equations (42) and (43) together, we conclude that

$$|c_3| \le \frac{2|\varpi|(1-\tau)}{(\beta+2)(\beta+1)(1+2\eta)} + \frac{4|\varpi|^2(1-\tau)^2}{(\beta+1)^2(1+\eta)^2},$$

$$|c_3| \le \frac{2|\varpi|(1-\tau)}{(\beta+2)(\beta+1)(1+2\eta)} + \frac{4|\varpi|(1-\tau)}{|(\beta+2)(\beta+1)(1+2\eta)-2(\beta+1)^2(1-\eta)|}.$$

In the order mentioned, proof is now finished.

4. Conclusions

In this study, we have introduced and examined two specific subclasses of analytic bi-univalent functions associated with the Ruscheweyh derivative in this research. Our main goal was establish the first limits for the coefficients of functions that fall within subclasses. The study of the results shows significant improvements, and corrections compared to previous investigations. Moreover, we have highlighted certain implication of these subclass by considering specific parameter specifications.

References

[1] S. Ruscheweyh, "New criteria for univalent functions", *Proceedings of the American Mathematical Society*, vol.49, no.1, pp. 109–115, 1975.

[2] P. L. Duren, "Univalent functions, Grundlehren der Mathematischen Wissenschaften", *Band 259, Springer Verlag, New York, Berlin, Heidelberg and Tokyo*, vol. 1, no. 6, 1983.

[3] M. Lewin, "On a coefficient problem for bi-univalent functions", *Proceedings of the American Mathematical Society*, vol.18, no. 1, pp. 63–68,1967.

[4] D. A. Braennan, J. G. Clunie, "Aspects of contemporary complex analysis, Proceedings London-New York: Academic Press of the NATO Advanced Study Institute held at the University of Durham" Durham; July pp.1-20, 1979, *London-New York:Academic Press*,1980.

[5] E. Natanyahu, "The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in |z| < 1", *Archive for Rational Mechanics and Analysis*, vol.32, no.2, pp. 100-112,1969.

[6] H. M. Srivasteva, A. K. Mishra, P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions", *Applied Mathematics Letters*, vol. 23, no.6, pp.1188–1192, 2010.

[7] D. A. Braennan, T. S. Taha, "On Some classes of bi-univalent functions", *Studia Universitatis Babes-Bolyai Mathematica*, vol.31, no.2, pp. 70–77,1986.

[8] K. Bilal, H. M. Srivasteva, M. Tahir, M. Darus, Q. Z. Ahmad, N. Khan, "Applications of a certain q-integral operator to the subclasses of analytic and bi-univalent functions", *American Institute of Mathematical Sciences*, vol.6, no1,pp. 1024–1039, 2020.

[9] A. M. Delphi and K. A. Jassim," A Class of Harmonic Multivalent Functionsfor Higher Derivatives Associated with General Linear Operator", *Iraqi Journal of Science*, vol.63, no.9, pp. 3867-3876, 2022.

[10] H. M. Srivasteva, A. Motamednezhad, E. A. Adegani, "Faber polynomial coefficient estimates for bi-univalent functions defined by using differential subordination and a certain fractional derivative operator", *Mathematics*, vol.8, no.2, pp .5 –12, 2020.

[11] D. Ali mohammadia, N. E. Cho, E. A. Adegani, "Coefficient bounds for subclasses of analytic and bi-univalent functions", *Filomat,* vol.34, no.14, pp.4709–4721, 2020.

[12] F. Ghanim and H. F. Al-Janaby,"An analytical study on Mittag- Leffler_ confluent hypergeometric functions with fractional integral operato", *Journal Mathematical Methods in the Applied Sciences*, vol.44, no. 5, pp.3605-3614, 2021.

[13] F. Ghanim, Hiba. F, Al-Janaby and O. Bazighifan,"Some new extensions on fractional differential and integral properties for Mittag-Leffler confluent hypergeometric functio", *Journal Fractal and Fractional*, vol.5, no .4, pp.143, 2021.

[14] A.K. Wanas and A.H. Majeed, "Second Hankel Determinant for a Subclass of λ-pseudo- Statlik Bi-Univalent Functions", *Iranian Journal of Mathematical Sciences and informatics,* vol. 16, no. 3 pp. 49-59, 2021.

[15] H. M. Srivastava, A. Motamednezhad, E. A. Adegani, "Faber polynomial coefficient estimates for bi-univalent functions defined by using differential subordination and a certain fractional derivative operator", *Mathematics*, vol.8, no.2 , Article ID 172,pp. 1–12, 2020.
[16] S. Bulut, Faber polynomial coefficient estimates for a subclass of analytic bi-univalent functions, *Filomat*, vol.30, no.6, pp. 1567–1575, 2016.
[17] B. A. Frasin, M.K. Aouf, "New subclasses of bi-univalent functions", *Applied Mathematics Letters*, vol. 24, no. 2, pp. 1569–1573, 2011.