DOI: 10.24996/ijs.2025.66.11.33

ISSN: 0067-2904

The Fear and Allee-Effect on The Dynamics of A Food Chain Model Involving Additional Food to Predators

Inaam Ibrahim Shawka[1*], Azhar Abbas Majeed[2]

[1]*General Directorate of Educational Planning, Ministry of Education, Baghdad, Iraq*
[2]*Department of Mathematics, College of Science, University of Baghdad, Iraq*

Received: 9/5/2024 Accepted: 10/11/2024 Published: 30/11/2025

Abstract

In this paper, a food chain model with fear, Allee effect and linear harvesting in prey population with additional food and nonlinear harvesting in predator's population have been advanced and studied. Furthermore, it is assumed that the predators are feed on prey using Crowley-Martin functional response. The existence, uniqueness and boundedness of the solution have been discussed. The conditions of local and global stability have been found. The aim of this study is to determine the fear, Allee and additional food effect on the stability of the proposed system, it has been shown that they have an important effect in controlling the stability of the system. Finally, numerical simulation carried out by using Mathematica to verify our analytical results.

Keywords: prey-predator, fear, Allee-effect, additional food, harvesting, stability.

تأثير آلي والخوف على ديناميكيات نموذج سلسلة غذائية متضمنة غذاء أضافي للمفترسات

انعام ابراهيم شوكة[1]*, ازهار عباس مجيد[2]

[1]المديرية العامة للتخطيط التربوي , وزارة التربية , بغداد , العراق
[2]قسم الرياضيات , كلية العلوم , جامعة بغداد , بغداد , العراق

الخلاصة

في هذا البحث تم اقتراح ودراسة نموذج سلسلة غذائية مع الخوف, تاثيرآلي والحصاد غير الخطي في مجتمع الفريسة, وبوجود غذاء اضافي وحصاد غير خطي في مجتمع المفترسات اضافة لذالك افترض ان المفترسات تتغذى على الفرائس بدلالة استجابة وظيفية من نوع كراولي–مارتن , وجود ووحدانية وقيود الحل للنموذج نوقشت , شروط الاستقرارية المحلية والشاملة وجدت, الهدف من هذه الدراسة معرفة تأثير الخوف, آلي والغذاء الاضافي على استقرارية النموذج المقترح, وتبين انه لها تأثير مهم في السيطرة على استقرارية النظام , واخيراً نفذت المحاكاة العددية باستخدام برنامج الماثماتيكا للتأكد من نتائجنا التحليلية.

*Email: enaam.ib@gmail.com

1. Introduction

Applied mathematicians, theoretical ecologists and other scientists usually use mathematical models may help them to understand the dynamic behaviors of real-world problems, it is an effective method to study the dynamic behaviors of the interaction between the predator and prey, the first prey – predator model was proposed by Lotka [1] in 1925 and Volterra [2] in 1926.

At first, Lotka developed an application to study the dynamics of a predator-prey system after proposing a model for the study of plant species and herbivorous animal species. In his stead, Volterra concurrently examined the same model to account for the observed rise in the proportion of predatory fish caught during World War I. Volterra was interested in learning more about this situation because it seemed confusing that there had been less fishing effort during those years, many mathematicians and ecologists extended the model of Lotka-Volterra [3-5] and the references therein.

Actually, since the study of predator-prey models has long been a popular area in biomathematics, a large body of work has been done on the important subjects in ecology, our goal in this paper is to show a predator–prey model with the presence of fear effect with Crowely-Martin functional responses as well as with Allee effect, additional food and harvesting. On the other hand, many authors have focus in their study on the hunting cooperation, delay and prey refuge or study combination between more than one characteristic. As a suggestion for interested readers we propose the following references [6-12].

The predator's intake rate in relation to prey density is referred to in ecology as a functional response, Holling [13,14] proposed three types of functional response, and many authors [15-20] used Holling type I, type II, and type III in their study which is dependent on prey density only, there are other types of functional responses, for example, Holling type IV [21] functional response, Beddington-DeAngelis [22] and Crowely-Martin [23] which is dependent not only the prey but also on the predator, other models consider a functional response of ratio-dependent type [24].

In 1931 Allee [25] was the first who describe when the density of a prey population is too small, making it difficult for it to reproduce or survive, there are two types of Allee effect strong Allee effect and weak Allee effect and they are depending on the nature of density dependence at low densities. The weak Allee effect occurs when per capita growth rates are lower at low densities than they are at high densities, but growth rates are still positive in this scenario of a small population. As an illustration of the strong Allee effect, consider the following: a population's passage through the threshold curve, the per capita rate of population growth rapidly declines and then accelerates to negative until the population reaches extinction, as a suggestion for interested readers we propose the following references [26, 27].

Predation anxiety has been shown to reduce prey growth rates in addition to directly killing them over time, according to a number of experimental studies. The first study on the species fear effect was conducted by [28] on female song sparrows that were free-living and observed to react to the sound of predators by developing anti-predator defenses and fleeing their nests at the first sign of danger. By broad casting the predator sound of some song sparrow populations, while no predator sounds were heard for the others, the author studied this impact. During that period every nest in the population under investigation was protected from direct killing, which was obviously necessary because fear could have an adverse influence on reproductions as a result, they found that 40% reduction in the number of off springs produced per year by wild, in addition to if the prey feel fear of predators and act according this will make hunting more difficult for the predators. Therefore, many authors study the theoretical arguments regarding the impact of fear behaviors [29-33] and the references therein.

Recently, the study of ecosystems in which the predator has access to additional food sources in addition to its target prey has gained importance and is now a key area of research for mathematicians, ecologists, and biologists. This is due to the fact that providing more food has been shown to be very effective in both controlling invasive or harmful species and conserving endangered species [34-36]. Also, many authors proposed and studied the food chain model with the effect of additional food [37, 38] and the references therein.

Harvesting is a crucial area of study for ecologists and biologists, because unscientific and ongoing harvesting is driving thousands of species toward extinction. Harvesting is also beneficial to a nation's economic growth. Therefore, it is imperative that an appropriate optimal harvesting policy be investigated. A report has been submitted on combined harvesting which has an impact on the populations of prey and predators, [39]. After that, [40] studied the effects of predator harvesting of a two prey one predator system. Then [41] studied the effects of nonlinear predator harvesting in a complex predator–prey system. After that, [42] investigated the impact of harvesting for prey populations within a reserved area in the presence of toxicity. Therefore, many authors study the impact of harvesting like [43-45] and the references therein.

The goal of this paper is to study the food chain model in the presence of fear and Allee effect as well as additional food and non-linear harvesting with Crowely-Martin type of functional response it is observed that the additional food supplied to predators and Allee effect have an important role to control the instability of our system.

2. Mathematical model

In this paper, a model of (Prey, Predator and Top predator) with Crowley-Martin functional response has been proposed for study. Whose total population density at time T is denoted by $N(T), M_1(T), M_2(T)$, respectively.

$$\begin{aligned}
\frac{dN}{dT} &= bN\left(1-\frac{N}{K}\right)\left(\frac{1}{fM_1+1}\right)\left(\frac{N}{N+A}\right)-\frac{B_1NM_1}{(1+n_1N)(1+n_2M_1)}-h_1N,\\
\frac{dM_1}{dT} &= \frac{l_1NM_1}{(1+n_1N)(1+n_2M_1)}-\frac{B_2M_1M_2}{(1+n_3M_1)(1+n_4M_2)}+(1-\eta)\beta M_1-d_1M_1-h_2M_1^2, \qquad (1)\\
\frac{dM_2}{dT} &= \frac{l_2M_1M_2}{(1+n_3M_1)(1+n_4M_2)}+\eta\beta M_2-d_2M_2-h_3M_2^2\,.
\end{aligned}$$

With the following initial condition $N(0) \geq 0$, $M_1(0) \geq 0$ and $M_2(0) \geq 0$. While the biological meaning of the parameters in system (1) illustrated in Table 1:

Table 1: Biological meaning of the parameters of system 1

Parameters	Biological meaning
$b > 0$	The intrinsic growth rate of the prey population
$K > 0$	The carrying capacity of the prey population
$f > 0$	The fear rate of the prey species from predator
$A > 0$	The Allee effect parameter
$B_i > 0, i = 1,2$	The attack rate of the prey and the predator respectively
$l_i > 0, l_i \le B_i$	The conversion rate of food to the predator and top predator respectively
$n_j > 0, j = 1,3$	The handling time of prey and predator respectively
$n_m > 0, m = 2,4$	The magnitude of interference among predator and top predator respectively
$\beta > 0$	The additional food for top predator
$(1-\eta)\beta, 0 < \eta < 1$	The benefited of predator by some portion of the additional food of top predator
$h_r > 0, r = 1,2,3$	The harvesting rates of prey, predator and top predator respectively
$d_i > 0$	The natural death rates of predator and top predator respectively

It is easy to verify that all the interaction functions of system (1) are continuous and have continuous partial derivatives on R_+^3 with respect to dependent variables N, M_1 and M_2. Accordingly, they are Lipschitzian functions and hence system (1) has a unique solution for each non-negative initial condition. Further the boundedness of the system is shown in the following theorem.

Theorem 2.1: All the solutions of system (1) which initiate in R_+^3 are uniformly bounded.
Proof: Let $(N(t), M_1(t), M_2(t))$ be any solution of the system (1) with non-negative initial condition$(N(0), M_1(0), M_2(0)) \in R_+^3$.
Assume that $W(t) = N(t) + M_1(t) + M_2(t)$ then taken the time derivative of $W(t)$ along the solution of the system (1), we get

$$\frac{dW}{dt} = bN\left(1-\frac{N}{K}\right)\left(\frac{1}{fM_1+1}\right)\left(\frac{N}{N+A}\right) - \frac{B_1NM_1}{(1+n_1N)(1+n_2M_1)} - h_1N - \frac{B_2M_1M_2}{(1+n_3M_1)(1+n_4M_2)} + (1-\eta)\beta M_1 - d_1M_1 - h_2M_1^2 + \frac{l_2M_1M_2}{(1+n_3M_1)(1+n_4M_2)} + \eta\beta M_2 - d_2M_2 - h_3M_2^2.$$

Now, from the biological point of view, always $l_i < B_i$, $i = 1,2$ hence it is obtained that

$$\frac{dW}{dt} \le bN\left(1-\frac{N}{K}\right) + (1-\eta)\beta M_1 - h_2 M_1^2 + \eta\beta M_2 - h_3M_2^2 - h_1N - d_1 M_1 - d_2 M_2.$$

Thus, using comparison theorem, we have

$$\frac{dW}{dt} \le bN\left(1-\frac{N}{K}\right) + (1-\eta)\beta M_1\left(1-\frac{h_2M_1}{(1-\eta)\beta}\right) + \eta\beta M_2\left(1-\frac{h_3M_2}{\eta\beta}\right) - HW,$$

where $H = \min\{h_1, d_1, d_2\}$.
Now since the function $g(N) = bN\left(1-\frac{N}{K}\right)$ is logistic function with respect to N and hence it is bounded above by the constant $\frac{bK}{4}$.
Also, since the function $g_1(M_1) = (1-\eta)\beta M_1\left(1-\frac{h_2M_1}{(1-\eta)\beta}\right)$ is logistic function with respect to M_1 and hence it is bounded above by the constant $\frac{(1-\eta)^2\beta^2}{4h_2}$.
Finally, since the function $g_2(M_2) = \eta\beta M_2\left(1-\frac{h_3M_2}{\eta\beta}\right)$ is logistic function with respect to M_2 and hence it is bounded above by the constant $= \frac{(\eta\beta)^2}{4h_3}$, it is observed that

$$\frac{dW}{dt} - HW \le L, \text{ where } L = \left(\frac{bK}{4} + \frac{(1-\eta)^2\beta^2}{4h_2} + \frac{\eta^2\beta^2}{4h_3}\right).$$

Then by solving the above differential inequality it is obtained that

$$W(t) \le \frac{L}{H} + \left(W_0 - \frac{L}{H}\right) e^{-Ht}.$$

Then,

$$\lim_{t\to\infty} W(t) \le \frac{L}{H}.$$

So, $0 \le W(t) \le \frac{L}{H}$. That the solutions are uniformly bounded.

3. Existence of equilibrium points (EPs):

In this section, the existence of all possible (EPs) of system (1) has been discussed. Notice that, system (1) has ten (EPs):

- The trivial (EP) $E_0 = (0, 0, 0)$. Cleary, it is always exist.
- The (EPs) $E_1 = (\bar{N}, 0, 0)$ and $E_2 = (\bar{\bar{N}}, 0, 0)$ exists if the following equation has positive solution

$$bN^2 - K(b - h_1)N + h_1KA = 0. \tag{2}$$

Clearly, according to the Descartes's rule, either Equation(2) has no root or, there are two roots, if the following condition holds

$$b > h_1 \tag{3}$$

- The (EP) $E_3 = (0, \widehat{M}_1, 0)$, where $\widehat{M}_1 = \frac{(1-\eta)\beta - d_1}{h_2}$ exists if the following condition holds

$$(1-\eta)\beta > d_1 \tag{4}$$

- The (EP) $E_4 = (0, 0, \breve{M}_2)$, where $\breve{M}_2 = \frac{\eta\beta - d_2}{h_3}$ exists if the following condition holds

$$\eta\beta > d_2 \tag{5}$$

- The (EP) $E_5 = (\widetilde{N}, \widetilde{M}_1, 0)$ exists if and only if the following equations have a positive solution

$$b\left(1 - \frac{N}{K}\right)\left(\frac{1}{fM_1+1}\right)\left(\frac{N}{N+A}\right) - \frac{B_1M_1}{(1+n_1N)(1+n_2M_1)} - h_1 = 0, \tag{6.1}$$

$$\frac{l_1N}{(1+n_1N)(1+n_2M_1)} - \frac{B_2M_2}{(1+n_3M_1)(1+n_4M_2)} + (1-\eta)\beta - d_1 - h_2M_1 = 0. \tag{6.2}$$

From Equation(6.2), we have

$$N = \frac{(1+n_2M_1)(h_2M_1 + d_1 - (1-\eta)\beta)}{l_1 - n_1(1+n_2M_1)(h_2M_1 + d_1 - (1-\eta)\beta)}. \tag{6.3}$$

Now, by substituting Equation (6.3) in Equation (6.1), we obtain the following equation

$$\gamma_1M_1^8 + \gamma_2M_1^7 + \gamma_3M_1^6 + \gamma_4M_1^5 + \gamma_5M_1^4 + \gamma_6M_1^3 + \gamma_7M_1^2 + \gamma_8M_1 + \gamma_9 = 0,$$

where

$$\gamma_1 = fh_2n_2\left((1+n_1A)(h_2^2n_2^3 + S_4)S_8 + Kh_2^2n_1n_2^3(h_1n_1 + A)\right),$$

$$\gamma_2 = h_2n_2\Big[h_2n_1n_2^2\left(h_2\left[S_6 - \frac{bn_2 - Kfh_1n_1}{n_2}\right] - 3Kfh_1n_1S_3 + Kh_2n_2A(S_8 + h_1f)\right) - fh_2S_3[KAn_1(n_2^2+2) - 3n_2^2(1+n_1A)S_8] - (h_2^2n_2^3 + S_4)S_7\Big],$$

$$\gamma_3 = h_2n_2\Bigg[h_2n_1n_2^2\left(S_6 - \frac{b(h_2 - S_3)}{h_2}\right) - 3n_2S_3\left(h_2n_2S_7 + n_1(S_6n_2 + kfh_1n_1)\right) - fn_2(1+n_1A)(2S_3^2 - h_2S_1)S_8 - K\left[fh_1n_1n_2\left(S_5 - \frac{n_1[h_2(h_2 - 4n_2S_1) + n_2^2S_1^2 - 2S_3]}{h_2n_2}\right) + Ah_2^2h_1n_1n_2^3 + fA\left(S_5 - \frac{n_1(S_1 + 2S_3^2)}{h_2n_2^2}\right) + Ah_2n_1(n_2^2+2)(S_8 + fh_1)S_3\right]\Bigg],$$

$\gamma_4 = 3h_2n_1n_2^2\left[b\left(\frac{h_2S_3-h_2^2-n_2^2S_1^2}{3h_2n_2}+S_1\right)-h_2S_3S_6\right]-f(1+n_1A)S_3S_8[2h_2n_2(n_1l_1+2n_2S_1)-S_4]+h_2n_2^2S_7(2S_3^2-h_2n_2S_1)-h_2n_2(n_2S_6+Kfh_1n_1)[S_5-n_1(h_2(h_2-4n_2S_1)+n_2^2S_1-2S_3)]-K\left[3h_2^3n_2^3\big(b-h_1(1+n_1A)\big)S_3+h_2n_2A(S_8+h_1f)\left(S_3^2-\frac{h_2n_2^2S_4-n_1S_4}{2n_1}\right)+h_2^2h_1n_1n_2A(n_2^2+2)S_3+fA\left(\frac{n_1S_1S_4-2h_2n_2\big(n_1S_3(n_1l_1+n_2S_1)\big)}{2h_2n_2^2}+S_5\right)\right]-Kf\left(nS_3\left(\frac{h_2(h_2-2n_2S_3S_4)+n_2^2S_1^2}{2h_2n_2}-S_1\right)\right),$

$\gamma_5 = -bh_2n_1n_2\big(4h_2^2+n_2^4S_1^4-n_2S_1(17h_2+6n_2S_1+1)\big)-S_6h_2n_2\left(\frac{\big(4h_2n_2S_1-n_1(h_2^2+n_2^2S_1^2-2S_3)\big)}{h_2n_2}-(n_2S_6+Kfh_1n_1)S_3\big(n_1(h_2^2+n_2^2S_1^2)-2h_2n_2(n_1S_1+S_5)\big)-K\left[fh_1n_1n_2\left(n_1n_2h_2S_1^3+S_5\big(h_2^2+n_2^2S_1^2+2S_1(2h_2n_2+n_1S_3)\big)\right)-(b-h_1(1+n_1)A)h_2n_2(2n_2S_3^2-h_2n_2^2S_1)+S_3S_7(2)+fS_8(1+n_1A)(S_1S_4+2(n_1l_1+n_2S_1)S_3^2+h_2n_2S_2)-KA\big(f(n_1(2(n_1l_1+n_2S_1)S_3^2-h_2n_2S_2)+S_4S_5)+(S_8+h_1f)\big(n_1S_1S_4-2h_2n_2(n_1S_3(n_1l_1+n_2S_1)+n_2S_5)\big)+h_1h_2n_2\big(h_2n_2^2S_5-(n_1S_4+2n_1S_4+2n_1S_3^2)\big)\right],$

$\gamma_6 = -bn_1\left[h_2\left(\frac{h_2^2-9n_2^2S_1^2}{n_2(8h_2+1)}-S_1\right)-n_2^3S_1^3+n_2\left(\frac{-3\big(h_2^2+n_2^2S_1^2\big)}{7h_2}+S_1\right)\right]-S_3S_6\left[\frac{n_1\big(h_2^2+n_2^2S_1^2\big)+2h_2n_2n_1S_1}{2h_2n_2}-S_5\right]-KS_3\left[fh_1n_1n_2S_1(l_1+S_5)-\big(b-h_1(1+n_1A)\big)\Big(2h_2n_2\big((n_1l_1+n_2S_1)+n_2S_1\big)\Big)\right]-KA\left[h_1\left(\left(\frac{n_1S_1S_4-2h_2n_2\big(n_1S_3(n_1l_1+n_2S_1)\big)}{n_2}+S_5\right)+(n_1(2(n_1l_1+n_2S_1)S_3^2-h_2n_2S_2)+S_4S_5)(S_8+h_1f)+fA(2(n_1l_1+n_2S_1)S_5+n_1S_2)S_3\right]-[S_1S_4+2(n_1l_1+n_2S_1)S_3^2+h_2n_2S_2]S_7-[S_6n_2+Kfh_1n_1][n_1n_2h_2S_1^2+S_5(h_2+4h_2n_2S_1+n_2^2S_1^2+2n_1S_1S_3)]+2f[n_1l_1+n_2S_1][1+n_1A]S_1S_3S_8$

$\gamma_7 = S_1\left[bn_1\left(\frac{3\big(h_2^2+n_2^2S_1^2\big)}{n_2(7h_2-S_3)}-S_1\right)-S_3\big((n_2S_6+Kfh_1n_1)(l_1+S_5)+2(n_1l_1+n_2S_1)S_7\big)+f(1+n_1A)S_2S_8-2n_1S_3S_6\right]-S_6\big(n_1n_2h_2S_1^2+S_5(h_2^2+4h_2n_2S_1+n_2^2S_1^2)\big)-K\left[\big(b-h_1(1+n_1A)\big)\left(S_3^2+\frac{S_1S_4+S_2(h_2n_2+fAS_5)}{2l_1(n_1+n_2)}\right)+A[2h_1n_1l_1(n_1+n_2)S_3^2+fS_2S_5-(h_1h_2n_2S_2-S_4S_5)+(2(n_1l_1+n_2S_1)S_5+n_1S_2)S_3S_8]\right]$

$\gamma_8 = S_1[(n_2S_6+kfh_1n_1)S_1S_5-(l_1+S_5)S_3S_6-2kS_3\big(b-h_1(1+n_1A)\big)(n_1l_1+n_2S_1)-(bh_2n_1S_1+S_2S_7)]-\left[\frac{n_1h_1KAS_2}{(S_8+h_1f)S_2-2h_1KA(n_1l_1+n_2S_1)}-S_5\right]$

$\gamma_9 = -\left[S_1^2(bn_1S_1+S_5S_6)+KS_2\big((b-h_1(1+n_1A))S_1+h_1AS_5\big)\right],$

with

$S_1 = (1-n)\beta-d_1,$

$S_2 = l_1^2+2l_1n_1S_1+n_2S_1^2,$

$S_3 = n_2S_1-h_2,$

$S_4 = h_2n_2(h_2-2l_1n_1)+n_2^2S_1(S_3-3h_2),$

$S_5 = l_1+n_1S_1,$

$S_6 = b-kn_1(\mathrm{b}-h_1),$

$S_7 = kn_2\big(b - h_1(1 + n_1A)\big) - B_1 + h_1f(1 + n_1A),$
$S_8 = B_1 + h_1n_1 > 0.$
Which has positive solution if the next conditions with conditions (3 and 4) are hold

$$(1 - n)\beta < d_1 + \frac{h_2}{n_2}, \quad (6.4)$$
$$\text{b} > \text{Max}\{h_1(1 + n_1A),\ Kn_1(\text{b} - h_1)\}, \quad (6.5)$$
$$l_1 > \frac{h_2}{2n_1}, \quad (6.6)$$
$$Kn_2(b - h_1(1 + n_1A) + h_1f(1 + n_1A) > B_1, \quad (6.7)$$
$$bn_2 > kfh_1n_1, \quad (6.8)$$
$$KAn_1(n_2^2 + 2) > 3n_2^2(1 + n_1A)S_8, \quad (6.9)$$
$$h_2^2n_2^3 + S_4 < 0, \quad (6.10)$$
$$h_2 > 2n_2S_3S_4, \quad (6.11)$$
$$S_6 > \text{Max}\left\{\frac{b(h_2 - S_3)}{h_2}, \frac{bn_2 - kfh_1n_1}{n_2}\right\}, \quad (6.12)$$
$$\text{Max}\left\{h_2n_2S_1,\ \frac{h_2n_2^2S_5 - n_1S_4}{2n_1}\right\} < S_3^2 < \text{Min}\left\{\frac{h_2S_1}{2}, \frac{-\big(S_1S_4 + S_2(h_2n_2 + fAS_5)\big)}{2l_1(n_1 + n_2)}\right\}, \quad (6.13)$$
$$\text{Max}\left\{\frac{n_1S_1S_4 - 2h_2n_2n_1S_3(n_1l_1 + n_2S_1)}{2h_2n_2^2}, \frac{n_1h_1KAS_2}{(S_8 + h_1f)S_2 - 2h_1KA(n_1l_1 + n_2S_1)}\right\} < S_5 <$$
$$\text{Min}\left\{\frac{n_1\big(h_2(h_2 - 4n_2S_1) + n_2^2S_1^2 - 2S_3\big)}{h_2n_2}, \frac{n_1(S_1 + 2S_3^2)}{h_2n_2^2}, n_1\big(h_2(h_2 - 4n_2S_1) + n_2^2S_1 - 2S_3\big),\right.$$
$$\left.\frac{2h_2n_2\big(n_1S_3(n_1l_1 + n_2S_1)\big) - n_1S_1S_4}{n_2}, \frac{n_1\big(h_2^2 + n_2^2S_1^2 - 2h_2n_2S_1\big)}{2h_2n_2}\right\}, \quad (6.14)$$
$$\text{Max}\left\{\frac{h_2(h_2 - S_3) + n_2^2S_1^2}{3n_2h_2}, \frac{h_2(h_2 - 2n_2S_3S_4) + n_2^2S_1^2}{2n_2h_2}, \frac{h_2^2 - 9n_2^2S_1^2}{n_2(8h_2 + 1)}\right\} < S_1 <$$
$$\text{Min}\left\{\frac{h_2}{3n_2}, \frac{3\big(h_2^2 + n_2^2S_1^2\big)}{7h_2}, \frac{3\big(h_2^2 + n_2^2S_1^2\big)}{n_2(7h_2 - S_3)}\right\} \quad (6.15)$$
$$4h_2^2 + n_2^2S_1^4 > n_2S_1(17h_2 + 6n_2S_1 + 1), \quad (6.16)$$
$$fh_1n_1n_2S_1(l_1 + S_5) > \big(b - h_1(1 + n_1A)\big)\Big(2h_2n_2\big((n_1l_1 + n_2S_1) + n_2S_1\big)\Big), \quad (6.17)$$
$$bn_1\big(3(h_2^2 + n_2^2S_1^2) - n_2S_1(7h_2 - S_3)\big) - S_3\big((n_2S_6 + Kfh_1n_1)(l_1 + S_5) + 2(n_1l_1 + n_2S_1)S_7\big) + f(1 + n_1A)S_2S_8 - 2n_1S_3S_6 > S_6\big(n_1n_2h_2S_1^2 + S_5(h_2^2 + 4h_2n_2S_1 + n_2^2S_1^2)\big), \quad (6.18)$$
$$2h_1n_1l_1(n_1 + n_2)S_3^2 + fS_2S_5 < h_1h_2n_2S_2 - S_4S_5 - (2S_5(n_1l_1 + n_2S_1) + n_1S_2)\ S_3S_8, \quad (6.19)$$
$$(n_2S_6 + Kf_1n_1)S_1S_5 - (l_1 + S_5)S_3S_6 - 2kS_3\big(b - h_1(1 + n_1A)\big)\ (n_1l_1 + n_2S_1) > bh_1n_1S_1 + S_2S_7. \quad (6.20)$$

So, the (EP) $E_5 = \big(\tilde{N}, \tilde{M}_1, 0\big)$ where $\tilde{N} = N\big(\tilde{M}_1\big)$ exist if in addition to conditions (3 , 4, 6.4 − 6.20) the following conditions hold

$$(1 - \eta)\beta < h_2\tilde{M}_1 + d_1, \quad (6.21)$$
$$l_1 > n_1\big(1 + n_2\tilde{M}_1\big)\big(h_2\tilde{M}_1 + d_1 - (1 - \eta)\beta\big). \quad (6.22)$$

- The (EPs) $E_6 = \big(\bar{N}, 0, \breve{M}_2\big)$ and $E_7 = \big(\bar{\bar{N}}, 0, \breve{M}_2\big)$ exist if and only if the next equations have a positive solution

$$b\left(1 - \frac{N}{K}\right)\left(\frac{1}{fM_1 + 1}\right)\left(\frac{N}{N + A}\right) - h_1 = 0, \quad (7.1)$$
$$\eta\beta - d_2 - h_3M_2 = 0. \quad (7.2)$$

From Equation (7.1), we have Equation (2) which has no root or, has two roots if condition (3) holds while from Equation (7.2), we have $\breve{M}_2 = \frac{\eta\beta - d_2}{h_3}$ which is positive if condition (5) holds

- The (EP) $E_8 = (0\,, \dot{M}_1, \dot{M}_2\,)$ exists if and only if the next equations have a positive solution

$$\frac{-B_2 M_2}{(1+n_3 M_1)(1+n_4 M_2)} + (1-\ \eta)\beta - d_1 - h_2 M_1 = 0, \quad (8.1)$$

$$\frac{l_2 M_1}{(1+n_3 M_1)(1+n_4 M_2)} + \eta\beta - d_2 - h_3 M_2 = 0, \quad (8.2)$$

from Equation(8.2), we have

$$M_1 = \frac{(1+n_4 M_2)(h_3 M_2 - (\eta\beta - d_2))}{l_2 - n_3(1+n_4 M_2)(h_3 M_2 - (\eta\beta - d_2))}, \quad (8.3)$$

Now, by substituting Equation(8.3) in Equation(8.2)we have

$$\alpha_1 M_2^5 + \alpha_2 M_2^4 + \alpha_3 M_2^3 + \alpha_4 M_2^2 + \alpha_5 M_2 + \alpha_6 = 0, \quad (8.4)$$

where

$$\alpha_1 = -n_3 n_4^3 h_3^2 [n_3((1-\eta)\beta - d_1)],$$

$$\alpha_2 = n_3 n_4 \Big[n_3 n_4 h_3^2 \Big((1+n_4)((1-\eta)\beta - d_1) - B_2\Big) + \big(n_3((1-\eta)\beta - d_1) - h_2\big) \big((n_4(\eta\beta - d_2) - h_3)(n_4 + h_3)\big) + n_4 h_2 (2n_4(\eta\beta - d_2) - 3h_3) - n_3 h_3\Big],$$

$$\alpha_3 = n_3 n_4 \big[(n_4(\eta\beta - d_2) - h_3)\big[(n_3((1-\eta)\beta - d_1) - h_2)\big((n_4(\eta\beta - d_2) - h_3 - n_3 h_3) + h_2 h_3 - 2n_3 h_3\big((1-\eta)\beta - d_1\big) - n_4 h_2(\eta\beta - d_2)\big] + 2n_4 h_3 h_2(\eta\beta - d_2)\big] - n_4 h_3\big[\big(n_3\big((1-\eta)\beta - d_1\big) - h_2\big)\big((2n_3 n_4(\eta\beta - d_2) - h_3) + n_4(l_2 + n_3(\eta\beta - d_2))\big)\big],$$

$$\alpha_4 = n_3\big((1-\eta)\beta - d_1\big)\big[n_3(n_4(\eta\beta - d_2) - h_3)^2 - 2n_4 h_3\big(l_2 + n_3(\eta\beta - d_2)\big)\big] + n_3\big(n_4\big((1-\eta)\beta - d_1\big) - B_2\big)\big[n_4 h_3\big(l_2 + n_3(\eta\beta - d_2)\big) + n_3(n_4(\eta\beta - d_2) - h_3)\big] - n_4\big(n_3\big((1-\eta)\beta - d_1\big) - h_2\big)\big[n_3^2(2(\eta\beta - d_2) - h_3)(n_4(\eta\beta - d_2) - h_3) + \big((n_n(\eta\beta - d_2) - h_3) - n_3 h_3\big)\big(l_2 + n_3(\eta\beta - d_2)\big)\big] - n_3 h_3(n_4(\eta\beta - d_2) - h_3)\big[\big((n_4(\eta\beta - d_2) - h_3) + 2n_4\big) + 2n_3 h_2 n_4 h_3(\eta\beta - d_2)\big],$$

$$\alpha_5 = \big(l_2 + n_3(\eta\beta - d_2)\big)\Big[n_3(n_4(\eta\beta - d_2) - h_3)\Big((2+n_4)\big((1-\eta)\beta - d_1\big) - B_2\Big) - n_4\big((n_3(1-\eta)\beta - d_1) - h_2\big)(2(\eta\beta - d_2) - h_3)\Big] - n_3(n_4(\eta\beta - d_2) - h_3)\big[(\eta\beta - d_2)\big(n_3\big((1-\eta)\beta - d_1\big) - h_2\big) - 2h_2(\eta\beta - d_2)\big] - n_3 n_4 h_2(\eta\beta - d_2)^2,$$

$$\alpha_6 = \big(l_2 + n_3(\eta\beta - d_2)\big)\Big[\big(l_2 + n_3(\eta\beta - d_2)\big)\Big((1+n_4)\big((1-\eta)\beta - d_1\big) - B_2\Big) - (\eta\beta - d_2)\big(n_3\big((1-\eta)\beta - d_1\big) + h_2\big)\Big] - n_3 h_2(\eta\beta - d_2)^2.$$

Clearly, by Descartes's rule, Equation(8.4) has a unique positive root, if in addition to condition (4 and 5) the following conditions hold

$$\text{Max}\left\{d_1 + \frac{h_2}{n_3}, d_1 + \frac{B_2}{(1+n_4)}\right\} < (1-\eta)\beta < d_1 + \frac{B_2}{n_4}, \quad (8.5)$$

$$d_2 + \frac{h_3}{2} < \eta\beta < d_2 + \frac{h_3}{n_4}, \quad (8.6)$$

$$n_3 n_4 h_3^2 \Big((1+n_4)\big((1-\eta)\beta - d_1\big) - B_2\Big) < n_3 h_3 - \big(n_3\big((1-\eta)\beta - d_1\big) - h_2\big) \big((n_4(\eta\beta - d_2) - h_3)(n_4 + h_3)\big) + n_4 h_2(2n_4(\eta\beta - d_2) - 3h_3), \quad (8.7)$$

$$n_3 n_4\big[(n_4(\eta\beta - d_2) - h_3)\big[(n_3((1-\eta)\beta - d_1) - h_2)\big((n_4(\eta\beta - d_2) - h_3 - n_3 h_3) + h_2 h_3 - 2n_3 h_3\big((1-\eta)\beta - d_1\big) - n_4 h_2(\eta\beta - d_2)\big] + 2n_4 h_3 h_2(\eta\beta - d_2)\big] >$$

$n_4h_3\big[\big(n_3\big((1-\eta)\beta-d_1\big)-h_2\big)\big((2n_3n_4(\eta\beta-d_2)-h_3)+n_4(l_2+n_3(\eta\beta-d_2))\big)\big],$ (8.8)

$n_3(n_4(\eta\beta-d_2)-h_3)^2>2n_4h_3\big(l_2+n_3(\eta\beta-d_2)\big),$ (8.9)

$n_4h_3\big(l_2+n_3(\eta\beta-d_2)\big)<-n_3(n_4(\eta\beta-d_2)-h_3),$ (8.10)

$n_3^2(2(\eta\beta-d_2)-h_3)(n_4(\eta\beta-d_2)-h_3)<-\big((n_4(\eta\beta-d_2)-h_3)-n_3h_3\big)\big(l_2+n_3(\eta\beta-d_2)\big),$ (8.11)

$2n_4(n_3h_2h_3(\eta\beta-d_2)+1)>-\big((n_4(\eta\beta-d_2)-h_3)\big),$ (8.12)

$(\eta\beta-d_2)\big(n_3\big((1-\eta)\beta-d_1\big)-h_2\big)>2h_2(\eta\beta-d_2),$ (8.13)

$\big(l_2+n_3(\eta\beta-d_2)\big)\Big[n_3(n_4(\eta\beta-d_2)-h_3)\Big((2+n_4)\big((1-\eta)\beta-d_1\big)-B_2\Big)-n_4\big((n_3(1-\eta)\beta-d_1)-h_2\big)(2(\eta\beta-d_2)-h_3)\Big]-n_3n_4h_2(\eta\beta-d_2)^2<n_3(n_4(\eta\beta-d_2)-h_3)\big[(\eta\beta-d_2)\big(n_3\big((1-\eta)\beta-d_1\big)-h_2\big)-2h_2(\eta\beta-d_2)\big],$ (8.14)

$\big(l_2+n_3(\eta\beta-d_2)\big)\Big((1+n_4)\big((1-\eta)\beta-d_1\big)-B_2\Big)>(\eta\beta-d_2)\big(n_3\big((1-\eta)\beta-d_1\big)+h_2\big)+\frac{n_3h_2(\eta\beta-d_2)^2}{(l_2+n_3(\eta\beta-d_2))}.$ (8.15)

So, the (EP) $E_8=\big(0\,,\dot{M}_1,\dot{M}_2\,\big)$ where $\dot{M}_1=M_1\big(\dot{M}_2\big)$ exists, if in addition to conditions $(4,5\text{ and }8.5-8.15)$, the following conditions hold

$\eta\beta<h_3\dot{M}_2+d_2,$ (8.16)

$l_2>n_3\big(1+n_4\dot{M}_2\big)\big(h_3\dot{M}_2-(\eta\beta-d_2)\big),$ (8.17)

- The (EP) $E_9=(N^*,M_1^*,M_2^*)$ exists if and only if the next equations have a positive solution

$b\Big(1-\frac{N}{K}\Big)\Big(\frac{1}{fM_1+1}\Big)\Big(\frac{N}{N+A}\Big)-\frac{B_1M_1}{(1+n_1N)(1+n_2M_1)}-h_1=0,$ (9.1)

$\frac{l_1N}{(1+n_1N)(1+n_2M_1)}-\frac{B_2M_2}{(1+n_3M_1)(1+n_4M_2)}+(1-\eta)\beta-d_1-h_2M_1=0,$ (9.2)

$\frac{l_2M_1}{(1+n_3M_1)(1+n_4M_2)}+\eta\beta-d_2-h_3M_2=0,$ (9.3)

from Equation(9.3), we have

$$M_1=\frac{(1+n_4M_2)(h_3M_2-(\eta\beta-d_2))}{l_2-n_3(1+n_4M_2)(h_3M_2-(\eta\beta-d_2))}. \quad (9.4)$$

Now, by substituting Equation(9.4) in both Equation(9.1) and Equation(9.2), we have

$$L_1(N,M_2)=\big[-bn_1N^3+(Kn_1(b-h_1)-b)N^2+K\big(b-h_1(1+n_1A)\big)N-Kh_1A\big](l_2+n_3P_2)^2-\Big[bn_1n_2N^3-(Kn_1n_2(b-h_1)-bn_2-Kfh_1n_1)N^2-K\Big(n_2\big(b-h_1(1+n_1A)\big)-\big(B_1+h_1f(1+n_1A)\big)N-KA(B_1+h_1f+h_1n_2)\Big)\Big]\big((l_2+n_3P_2)P_2\big)-Kf[h_1n_1n_2N^2+(B_1+h_1n_2)(N+A)](-P_2)^2=0, \quad (9.5)$$

$$L_2(N,M_2)=\big[P+(l_1+n_1P)N+\big((1+n_1N)P_3+l_1n_4N\big)M_2\big](l_2+n_3P_2)^2-\Big[\big(n_2P+n_3(P-h_2M_2)\big)+\big(n_3\big(l_1+n_1(P-h_2M_2)\big)+n_1n_2P\big)N+\Big(n_2\big(P_3+n_3n_4(P-h_2M_2)\big)\Big)M_2-\big(n_1\big(n_2P_3+n_3n_4(P-h_2M_2)\big)+l_2n_3n_4\big)NM_2\Big](l_2+n_3P_2)P_2+[n_3n_4(P-h_2M_2)(1+n_1N)(1+n_4M_2)](-P_2)^2=0, \quad (9.6)$$

where

$P = (1-\eta)\beta - d_1,$

$P_1 = \eta\beta - d_2,$

$P_2 = (1 + n_4 M_2)\big((\eta\beta - d_2) - h_3 M_2\big),$

$P_3 = n_4\big(((1-\eta)\beta - d_1) - h_2 M_2\big) - (\beta + h_2).$

Now from Equation(9.5), we notice that, when $M_2 \to 0$, then $N \to N^{*1}$ is a positive root of the equation that follows

$$f(N) = \sigma_1 N^3 + \sigma_2 N^2 + \sigma_3 N + \sigma_4 = 0 \ , \quad (9.7)$$

where

$\sigma_1 = -bn_1\big(l_2 + n_3(\eta\beta - d_2)\big)\big(l_2 + (\eta\beta - d_2)(n_3 + n_2)\big),$

$\sigma_2 = \big(l_2 + (\eta\beta - d_2)(n_3 - n_2)\big)\Big((Kn_1(b - h_1) - b)\big(l_2 + n_3(\eta\beta - d_2)\big) - Kfh_1n_1(\eta\beta - d_2)\Big),$

$\sigma_3 = K\Big(\big(b - h_1(1 + n_1A)\big)\big(l_2 + n_3(\eta\beta - d_2)\big)\big(l_2 + (\eta\beta - d_2)(n_3 - n_2)\big) + (\eta\beta - d_2)\big(B_1 + h_1f(1 + n_1A)\big)\big(l_2 + (\eta\beta - d_2)\big)\big(n_3 - f(B_1 + h_1n_1)\big)\Big),$

$\sigma_4 = KA\Big(\big(l_2 + (\eta\beta - d_2)(n_3 - f)\big)\Big((\eta\beta - d_2)\big(B_1 - h_1(n_3 - n_1)\big) - h_1 l_2\Big)\Big).$

Moreover, from Equation (9.5) we have $\frac{dN}{dM_2} = -\left(\left(\frac{\partial L_1}{\partial M_2}\right) / \left(\frac{\partial L_1}{\partial N}\right)\right)$. So , $\frac{dN}{dM_2} < 0$ if one of the following sets of conditions hold

$$\left(\frac{\partial L_1}{\partial M_2}\right) > 0, \left(\frac{\partial L_1}{\partial N}\right) > 0 \ \ OR \ \ \left(\frac{\partial L_1}{\partial M_2}\right) < 0, \left(\frac{\partial L_1}{\partial N}\right) < 0. \quad (9.8)$$

As well as, from Equation(9.6) we notice that, when $M_2 \to 0$, then $N \to N^{*2}$,
where $N^{*2} = \frac{P(l_2 + P_1(n_3 - n_2))}{l_1(l_2 + n_3P_1) + n_1P(l_2 - n_2P_1))}$.

Furthermore, from Equation(9.6), we have $\frac{dN}{dM_2} = -\left(\left(\frac{\partial L_2}{\partial M_2}\right) / \left(\frac{\partial L_2}{\partial N}\right)\right)$. So, $\frac{dN}{dM_2} > 0$ if one of the following sets of conditions hold

$$\left(\frac{\partial L_2}{\partial M_2}\right) > 0, \left(\frac{\partial L_2}{\partial N}\right) < 0 \ \ OR \ \ \left(\frac{\partial L_2}{\partial M_2}\right) < 0, \left(\frac{\partial L_2}{\partial N}\right) > 0. \quad (9.9)$$

Then the two isoclines (9.5) and (9.6) intersect at a unique positive point (N^*, M_2^*) in the $Int.R_+^2$ of NM_2- plane .

So, the (EP) $E_9 = (N^*, M_1^*, M_2^*)$ where $M_1^* = M_1(M_2^*)$ exists if in addition to conditions (3, 5, 6.5) the following conditions hold

$$f > n_3 > \text{Max}\ \{n_1, n_2, f(B_1 + h_1n_1)\} \quad (9.10)$$

$$B_1 < h_1(n_3 - n_2), \quad (9.11)$$

$$(1-n)\beta < d_1 + \frac{l_2}{n_2}\ , \quad (9.12)$$

$$N^{*1} > N^{*2}, \quad (9.13)$$

$$h_3M_2^* + d_2 > \eta\beta, \quad (9.14)$$

$$l_2 > n_3(1 + n_4M_2^*)(h_3M_2^* - (\eta\beta - d_2)). \quad (9.15)$$

4. Local stability analysis (LSA):

In this section, the (LSA) around each of the equilibrium points is discussed through computing the Jacobian matrix (JM), $J(N, M_1, M_2)$ and found the eigenvalues of system (1) at each of them.

Note that, we used the symbols λ_{iN} , λ_{iM_1} and λ_{iM_2} to represent the eigenvalues of (JM) J_i; $i = 0, 1, \ldots 9$ that describe the dynamics in N-direction , M_1-direction and M_2-direction respectively, where the Jacobian matrix $J(N, M_1, M_2)$ of the system (1) at each of them can be written as $J = [a_{ij}]_{3\times3}$,

$$a_{11} = \frac{bN(N(K-2N)+A(2K-3N))}{K(fM_1+1)(N+A)^2} - \frac{B_1M_1}{(1+n_1N)^2(1+n_2M_1)} - h_1,$$
$$a_{12} = \frac{fbN^2(N-K)}{K(fM_1+1)^2(N+A)} - \frac{B_1N}{(1+n_1N)(1+n_2M_1)^2}, \quad a_{13} = 0, \quad a_{21} = \frac{l_1M_1}{(1+n_1N)^2(1+n_2M_1)},$$
$$a_{22} = \frac{l_1N}{(1+n_1N)(1+n_2M_1)^2} - \frac{B_2M_2}{(1+n_3M_1)^2(1+n_4M_2)} + (1-\eta)\beta - d_1 - 2h_2M_1,$$
$$a_{23} = \frac{-B_2M_1}{(1+n_3M_1)(1+n_4M_2)^2}, \quad a_{31} = 0, \quad a_{32} = \frac{l_2M_2}{(1+n_3M_1)^2(1+n_4M_2)},$$
$$a_{33} = \frac{l_2M_1}{(1+n_3M_1)(1+n_4M_2)^2} + \eta\beta - d_2 - 2h_3M_2.$$

- It is easy to verify that, the (JM) of system (1) at $E_0 = (0,0,0)$ can be written as:

$$J_0 = J(E_0) = \begin{bmatrix} -h_1 & 0 & 0 \\ 0 & (1-\eta)\beta - d_1 & 0 \\ 0 & 0 & \eta\beta - d_2 \end{bmatrix}. \quad (10)$$

Then the characteristic equation of J_0 is given by

$$(-h_1 - \lambda)\big((1-\eta)\beta - d_1 - \lambda\big)(\eta\beta - d_2 - \lambda) = 0$$

then the eigenvalues of J_0 are

$\lambda_{0N} = -h_1 < 0$, $\lambda_{0M_1} = (1-\eta)\beta - d_1$ and $\lambda_{0M_2} = \eta\beta - d_2$, so the equilibrium point E_0 is locally asymptotically stable (LAS) in the R^3_+ , by reversing conditions (4) and (5) and it is unstable otherwise.

- The (JM) of system (1) at $E_1 = (\bar{N}, 0, 0)$ can be written as

$$J_1 = J(E_1) = \begin{bmatrix} \frac{b\bar{N}(\bar{N}(K-2\bar{N})+A(2K-3\bar{N}))}{K(\bar{N}+A)^2} - h_1 & \frac{fb\bar{N}^2(\bar{N}-K)}{K(\bar{N}+A)} - \frac{B_1\bar{N}}{(1+n_1N)} & 0 \\ 0 & \frac{l_1\bar{N}}{(1+n_1\bar{N})} + (1-\eta)\beta - d_1 & 0 \\ 0 & 0 & \eta\beta - d_2 \end{bmatrix}. \quad (11)$$

Then the characteristic equation of J_1 is given by

$$\left(\frac{b\bar{N}(\bar{N}(K-2\bar{N})+A(2K-3\bar{N}))}{K(\bar{N}+A)^2} - h_1 - \lambda\right)\left(\frac{l_1\bar{N}}{(1+n_1\bar{N})} + (1-\eta)\beta - d_1 - \lambda\right)(\eta\beta - d_2 - \lambda) = 0,$$

then the eigenvalues of J_1 are:

$$\lambda_{1N} = \frac{b\bar{N}(\bar{N}(K-2\bar{N})+A(2K-3\bar{N}))}{K(\bar{N}+A)^2} - h_1 ,$$
$$\lambda_{1M_1} = \frac{l_1\bar{N}}{(1+n_1\bar{N})} + \big((1-\eta)\beta - d_1\big) \quad \text{and} \quad \lambda_{1M_2} = \eta\beta - d_2.$$

Then E_1 is (LAS) by reversing conditions (4,5) with next conditions

$$\bar{N} < \frac{1}{2}K, \quad (12)$$
$$\frac{b\bar{N}(\bar{N}(K-2\bar{N})+A(2K-3\bar{N}))}{K(\bar{N}+A)^2} < h_1, \quad (13)$$
$$\frac{l_1\bar{N}}{(1+n_1\bar{N})} < (d_1 - (1-\eta)\beta). \quad (14)$$

otherwise E_1 is unstable, similarly for the equilibrium point E_2.

- The (JM) of system (1) at $E_3 = \left(0\,, \widehat{M}_1, 0\right)$ where $\widehat{M}_1 = \frac{(1-\eta)\beta - d_1}{h_2}$ can be written as $J_3 = J(E_3) =$

$$\begin{bmatrix} \frac{-B_1\widehat{M}_1}{(1+n_2\widehat{M}_1)} - h_1 & 0 & 0 \\ \frac{l_1\widehat{M}_1}{(1+n_2\widehat{M}_1)} & -\left((1-\eta)\beta - d_1\right) & \frac{-B_2\widehat{M}_1}{(1+n_3\widehat{M}_1)} \\ 0 & 0 & \frac{l_2\widehat{M}_1}{(1+n_3\widehat{M}_1)} + (\eta\beta - d_2) \end{bmatrix}. \quad (15)$$

Then the characteristic equation of J_3 is given by

$$\left(\frac{-B_1\widehat{M}_1}{(1+n_2\widehat{M}_1)} - h_1 - \lambda\right)\left(-\left((1-\eta)\beta - d_1\right) - \lambda\right)\left(\frac{l_2\widehat{M}_1}{(1+n_3\widehat{M}_1)} + (\eta\beta - d_2) - \lambda\right) = 0,$$

then the eigenvalues of J_3 are

$$\lambda_{3N} = \frac{-B_1\widehat{M}_1}{(1+n_2\widehat{M}_1)} - h_1 < 0\,, \quad \lambda_{3M_1} = -\left((1-\eta)\beta - d_1\right) \quad \text{and} \quad \lambda_{3M_2} = \frac{l_2\widehat{M}_1}{(1+n_3\widehat{M}_1)} + \eta\beta - d_2,$$

then E_3 is (LAS) by condition (4) and reversing condition (5) with condition:

$$\frac{l_2\widehat{M}_1}{(1+n_3\widehat{M}_1)} < (d_2 - \eta\beta), \quad (16)$$

otherwise E_3 is unstable.

- The (JM) of system (1) at $E_4 = \left(0\,, 0\,, \breve{M}_2\right)$ where $\breve{M}_2 = \frac{\eta\beta - d_2}{h_3}$ can be written as:

$$J_4 = J(E_4) = \begin{bmatrix} -h_1 & 0 & 0 \\ 0 & \frac{-B_2\breve{M}_2}{(1+n_4\breve{M}_2)} + \left((1-\eta)\beta - d_1\right) & 0 \\ 0 & \frac{l_2\breve{M}_2}{(1+n_4\breve{M}_2)} & -(\eta\beta - d_2) \end{bmatrix}. \quad (17)$$

Then the characteristic equation of J_4 is given by:

$$(-h_1 - \lambda)\left(\frac{-B_2\breve{M}_2}{(1+n_4\breve{M}_2)} + \left((1-\eta)\beta - d_1\right) - \lambda\right)\left(-(\eta\beta - d_2) - \lambda\right) = 0,$$

then the eigenvalues of J_4 are

$$\lambda_{4N} = -h_1 < 0\,, \quad \lambda_{4M_1} = \frac{-B_2\breve{M}_2}{(1+n_4\breve{M}_2)} - \left(d_1 - (1-\eta)\beta\right) \quad \text{and} \quad \lambda_{4M_2} = -(\eta\beta - d_2),$$

then E_4 is (LAS) by condition (5) and reversing condition (4), otherwise E_4 is unstable.

- The (JM) of system (1) at $E_5 = \left(\widetilde{N}\,, \widetilde{M}_1\,, 0\right)$ can be written as

$$J_5 = J(E_5) = \left[\tau_{ij}\right]_{3\times 3}. \quad (18)$$

$$\tau_{11} = \frac{b\widetilde{N}(\widetilde{N}(K-2\widetilde{N}) + A(2K-3\widetilde{N}))}{K(f\widetilde{M}_1+1)(\widetilde{N}+A)^2} - \frac{B_1\widetilde{M}_1}{(1+n_1\widetilde{N})^2(1+n_2\widetilde{M}_1)} - h_1\,,$$

$$\tau_{12} = \frac{fb\widetilde{N}^2(\widetilde{N}-K)}{K(f\widetilde{M}_1+1)^2(\widetilde{N}+A)} - \frac{B_1\widetilde{N}}{(1+n_1\widetilde{N})(1+n_2\widetilde{M}_1)^2}\,, \quad \tau_{13} = 0\,, \quad \tau_{21} = \frac{l_1\widetilde{M}_1}{(1+n_1\widetilde{N})^2(1+n_2\widetilde{M}_1)}\,,$$

$$\tau_{22} = \frac{l_1\widetilde{N}}{(1+n_1\widetilde{N})(1+n_2\widetilde{M}_1)^2} + (1-\eta)\beta - d_1 - 2h_2\widetilde{M}_1\,, \quad \tau_{23} = \frac{-B_2\widetilde{M}_1}{(1+n_3\widetilde{M}_1)}\,,$$

$$\tau_{31} = 0\,, \quad \tau_{32} = 0\,, \quad \tau_{33} = \frac{l_2\widetilde{M}_1}{(1+n_3\widetilde{M}_1)} + \eta\beta - d_2.$$

Then the characteristic equation of J_5 is given by

$$(\tau_{33} - \lambda)[\lambda^2 - (\tau_{11} + \tau_{22})\lambda + \tau_{11}\tau_{22} - \tau_{12}\tau_{21}] = 0.$$

Then, either $(\tau_{33} - \lambda) = 0$, which gives $\lambda_{5M_2} = \frac{l_2\widetilde{M}_1}{(1+n_3\widetilde{M}_1)} + \eta\beta - d_2$.

Or $[\lambda^2 - (\tau_{11} + \tau_{22})\lambda + \tau_{11}\tau_{22} - \tau_{12}\tau_{21}] = 0$,

which gives

$$\lambda_{5N}+\lambda_{5M_1}=\left(\frac{b\tilde{N}\left(\tilde{N}(K-2\tilde{N})+A(2K-3\tilde{N})\right)}{K(f\tilde{M}_1+1)(\tilde{N}+A)^2}-\frac{B_1\tilde{M}_1}{(1+n_1\tilde{N})^2(1+n_2\tilde{M}_1)}-h_1\right)+\left(\frac{l_1\tilde{N}}{(1+n_1\tilde{N})(1+n_2\tilde{M}_1)^2}+\left((1-\eta)\beta-d_1\right)-2h_2\tilde{M}_1\right),$$

$$\lambda_{5N}.\lambda_{5M_1}=\left(\frac{b\tilde{N}(\tilde{N}(K-2\tilde{N})+A(2K-3\tilde{N}))}{K(f\widetilde{M_1}+1)(\tilde{N}+A)^2}-\frac{B_1\tilde{M}_1}{(1+n_1\tilde{N})^2(1+n_2\tilde{M}_1)}-h_1\right).\left(\frac{l_1\tilde{N}}{(1+n_1\tilde{N})(1+n_2\tilde{M}_1)^2}+(1-\eta)\beta-d_1-2h_2\tilde{M}_1\right)-\left(\frac{fb\tilde{N}^2(\tilde{N}-K)}{K(f\widetilde{M_1}+1)^2(\tilde{N}+A)}-\frac{B_1\tilde{N}}{(1+n_1\tilde{N})(1+n_2\tilde{M}_1)^2}\right)\left(\frac{l_1\tilde{M}_1}{(1+n_1\tilde{N})^2(1+n_2\tilde{M}_1)}\right),$$

then E_5 is (LAS) by condition (4) and the next conditions hold

$$\tilde{N}<\frac{1}{2}K, \tag{19}$$

$$\frac{b\tilde{N}(\tilde{N}(K-2\tilde{N})+A(2K-3\tilde{N}))}{K(f\tilde{M}_1+1)(\tilde{N}+A)^2}<\frac{B_1\tilde{M}_1}{(1+n_1\tilde{N})^2(1+n_2\tilde{M}_1)}+h_1, \tag{20}$$

$$\frac{l_2\tilde{M}_1}{(1+n_3\tilde{M}_1)}+\eta\beta<d_2, \tag{21}$$

$$\frac{l_1\tilde{N}}{(1+n_1\tilde{N})(1+n_2\tilde{M}_1)^2}+\left((1-\eta)\beta-d_1\right)<2h_2\tilde{M}_1, \tag{22}$$

$$\frac{fb\tilde{N}^2(\tilde{N}-K)}{K(f\widetilde{M}_1+1)^2(\tilde{N}+A)}<\frac{B_1\tilde{N}}{(1+n_1\tilde{N})(1+n_2\tilde{M}_1)^2}\,. \tag{23}$$

Otherwise E_5 is unusable.

- The (JM) of system (1) at $E_6=\left(\overline{N},0,\breve{M}_2\right)$ can be written as

$$J_6=J(E_6)=\left[\chi_{ij}\right]_{3\times 3}. \tag{24}$$

$$\chi_{11}=\frac{b\overline{N}\left(\overline{N}(K-2\overline{N})+A(2K-3\overline{N})\right)}{K(\overline{N}+A)^2}-h_1,\ \chi_{12}=\frac{fb\overline{N}^2(\overline{N}-K)}{K(\overline{N}+A)}-\frac{B_1\overline{N}}{(1+n_1N)},\quad \chi_{13}=0,$$

$$\chi_{21}=0,\quad \chi_{22}=\frac{l_1\overline{N}}{(1+n_1\overline{N})}-\frac{B_2\breve{M}_2}{(1+n_4\breve{M}_2)}+(1-\eta)\beta-d_1,\ \chi_{23}=0,$$

$$\chi_{31}=0,\quad \chi_{32}=\frac{l_2\breve{M}_2}{(1+n_4\breve{M}_2)},\ \chi_{33}=-(\eta\beta-d_2).$$

Then the characteristic equation of J_6 is given by

$$(\chi_{11}-\lambda)(\chi_{22}-\lambda)(\chi_{33}-\lambda)=0,$$

then the eigenvalues of J_6 are

$$\lambda_{6N}=\frac{b\overline{N}\left(\overline{N}(K-2\overline{N})+A(2K-3\overline{N})\right)}{K(\overline{N}+A)^2}-h_1,\qquad \lambda_{6M_1}=\frac{l_1\overline{N}}{(1+n_1\overline{N})}-\frac{B_2\breve{M}_2}{(1+n_4\breve{M}_2)}+(1-\eta)\beta-d_1$$

and

$$\lambda_{6M_2}=-(\eta\beta-d_2),$$

then E_6 is (LAS) by conditions $(5,12,13\text{ and }14)$ and reversing condition (4).

Similarly, $E_7=\left(\overline{\overline{N}},0,\breve{M}_2\right)$ is (LAS) if the conditions hold, otherwise E_6 and E_7 are unstable.

- The (JM) of system (1) at $E_8=\left(0,\dot{M}_1,\dot{M}_2\right)$ can be written as

$J_8=\left[\Omega_{ij}\right]_{3\times 3}$, were (25)

$$\Omega_{11}=\frac{-B_1\dot{M}_1}{(1+n_2\dot{M}_1)}-h_1<0,\ \Omega_{12}=0,\ \Omega_{13}=0,\ \Omega_{21}=\frac{l_1\dot{M}_1}{(1+n_2\dot{M}_1)},$$

$$\Omega_{22}=\frac{-B_2\dot{M}_2}{(1+n_3\dot{M}_1)^2(1+n_4\dot{M}_2)}+(1-\eta)\beta-d_1-2h_2\dot{M}_1,\ \Omega_{23}=\frac{-B_2\dot{M}_1}{(1+n_3\dot{M}_1)(1+n_4\dot{M}_2)^2},$$

$\Omega_{31} = 0,\ \Omega_{32} = \frac{l_2\dot{M}_2}{(1+n_3\dot{M}_1)^2(1+n_4\dot{M}_2)},\ \ \Omega_{33} = \frac{l_2\dot{M}_1}{(1+n_3\dot{M}_1)(1+n_4\dot{M}_2)^2} + \eta\beta - d_2 - 2h_3\dot{M}_2.$

Then the characteristic equation of J_8 is given by

$(\Omega_{11} - \lambda)[\lambda^2 - (\Omega_{22} + \Omega_{33})\lambda + \Omega_{22}\Omega_{33} - \Omega_{23}\Omega_{32}] = 0$.

Then, either $(\Omega_{11} - \lambda) = 0$,

which gives $\lambda_{8N} = \frac{-B_1\dot{M}_1}{(1+n_2\dot{M}_1)} - h_1 < 0$,

or, $[\lambda^2 - (\Omega_{22} + \Omega_{33})\lambda + \Omega_{22}\Omega_{33} - \Omega_{23}\Omega_{32}] = 0$,

which gives

$$\lambda_{8M_1} + \lambda_{8M_2} = \left(\frac{-B_2\dot{M}_2}{(1+n_3\dot{M}_1)^2(1+n_4\dot{M}_2)} + \big((1-\eta)\beta - d_1\big) - 2h_2\dot{M}_1\right) + \left(\frac{l_2\dot{M}_1}{(1+n_3\dot{M}_1)(1+n_4\dot{M}_2)^2} + (\eta\beta - d_2) - 2h_3\dot{M}_2\right),$$

$$\lambda_{8M_1} . \lambda_{8M_2} = \left(\frac{-B_2\dot{M}_2}{(1+n_3\dot{M}_1)^2(1+n_4\dot{M}_2)} + \big((1-\eta)\beta - d_1\big) - 2h_2\dot{M}_1\right) \left(\frac{l_2\dot{M}_1}{(1+n_3\dot{M}_1)(1+n_4\dot{M}_2)^2} + (\eta\beta - d_2) - h_3\dot{M}_2\right) \left(\frac{-B_2\dot{M}_1}{(1+n_3\dot{M}_1)(1+n_4\dot{M}_2)^2}\right)\left(\frac{l_2\dot{M}_2}{(1+n_3\dot{M}_1)^2(1+n_4\dot{M}_2)}\right),$$

then E_8 is (LAS) by conditions (4 and 5) and the next conditions hold

$$\big((1-\eta)\beta - d_1\big) < \frac{B_2\dot{M}_2}{(1+n_3\dot{M}_1)^2(1+n_4\dot{M}_2)} + 2h_2\dot{M}_1, \tag{26}$$

$$\frac{l_2\dot{M}_1}{(1+n_3\dot{M}_1)(1+n_4\dot{M}_2)^2} + (\eta\beta - d_2) < 2h_3\dot{M}_2, \tag{27}$$

otherwise E_8 is unstable.

- Finally, the (JM) of system (1) at $E_9 = (N^*, M_1^*, M_2^*)$ can be written as:

$J_9 = [\,\rho_{ij}\,]_{3\times3}$. (28)

Here

$\rho_{11} = \frac{bN^*\left(N^*(K-2N^*)+A(2K-3N^*)\right)}{K(fM_1^*+1)(N^*+A)^2} - \frac{B_1M_1^*}{(1+n_1N^*)^2(1+n_2M_1^*)} - h_1,$

$\rho_{12} = \frac{fbN^{*2}(N^*-K)}{K(fM_1^*+1)^2(N^*+A)} - \frac{B_1N^*}{(1+n_1N^*)(1+n_2M_1^*)^2},\ \rho_{13} = 0,\ \rho_{21} = \frac{l_1M_1^*}{(1+n_1N^*)^2(1+n_2M_1^*)},$

$\rho_{22} = \frac{l_1N^*}{(1+n_1N^*)(1+n_2M_1^*)^2} - \frac{B_2M_2^*}{(1+n_3M_1^*)^2(1+n_4M_2^*)} + \big((1-\eta)\beta - d_1\big) - 2h_2M_1^*,$

$\rho_{23} = \frac{-B_2M_1^*}{(1+n_3M_1^*)(1+n_4M_2^*)^2},\ \ \rho_{31} = 0, \rho_{32} = \frac{l_2M_2^*}{(1+n_3M_1^*)^2(1+n_4M_2^*)},$

$\rho_{33} = \frac{l_2M_1^*}{(1+n_3M_1^*)(1+n_4M_2^*)^2} + (\eta\beta - d_2) - 2h_3M_2^*.$

Then the characteristic equation of J_9 is given by

$\lambda^3 + I_1\lambda^2 + I_2\,\lambda + I_3 = 0,$

where

$I_1 = -(\rho_{11} + \rho_{22} + \rho_{33}),$

$I_2 = \rho_{11}(\rho_{22} + \rho_{33}) + \rho_{22}\rho_{33} - \rho_{12}\rho_{21} - \rho_{23}\rho_{32},$

$I_3 = \rho_{11}(\rho_{23}\rho_{32} - \rho_{22}\rho_{33}) + \rho_{12}\rho_{21}\rho_{33}.$

So, by Routh Hurwitz criterion all the eigenvalues, which represent the roots of Equation (28), have negative real parts if and only if $I_1 > 0,\ I_3 > 0$ and $\Delta = (I_1I_2 - I_3)I_3 > 0$.

Clearly, in addition to conditions (4, 5) we have $I_1 > 0$ and $I_3 > 0$ provided that

$$N^* < \frac{1}{2}\,K, \tag{29}$$

$$\frac{bN^*\left(N^*(K-2N^*)+A(2K-3N^*)\right)}{K(fM_1^*+1)(N^*+A)^2} < \frac{B_1M_1^*}{(1+n_1N^*)^2(1+n_2M_1^*)} + h_1, \tag{30}$$

$$\frac{l_1N^*}{(1+n_1N^*)(1+n_2M_1^*)^2}+\big((1-\eta)\beta-d_1\big)<\frac{B_2M_2^*}{(1+n_3M_1^*)^2(1+n_4M_2^*)}+2h_2M_1^*, \quad (31)$$

$$\frac{l_2M_1^*}{(1+n_3M_1^*)(1+n_4M_2^*)^2}+(\eta\beta-d_2)<2h_3M_2^*. \quad (32)$$

Therefore, under the above conditions, all of J_9's eigenvalues have negative real parts, and as a result E_9 is (LAS), otherwise E_9 is unstable.

5. Global stability analysis (GSA)

In this section the (GSA) for the (EPs), which are (LAS) of system (1) have been studied analytically by use the suitable of Lyapunov method as shown in the following theorems.

Theorem 5.1: The equilibrium point $E_0=(0,0,0)$ of system (1) is globally asymptotically stable (GAS) in the sub region $\psi_0\subset R_+^3$. In addition to reversing conditions (4 and 5) the following conditions hold

$$K>N, \quad (33)$$

$$\left(\frac{bN^2(K-N)}{K(fM_1+1)(N+A)}\right)<(d_1-(1-\eta)\beta)M_1+(d_2-\eta\beta)M_2. \quad (34)$$

Proof: Consider the following function $V_0(N,M_1,M_2)=N+M_1+M_2$.
Clearly, V_0: $R_+^3\to R$ is a C^1 positive definite function.
Now, by differentiating V_0 with regard to time t, and performing some algebraic manipulation, gives that

$$\frac{dV_0}{dt}=bN\left(1-\frac{N}{K}\right)\left(\frac{1}{fM_1+1}\right)\left(\frac{N}{N+A}\right)-\frac{(B_1-l_1)NM_1}{(1+n_1N)(1+n_2M_1)}-h_1N-\frac{(B_2-l_2)M_1M_2}{(1+n_3M_1)(1+n_4M_2)}+\big((1-\eta)\beta-d_1\big)M_1-h_2M_1^2+(\eta\beta-d_2)M_2-h_3M_2^2.$$

Now, due to the biological facts $l_i<B_i$, $i=1,2,$ so

$$\frac{dV_0}{dt}<\left(\frac{bN^2(K-N)}{K(fM_1+1)(N+A)}\right)-(d_1-(1-\eta)\beta)M_1-(d_2-\eta\beta)M_2,$$

hence $\frac{dV_0}{dt}<0$ in the region ψ_0, by conditions (33 and 34) then V_0 is strictly Lyapunov function. Therefore, E_0 is a (GAS) in the region $\psi_0\subset R_+^3$

Furthermore, it is not viable to use the Lyapunov function to examine the global stability of two equilibrium points (E_1 and E_2) that are located in the interior of R_+^3 and have distinct beginning points neighbourhood but the same local stability conditions. Consequently, as opposed to studying it analytically as demonstrated in the previous theorems, we shall investigate it numerically.

Theorem 5.2: The equilibrium point $E_3=\left(0,\widehat{M}_1,0\right)$ of system (1) is (GAS) in the sub region $\psi_1\subset R_+^3$, If In addition to condition (33) and reversing condition (5) the following condition holds

$$\widehat{Q}_1<\widehat{Q}_2, \quad (35)$$

where

$$\widehat{Q}_1=\left(\frac{bN^2(K-N)}{K(fM_1+1)(N+A)}+\frac{B_2\widehat{M}_1M_2}{(1+n_3M_1)(1+n_4M_2)}\right),$$

$$\widehat{Q}_2=\left(\frac{l_1N\widehat{M}_1}{(1+n_1N)(1+n_2M_1)}+h_2(M_1-\widehat{M}_1)^2+(d_2-\eta\beta)M_2\right).$$

Proof: Consider the following function $V_1(N, M_1, M_2) = N + \left(M_1 - \widehat{M}_1 - \widehat{M}_1 \ln\frac{M_1}{\widehat{M}_1} \right) + M_2$

Clearly, $V_1: R_+^3 \to R$ is a C^1 positive definite function.

Now, by differentiating V_1 with regard to time t, and performing some algebraic manipulation, gives that

$$\frac{dV_1}{dt} = \frac{bN^2(K-N)}{K(fM_1+1)(N+A)} - \frac{(B_1-l_1)NM_1}{(1+n_1N)(1+n_2M_1)} - h_1N - \frac{l_1N\widehat{M}_1}{(1+n_1N)(1+n_2M_1)} - \frac{(B_2-l_2)M_1M_2}{(1+n_3M_1)(1+n_4M_2)} + \frac{B_2\widehat{M}_1M_2}{(1+n_3M_1)(1+n_4M_2)} - h_2\left(M_1 - \widehat{M}_1\right)^2 + (\eta\beta - d_2)M_2 - h_3M_2^2.$$

Now, due to the biological facts $l_i < B_i$, $i = 1,2$, so

$$\frac{dV_1}{dt} < \frac{bN^2(K-N)}{K(fM_1+1)(N+A)} + \frac{B_2\widehat{M}_1M_2}{(1+n_3M_1)(1+n_4M_2)} - \left(\frac{l_1N\widehat{M}_1}{(1+n_1N)(1+n_2M_1)} + h_2\left(M_1 - \widehat{M}_1\right)^2 + (d_2 - \eta\beta)M_2\right)$$
$$= \widehat{Q}_1 - \widehat{Q}_2,$$

hence $\frac{dV_1}{dt} < 0$ in the region ψ_1, by reversing condition (5) and using conditions (33 and 35) then V_1 is strictly Lyapunov function. Therefore, E_3 is a (GAS) in the region $\psi_1 \subset R_+^3$.

Theorem 5.3: The (EP) $E_4 = \left(0, 0, \breve{M}_2\right)$ of system (1) is (GAS) in the sub region $\psi_2 \subset R_+^3$. In addition to condition (33) and reversing condition (4) the following condition holds:

$$\breve{Q}_1 < \breve{Q}_2, \tag{36}$$

where

$$\breve{Q}_1 = \left(\frac{bN^2(K-N)}{K(fM_1+1)(N+A)}\right),$$
$$\breve{Q}_2 = \left(\frac{l_2M_1\breve{M}_2}{(1+n_3M_1)(1+n_4M_2)} + h_1N + (d_1 - (1-\eta)\beta)\, M_1\right).$$

Proof: Consider the following function $V_2(N, M_1, M_2) = N + M_1 + \left(M_2 - \breve{M}_2 - \breve{M}_2 \ln\frac{M_2}{\breve{M}_2} \right)$.

Clearly, $V_2: R_+^3 \to R$ is a C^1 positive definite function.

Now, by differentiating V_2 with regard to time, t and performing some algebraic manipulation, gives that

$$\frac{dV_2}{dt} = \frac{bN^2(K-N)}{K(fM_1+1)(N+A)} - \frac{(B_1-l_1)NM_1}{(1+n_1N)(1+n_2M_1)} - h_1N - \frac{(B_2-l_2)M_1M_2}{(1+n_3M_1)(1+n_4M_2)} - \frac{l_2M_1\breve{M}_2}{(1+n_3M_1)(1+n_4M_2)} + \left((1-\eta)\beta - d_1\right) M_1 - h_2\, M_1^2 - h_3\left(\left(M_2 - \breve{M}_2\right)^2\right).$$

Now, due to the biological facts $l_i < B_i$, $i = 1,2$ so:

$$\frac{dV_2}{dt} < \frac{bN^2(K-N)}{K(fM_1+1)(N+A)} - \left(\frac{l_2M_1\breve{M}_2}{(1+n_3M_1)(1+n_4M_2)} + h_1N + (d_1 - (1-\eta)\beta)\, M_1\right)$$
$$= \breve{Q}_1 - \breve{Q}_2,$$

hence $\frac{dV_2}{dt} < 0$ in the region ψ_2, by reversing condition (4) using conditions (33, 36) then V_2 is strictly Lyapunov function. Therefore, E_4 is a (GAS) in the region $\psi_2 \subset R_+^3$.

Theorem 5.4: The (EP) $E_5 = \left(\widetilde{N}, \widetilde{M}_1, 0\right)$ of system (1) is (GAS) in the sub region $\psi_3 \subset R_+^3$ if In addition to condition (33) the following condition holds

$$\widetilde{Q}_1 < \widetilde{Q}_2, \tag{37}$$

where

$$\tilde{Q}_1 = \frac{bN(K-N)N}{K(fM_1+1)(N+A)} + \frac{B_1\tilde{N}M_1}{(1+n_1N)(1+n_2M_1)} + \frac{b\tilde{N}(K-\tilde{N})\tilde{N}}{K(f\tilde{M}_1+1)(\tilde{N}+A)} + \frac{B_1N\tilde{M}_1}{(1+n_1\tilde{N})(1+n_2\tilde{M}_1)} + \frac{B_2\tilde{M}_1M_2}{(1+n_3M_1)(1+n_4M_2)},$$

$$\tilde{Q}_2 = \frac{bN(K-N)\tilde{N}}{K(fM_1+1)(N+A)} + \frac{b\tilde{N}(K-\tilde{N})N}{K(f\tilde{M}_1+1)(\tilde{N}+A)} + \frac{l_1N\tilde{M}_1}{(1+n_1N)(1+n_2M_1)} + \frac{l_1\tilde{N}M_1}{(1+n_1\tilde{N})(1+n_2\tilde{M}_1)} + h_2\left(M_2-\tilde{M}_2\right)^2 + (d_2-n\beta)M_2 + h_2M_2^2.$$

Proof: Consider the following function

$$V_3(N, M_1, M_2) = \left(N - \tilde{N} - \tilde{N}\ ln\frac{N}{\tilde{N}}\right) + \left(M_1 - \tilde{M}_1 - \tilde{M}_1\ ln\frac{M_1}{\tilde{M}_1}\right) + M_2.$$

Clearly $V_3: R_+^3 \to R$ is a C^1 positive definite function.
Now by differentiating V_3 with regard to time t, and performing some algebraic manipulation, gives that

$$\frac{dV_3}{dt} = \frac{bN(K-N)(N-\tilde{N})}{K(fM_1+1)(N+A)} - \frac{(B_1-l_1)NM_1}{(1+n_1N)(1+n_2M_1)} + \frac{B_1\tilde{N}M_1}{(1+n_1N)(1+n_2M_1)} - \frac{b\tilde{N}(K-\tilde{N})(N-\tilde{N})}{K(f\tilde{M}_1+1)(\tilde{N}+A)} + \frac{B_1N\tilde{M}_1}{(1+n_1\tilde{N})(1+n_2\tilde{M}_1)} - \frac{(B_1-l_1)\tilde{N}\tilde{M}_1}{(1+n_1\tilde{N})(1+n_2\tilde{M}_1)} - \frac{l_1N\tilde{M}_1}{(1+n_1N)(1+n_2M_1)} - \frac{(B_2-l_2)M_1M_2}{(1+n_3M_1)(1+n_4\mu_2)} + \frac{B_2\tilde{M}_1M_2}{(1+n_3M_1)(1+n_4M_2)} - \frac{l_1\tilde{N}M_1}{(1+n_1\tilde{N})(1+n_2\tilde{M}_1)} - h_2\left(M_2-\tilde{M}_2\right)^2 + (nB-d_2)M_2 - h_2M_2^2.$$

Now, due to the biological facts $l_i < B_i$, $i = 1,2$, so

$$\frac{dV_3}{dt} < \left(\frac{bN(K-N)N}{K(fM_1+1)(N+A)} + \frac{B_1\tilde{N}M_1}{(1+n_1N)(1+n_2M_1)} + \frac{b\tilde{N}(K-\tilde{N})\tilde{N}}{K(f\tilde{M}_1+1)(\tilde{N}+A)} + \frac{B_1N\tilde{M}_1}{(1+n_1\tilde{N})(1+n_2\tilde{M}_1)} + \frac{B_2\tilde{M}_1M_2}{(1+n_3M_1)(1+n_4M_2)}\right)\left(\frac{bN(K-N)\tilde{N}}{K(fM_1+1)(N+A)} + \frac{b\tilde{N}(K-\tilde{N})N}{K(f\tilde{M}_1+1)(\tilde{N}+A)} + \frac{l_1N\tilde{M}_1}{(1+n_1N)(1+n_2M_1)} + \frac{l_1\tilde{N}M_1}{(1+n_1\tilde{N})(1+n_2\tilde{M}_1)} + h_2\left(M_2-\tilde{M}_2\right)^2 + (d_2-n\beta)M_2 + h_2M_2^2\right)$$

$$= \tilde{Q}_1 - \tilde{Q}_2,$$

hence $\frac{dV_3}{dt} < 0$ in the region ψ_3, by conditions $(33, 37)$ then V_3 is strictly Lyapunov function. Therefore, E_5 is a (GAS) in the region $\psi_3 \subset R_+^3$.

Also, it is not viable to use the Lyapunov function to examine the global stability of two equilibrium points (E_6 and E_7) that are located in the interior of R_+^3 and have distinct beginning points neighbourhood but the same local stability conditions. Consequently, as opposed to studying it analytically as demonstrated in the previous theorems, we shall investigate it numerically.

Theorem 5.5: The (EP) $E_8 = \left(0, \dot{M}_1, \dot{M}_2\right)$ of system (1) is (GAS) in the sub region $\psi_4 \subset R_+^3$, If In addition to condition (33) the following condition holds:

$$\dot{Q}_1 < \dot{Q}_2, \tag{38}$$

where
$$\dot{Q}_1 = \left(\frac{bN^2(K-N)}{K(fM_1+1)(N+A)} + \frac{B_2\dot{M}_1M_2}{(1+n_3M_1)(1+n_4M_2)} + \frac{\dot{M}_1(B_2M_1+l_2\dot{M}_2)}{(1+n_3\dot{M}_1)(1+n_4\dot{M}_2)}\right),$$
$$\dot{Q}_2 = \left(\frac{l_1N\dot{M}_1}{(1+n_1N)(1+n_2M_1)} + \frac{l_2M_1\dot{M}_2}{(1+n_3M_1)(1+n_4M_2)} + \frac{\dot{M}_1(B_2\dot{M}_1+l_2M_2)}{(1+n_3\dot{M}_1)(1+n_4\dot{M}_2)}\right).$$

Proof: Consider the following function

$$V_4(N, M_1, M_2) = N + \left(M_1 - \dot{M}_1 - \dot{M}_1\ ln\frac{M_1}{\dot{M}_1}\right) + \left(M_2 - \dot{M}_2 - \dot{M}_2\ ln\frac{M_2}{\dot{M}_2}\right)$$

Clearly, $V_4: R_+^3 \to R$ is a C^1 positive definite function.
Now, by differentiating V_4 with regard to time t, and performing some algebraic manipulation, gives that

$$\frac{dV_4}{dt} = \frac{bN^2(K-N)}{K(fM_1+1)(N+A)} - \frac{(B_1-l_1)NM_1}{(1+n_1N)(1+n_2M_1)} - h_1M_1 - \frac{l_1N\dot{M}_1}{(1+n_1N)(1+n_2M_1)} - \frac{(B_2-l_2)M_1M_2}{(1+n_3M_1)(1+n_4M_2)} + \frac{B_2\dot{M}_1M_2}{(1+n_3M_1)(1+n_4M_2)} - h_2\left(M_1-\dot{M}_1\right)^2 + \frac{B_2\dot{M}_1(M_1-\dot{M}_1)}{(1+n_3\dot{M}_1)(1+n_4\dot{M}_2)} - \frac{l_2M_1\dot{M}_2}{(1+n_3M_1)(1+n_4M_2)} - h_3\left(M_2-\dot{M}_2\right)^2 - \frac{l_2\dot{M}_1(M_2-\dot{M}_2)}{(1+n_3\dot{M}_1)(1+n_4\dot{M}_2)}.$$

Now, due to the biological facts $l_i < B_i$, $i = 1,2$ so:

$$\frac{dV_4}{dt} < \left(\frac{bN^2(K-N)}{K(fM_1+1)(N+A)} + \frac{B_2\dot{M}_1M_2}{(1+n_3M_1)(1+n_4M_2)} + \frac{\dot{M}_1(B_2M_1+l_2\dot{M}_2)}{(1+n_3\dot{M}_1)(1+n_4\dot{M}_2)}\right) - \left(\frac{l_1N\dot{M}_1}{(1+n_1N)(1+n_2M_1)} + \frac{l_2M_1\dot{M}_2}{(1+n_3M_1)(1+n_4M_2)} + \frac{\dot{M}_1(B_2\dot{M}_1+l_2M_2)}{(1+n_3\dot{M}_1)(1+n_4\dot{M}_2)}\right)$$
$$= \dot{Q}_1 - \dot{Q}_2,$$

hence $\frac{dV_4}{dt} < 0$ in the region ψ_4 , by conditions (33 and 38) then V_4 is strictly Lyapunov function. Therefore, E_8 is a (GAS) in the region $\psi_4 \subset R^3_+$.

Theorem 5.6: The (EP) $E_9 = (N^*, M_1^*, M_2^*)$ of system (1) is (GAS) in the sub region $\psi_5 \subset R^3_+$, if in addition to condition (33) the following condition holds:

$$Q_1^* < Q_2^*, \tag{39}$$

where

$$Q_1^* = \left(\frac{bN^2(K-N)}{K(fM_1+1)(N+A)} + \frac{B_1N^*M_1}{(1+n_1N)(1+n_2M_1)} + \frac{bN^{*2}(K-N^*)}{K(fM_1^*+1)(N^*+A)} + \frac{B_1NM_1^*}{(1+n_1N^*)(1+n_2M_1^*)} + \frac{B_2M_1^*M_2}{(1+n_3M_1)(1+n_4M_2)} + \frac{B_2M_1M_2^*}{(1+n_2M_1^*)(1+n_4M_2^*)}\right),$$

$$Q_2^* = \left(\frac{bNN^*(K-N)}{K(fM_1+1)(N+A)} + \frac{bNN^*(K-N^*)}{K(fM_1^*+1)(N^*+A)} + \frac{l_1NM_1^*}{(1+n_1N)(1+n_2M_1)} + \frac{l_1N^*M_1}{(1+n_1N^*)(1+n_2M_1^*)} + \frac{l_2M_1M_2^*}{(1+n_3M_1)(1+n_4M_2)} + \frac{l_2M_1^*M_2}{(1+n_3M_1^*)(1+n_4M_1)}\right).$$

Proof: Consider the following function

$$V_5(N, M_1, M_2) = \left(N - N^* - N^* \ln\frac{N}{N^*}\right) + \left(M_1 - M_1^* - M_1^* \ln\frac{M_1}{M_1^*}\right) + \left(M_2 - M_2^* - M_2^* \ln\frac{M_2}{M_2^*}\right)$$

Clearly $V_5: R^3_+ \to R$ is a C^1 positive definite function.

Now, by differentiating V_5 with regard to time t, and performing some algebraic manipulation, gives that

$$\frac{dV5}{dt} = \frac{bN(K-N)(N-N^*)}{K(fM_1+1)(N+A)} - \frac{(B_1-l_1)NM_1}{(1+n_1N)(1+n_2M_1)} + \frac{B_1N^*M_1}{(1+n_1N)(1+n_2M_1)} - \frac{bN^*(K-N^*)(N-N^*)}{K(fM_1^*+1)(N^*+A)} + \frac{B_1M_1^*N}{(1+n_1N^*)(1+n_2M_1^*)} - \frac{(B_1-l_1)N^*M_1^*}{(1+n_1N^*)(1+n_2M_1^*)} - \frac{l_1NM_1^*}{(1+n_1N)(1+n_2M_1)} - \frac{(B_2-l_2)M_1M_2}{(1+n_2M_1)(1+n_4M_2)} + \frac{B_2M_1^*M_2}{(1+n_3M_1)(1+n_4M_2)} - \frac{l_1N^*M_1}{(1+n_1N^*)(1+n_2M_1^*)} + \frac{B_2M_1M_2^*}{(1+n_3M_1^*)(1+n_4M_2^*)} - \frac{(B_2-l_2)M_1^*M_2^*}{(1+n_3M_1^*)(1+n_4M_2^*)} - h_2(M_1-M_1^*)^2 - \frac{l_2M_1M_2^*}{(1+n_3M_1)(1+n_4M_2)} - h_3(M_2-M_2^*)^2 - \frac{l_2M_1^*M_2}{(1+n_3M_1^*)(1+n_4M_2^*)}.$$

Now, due to the biological facts $l_i < B_i$, $i = 1,2$, so

$$\frac{dV_5}{dt} < \left(\frac{bN^2(K-N)}{K(fM_1+1)(N+A)} + \frac{B_1N^*M_1}{(1+n_1N)(1+n_2M_1)} + \frac{bN^{*2}(K-N^*)}{K(fM_1^*+1)(N^*+A)} + \frac{B_1NM_1^*}{(1+n_1N^*)(1+n_2M_1^*)} + \frac{B_2M_1^*M_2}{(1+n_3M_1)(1+n_4M_2)} + \frac{B_2M_1M_2^*}{(1+n_2M_1^*)(1+n_4M_2^*)}\right) - \left(\frac{bNN^*(K-N)}{K(fM_1+1)(N+A)} + \frac{bNN^*(K-N^*)}{K(fM_1^*+1)(N^*+A)} + \frac{l_1NM_1^*}{(1+n_1N)(1+n_2M_1)} + \frac{l_1N^*M_1}{(1+n_1N^*)(1+n_2M_1^*)} + \frac{l_2M_1M_2^*}{(1+n_3M_1)(1+n_4M_2)} + \frac{l_2M_1^*M_2}{(1+n_3M_1^*)(1+n_4M_1)}\right) = Q_1^* - Q_2^*,$$

hence $\frac{dV_5}{dt} < 0$ in the region ψ_5 , by conditions (33 and 39) then V_5 is strictly Lyapunov function. Therefore, E_9 is a (GAS) in the region $\psi_5 \subset R_+^3$

6. Numerical simulation:

In this section, the dynamical behaviour of system (1) is studied numerically using Mathematica in order to investigate the effect of varying the value of each parameter on the dynamical behaviour of the system as well as to confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of the positive equilibrium point, system (1) has a (GAS) positive equilibrium point as shown in Figure (1).

$$\left.\begin{array}{l} b = 2,\ K = 3, f = 0.1,\ A = 0.01,\ B_1 = 0.4,\ l_1 = 0.2,\ n_1 = 0.01,\ n_2 = 0.01, \\ h_1 = 0.01,\ B_2 = 0.09,\ l_2 = 0.08,\ n_3 = 0.03,\ n_4 = 0.01,\ \eta = 0.7,\ \beta\ 0.1, \\ h_2 = 0.04\ , d_1 = 0.01\ , h_3 = 0.03\ , d_2 = 0.01. \end{array}\right\} \quad (40)$$

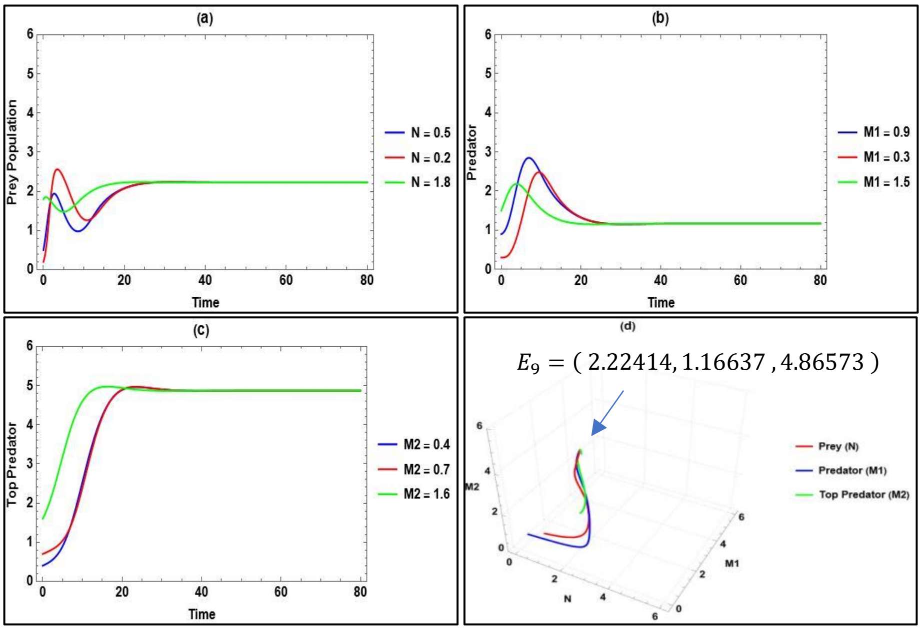

Figure 1: The trajectories of system (1) that started from three different initial points $(0.5, 0.9, 0.4)$, $(0.2, 0.3, 0.7)$ and $(1.8, 1.5, 1.6)$ for the data given in (40). (a) the trajectory of N as a function of time, (b) trajectory of M_1 as a function of time, (c) trajectory of M_2 as a function of time, (d) the trajectory approaches $E_9 = (2.22414, 1.16637, 4.86573)$.

Clearly, Figure (1) shows that system (1) has a (GAS) as the solution of system (1) approaches asymptotically to the positive equilibrium point $E_9 = (2.22414, 1.16637, 4.86573)$.

Now, in order to discuss the effect of the parameters values on the dynamical behaviour of the system, we varying one parameter at each time keeping other parameters as a data given in (40) and the obtained results show in Table

Table 2: The dynamical behaviour of system (1) at each parameter of the system

Range of Parameter	Stable Point
$0.01 \leq b < 0.03377815$, $0.03377815 \leq b < 2$.	E_4 E_9
$0.51 \leq K < 0.803$, $K \geq 0.803$.	E_6 E_9
$0.1 \leq f < 1$.	E_9
$0.01 \leq A < 2.1669$, $2.1669 \leq A < 2.186$, $2.186 \leq A < 3$,	E_9 E_9 E_4
$0.2 \leq B_1 < 2.37289$, $2.37289 \leq B_1 < 2.37291$, $B_1 \geq 37291$.	E_9 E_6 E_4
$0.01 \leq l_1 < 0.05466$, $0.05466 \leq l_1 \leq 0.4$.	E_6 E_9
$0.08 \leq B_2 < 0.302$, $B_2 \geq 0.302$.	E_9 E_6
$0 < l_2 < 0.09$.	E_9
$0.01 \leq n_1 < 0.9303$, $n_1 \geq 0.9303$.	E_9 E_6
$0.01 \leq n_2 < 2$.	E_9
$0.01 \leq n_3 < 2$.	E_9
$0.01 \leq n_4 < 2$	E_9
$0 < \beta < 0.3625$, $0.3625 \leq \beta < 1$.	E_9 E_6
$0 < \eta < 1$.	E_9
$0 < h_1 < 1$.	E_9
$0 < h_2 < 1$.	E_9
$0 \leq h_3 < 0.0086$, $0.0086 \leq h_3 < 1$.	E_9 E_6
$0 < d_1 < 0.43021$, $0.43021 \leq d_1 < 1$.	E_9 E_6
$0 \leq d_2 < 0.30326$, $0.30326 \leq d_2 < 1$.	E_9 E_5

The effect of varying the growth rate of the prey population in the range $0.01 \leq b < 0.03377815$ is studied, it is observed that system (1) approach to E_4, however increasing this parameter further $0.03377815 \leq b \leq 2$ the system approach to E_9 as shown in Figure (2).

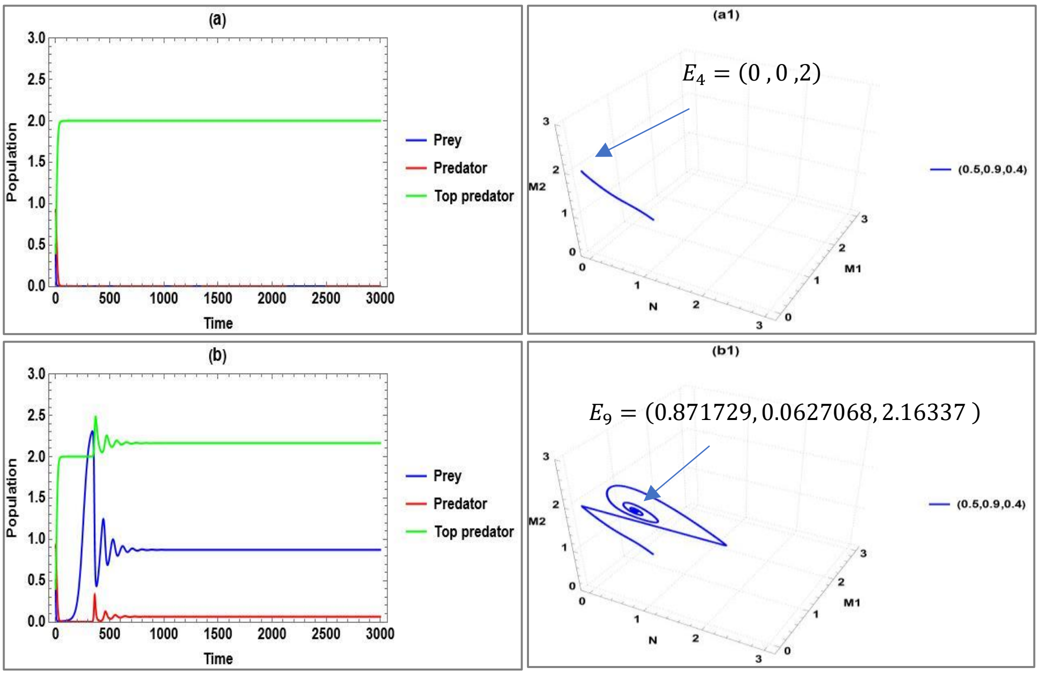

Figure 2: (a) Time series of the solution of system (1) which approaches to $E_4 = (0 ,0 ,2)$ when b = 0.01, $(a1)$ 3D phase portrait of (a), (b) Time series of the solution of system (1) which approaches $E_9 = (0.871729, 0.0627068, 2.16337)$ when b = 0.05, $(b1)$ 3D phase portrait of (b).

Furthermore varying the Allee effect in the range $0.01 \leq A < 2.16693$ was studied, it is observed that system (1) still approach to E_9, however increasing this parameter further $2.16693 \leq$ A < 2.186 the system approach to E_6, further increasing the parameter in the range $2.16693 \leq A < 3$ then the system approach to E_4 as shown in Figure (3).

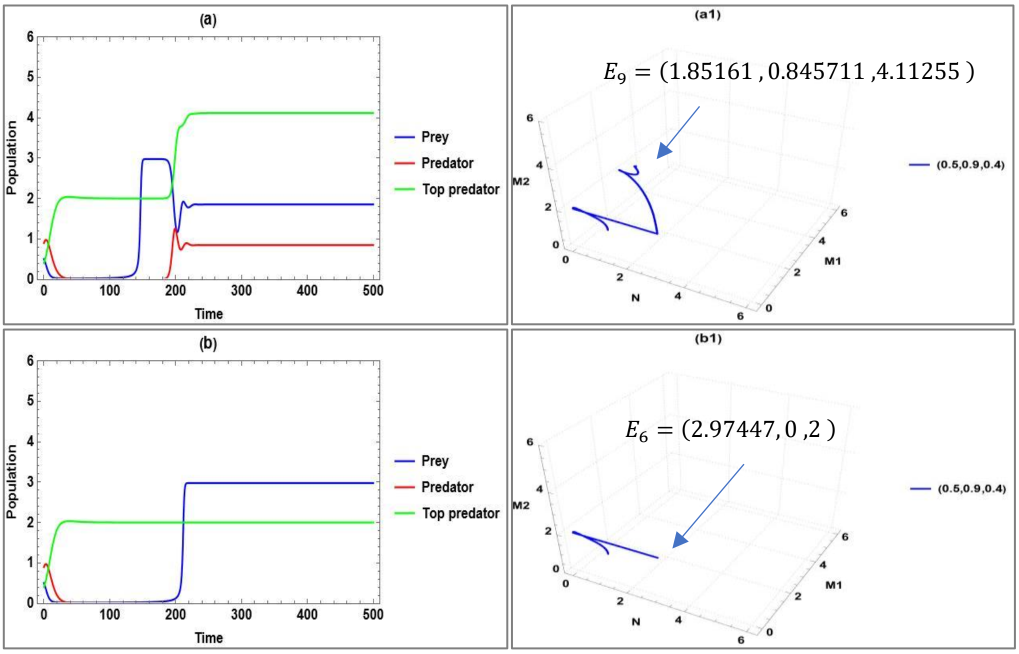

Figure 3: (a) Time series of the solution of system (1) which approaches to $E_9 = (1.85161 , 0.845711 ,4.11255)$ when A = 2, $(a1)$ 3D phase portrait of (a), (b) Time series of the solution of system (1) which approaches to $E_6 = (2.97447, 0 ,2)$ when A = 2.168, $(b1)$ 3D phase portrait of (b).

The effect of varying the additional food for top predator in the range $0 \leq \beta < 0.3625$ was studied, it is observed that system (1) still approach asymptotically to the positive equilibrium point E_9, however increasing this parameter further $0.3625 \leq \beta < 1$ causes extinction in the predator and the system will approach to E_6 as shown in Figure (4).

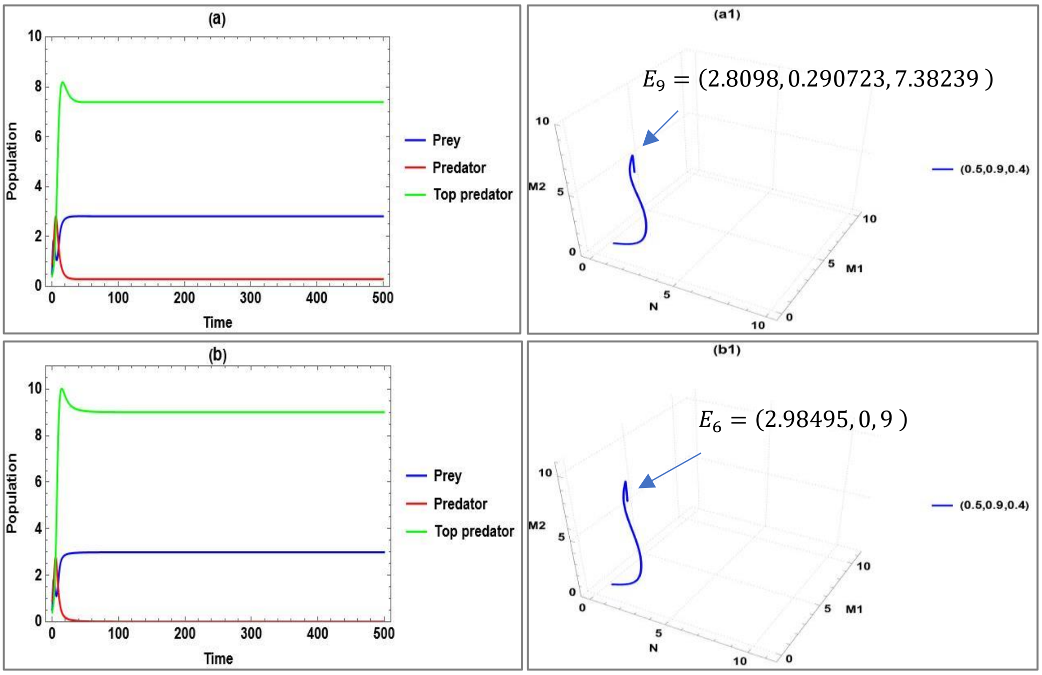

Figure 4: (a) Time series of the solution of system (1) which approaches to $E_9 = (2.8098,\ 0.290723,\ 7.38239)$ when $\beta = 0.3$, $(a1)$ 3D phase portrait of (a), (b) Time series of the solution of system (1) which approaches to $E_6 = (2.98495,\ 0,\ 9)$ when $\beta = 0.4$, $(b1)$ 3D phase portrait of (b).

The effect of varying the growth rate of prey population and the death rate of the predators was studied it is observed that system (1) will approach to the trivial equilibrium point $E_0 = (0,0,0)$ as it shows in Figure (5).

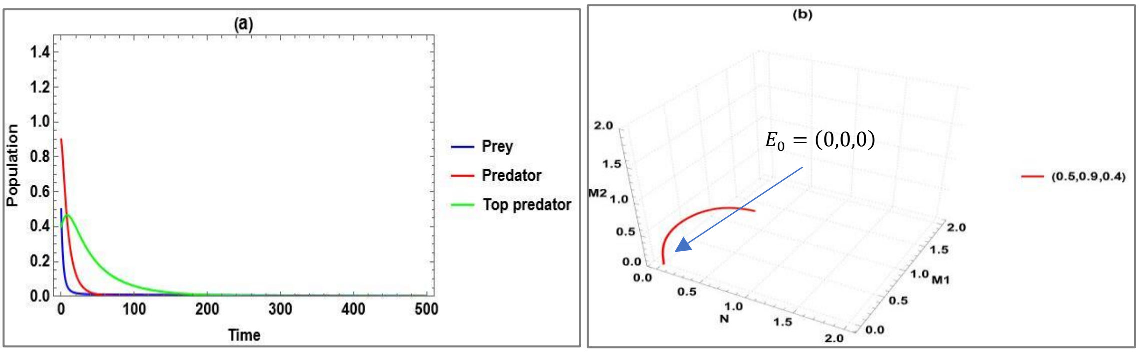

Figure 5: (a)-Time series of solution of system (1) for the data given in (40) approaches to $E_0 = (0,0,0)$ when $b = 0.016$, $d_1 = 0.09$ and $d_2 = 0.09$, (b)- 3D phase portrait of (a).

The effect of varying the growth rate of prey population, the death rate of the predators and harvesting rate of the prey population were studied, it is observed that system (1) will approach to the predators free equilibrium point E_1 as it show in Figure (6).

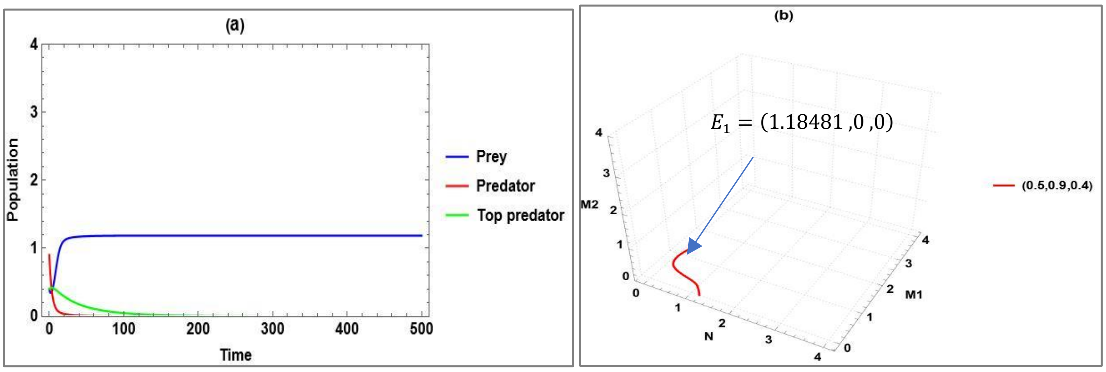

Figure 6: (a)-Time series of solution of system (1) for the data given in (40) approaches to free equilibrium point $E_1 = (1.18481, 0, 0)$ when $b = 1$, $d_1 = 0.3, d_2 = 0.09$ and $h_1 = 0.6$, (b)- 3D phase portrait of (a).

The effect of varying the death rate of the top predator and the additional food of the predators (η , β) were studied, it is observed that system (1) will approach to the predators free equilibrium point E_3 as it show in Figure (7).

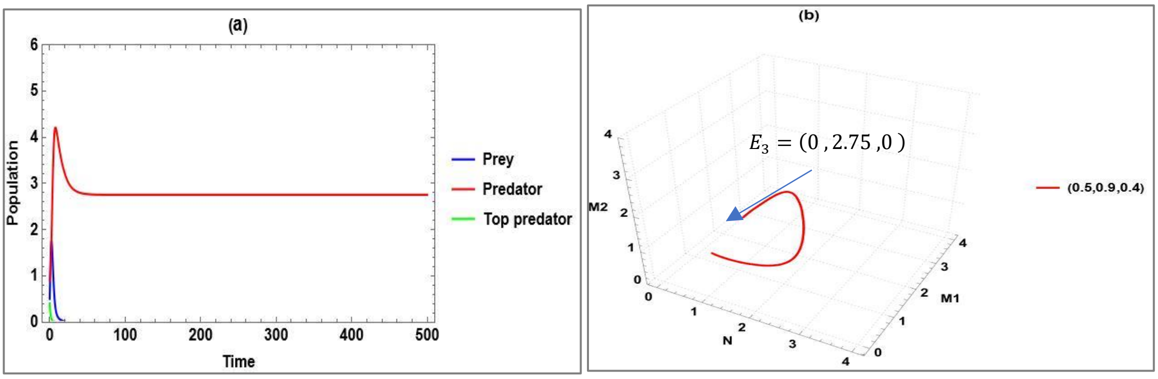

Figure 7: (a)-Time series of solution of system (1) for the data given in (40) approaches to $E_3 = (0, 2.75, 0)$ when $d_2 = 0.9$, $\eta = 0.4$ and $\beta = 0.2$, (b)- 3D phase portrait of (a).

The effect of varying the harvesting rate, handling time of the predator population and (η) were studied, it is observed that system (1) will approach to the prey free equilibrium point E_8 as it shows in Figure (8).

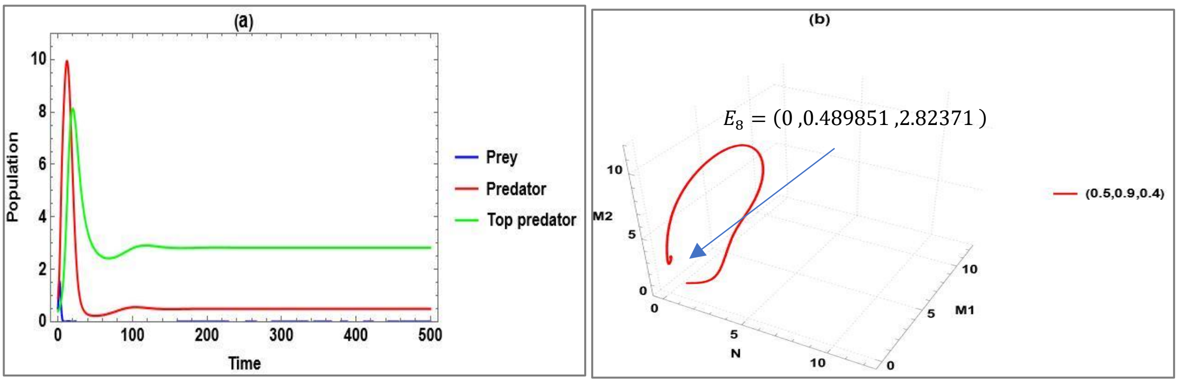

Figure 8: (a)-Time series of solution of system (1) for the data given in (40) approaches to prey free equilibrium point $E_8 = (0\,,0.489851\,,2.82371\,)$ when $h_2 = 0.01$, $n_3 = 0.2$, $\eta = 0.2$ and $\beta = 0.3$, (b)- 3D phase portrait of (a).

7.Conclutions and Discussions:

In our present work, we proposed and analysed a predator-prey model consisting of three species the prey, predator and top predator, it is assumed that the prey growth logistic with fear of the predator, Allee effect and linear harvesting, while we are proposed a non-liner harvesting in the predator's population as well as with additional food, the objectives of our work are to study:

- The impact of fear, Allee effect and harvesting in prey population.
- The impact of harvesting in predator population.
- The impact of suppled additional food in the harvesting of top predator population.

It is observed that system (1) has ten equilibria all of them are locally asymptotically stable under suitable conditions, six equilibria are studied globally because four of them having the same local stability conditions but with different neighbourhood of starting points. Finally, we verified our analytic result numerically for the data given in (40) and which are summarized

1. The parameters b, K, A, B_1, l_1, B_2, n_1, β, h_3, d_1 and d_2 have an important effect in controlling the stability of system (1).
2. The parameters f, l_2, n_2, n_3, n_4, η, h_1 and h_2 the solutions still approach to the positive equilibrium point.

References

[1] A. J. Lotka, *Elements of physical biology. Williams and Wilkins*. Baltimore, Md: Williams and Wilkins, 1925.

[2] V. Volterra, "Variazioni e fluttuazioni del numero d'individui in specie animali conviventi," *Memoria della Reale Accademia Nazionale del lincei*, 1926.

[3] R. M. May, "*Stability and complexity in model ecosystems,"* Princeton, NJ: Princeton University Press, 1973.

[4] K. R. Dixon and G. W. Cornwell, "A mathematical model for predator and prey populations," *Researches on Population Ecology*, vol. 12, no. 2, pp. 127–136, 1970.

[5] É. Diz-Pita and M. V. Otero-Espinar, "Predator–prey models: A review of some recent advances," *Mathematics*, vol. 9, no. 15, pp. 1783, 2021.

[6] A. A. Majeed and Z. K. Alabacy, "The persistence and bifurcation analysis of an ecological model with fear effect involving prey refuge and harvesting," 1^{st} *Samarra International Conference for Pure and Applied Sciences*, vol. 2394, no. 1, 2022.

[7] R. N. Shalan, "The Local Stability of an Eco-Epidemiological Model Involving a Harvesting on Predator Population", *International Journal of Advanced Science and Technology*, vol. 29, no. 1, pp. 946 - 955, 2020.

[8] Z. K. Alabacy and A. A. Majeed, "The local bifurcation analysis of two preys stage-structured predator model with anti-predator behavior," *Journal of Physics: Conference Series*, vol. 2322, no. 012061, pp.1-17, 2022.

[9] Y. Huang, Z. Zhu, and Z. Li, "Modeling the Allee effect and fear effect in predator–prey system incorporating a prey refuge," *Advances in Difference Equations*, vol. 2020, no. 1, 2020.

[10] I. K. Barasa, S.B.Apima and A.O.Wanjara"Mathematical Modelling of Plant-Pathogen-HerbivoreInteraction Incorporating Allee Effect and Harvesting," *Journal of Advances in Mathematics and Computer Science*, vol. 37, no. 9, pp. 23–38, 2022.

[11] A. Kumar and B. Dubey, "Dynamics of prey–predator model with strong and weak Allee effect in the prey with gestation delay," *Nonlinear Analysis Modelling and Control*, vol. 25, no. 3, pp. 417-442, 2020.

[12] N. Ali, "On the food chain model with prey refuge and fear effect," *International Journal of Nonlinear Analysis and Applications*, vol. 13, no. 2, pp. 2071-2086, 2022.

[13] C. S. Holling, "Some Characteristics of Simple Types of Predation and Parasitism," *Canadian Entomologist*, vol. 91, no. 7, pp. 385–398, 1959.

[14] C. S. Holling, "The Components of Predation as Revealed by a Study of Small-Mammal Predation of the European pine Sawfly," *Canadian Entomologist*, vol. 91, no. 5, pp. 293–320, 1959.

[15] X. Tian and R. Xu, "Global dynamics of a predator-prey system with Holling type II functional response," *Nonlinear Analysis Modelling and Control*, vol. 16, no. 2, pp. 242–253, 2011**.**

[16] O. A. Nev and H. A. van den Berg, "Holling Type I versus Holling Type II functional responses in Gram-negative bacteria," Transactions of Mathematics and Its Applications, vol. 2, no. 1, pp. 1–19, 2018.

[17] Q. Liu, D. Jiang, T. Hayat, and A. Alsaedi, "Dynamics of a stochastic predator–prey model with stage structure for predator and Holling type II functional response," *Journal of Nonlinear Science*, vol. 28, no. 3, pp. 1151–1187, 2018.

[18] Y. Xie, Z. Wang, B. Meng, and X. Huang, "Dynamical analysis for a fractional-order prey–predator model with Holling III type functional response and discontinuous harvest," *Applied Mathematics Letters*, vol. 106, no. 106342, pp. 1-8, 2020.

[19] X. Zou, Y. Zheng, L. Zhang, and J. Lv, "Survivability and stochastic bifurcations for a stochastic Holling type II predator-prey model," *Communications in Nonlinear Science and Numerical Simulation*, vol. 83, no. 105136, pp.1-45, 2020.

[20] H. Liu, Y. Ye, Y. Wei, M. Ma and J. Ye, "*Dynamic study of a predator-prey system with weak Allee effect and holling type- III functional response," Dynamic Systems and Applications,* vol. 27, no. 4, pp. 943-953, 2018.

[21] M. Rahman, M. S. Sabiar, and S. Islam, "Effects of prey refuge with Holling type IV functional response dependent prey predator model," *International Journal of Modelling and Simulation*, vol. 43, no.3, pp.1-29, 2023.

[22] R. S. Cantrell and C. Cosner, "On the dynamics of predator–prey models with the Beddington–DeAngelis functional response," *Journal of Mathematical Analysis and Applications*, vol. 257, no. 1, pp. 206–222, 2001.

[23] A. P. Maiti, C. Jana, and D. K. Maiti, "A delayed eco-epidemiological model with nonlinear incidence rate and Crowley–Martin functional response for infected prey and predator," *Nonlinear Dynamics*, vol. 98, no. 2, pp. 1137–1167, 2019.

[24] B. Chakraborty, H. Baek, and N. Bairagi, "Diffusion-induced regular and chaotic patterns in a ratio-dependent predator-prey model with fear factor and prey refuge," *Chaos*, vol. 31, no. 033128, pp.1-16, 2021.

[25] C. G. Dittmer and W. C. Allee, "Animal aggregations: A study in general sociology," *Journal* *of Educational Sociology*, vol. 5, no. 2, pp. 130, 1931.

[26] Z. AlSharawi, S. Pal, N. Pal, and J. Chattopadhyay, "A discrete-time model with non-monotonic functional response and strong Allee effect in prey," *Journal of Differences Equations and Applications*, vol. 26, no. 3, pp. 404–431, 2020.

[27] Z. Zhu, Y. Chen, Z. Li, and F. Chen, "Stability and Bifurcation in a Leslie–Gower Predator–Prey Model with Allee Effect," *International Journal of Bifurcation and Chaos*, vol. 32, no. 03, pp. 2250040-1-2250040-25, 2022.

[28] L. Y. Zanette, A. F. White, M. C. Allen, and M. Clinchy, "Perceived predation risk reduces the number of offspring songbirds produce per year," *Science*, vol. 334, no. 6061, pp. 1398–1401, 2011.

[29] K. Sarkar and S. Khajanchi, "Impact of fear effect on the growth of prey in a predator-prey interaction model," *Ecological Complexity*, vol. 42, no. 100826, pp. 1-17, 2020.

[30] V. Kumar and N. Kumari, "Controlling chaos in three species food chain model with fear effect," AIMS Mathematics, vol. 5, no. 2, pp. 828–842, 2019.

[31] S. K. Sasmal, S. Pal, N. Pal, and Y. Takeuchi, "Impact of fear on searching efficiency of prey: A prey-predator model with weak Allee effect," *International Journal of Bifurcation and Chaos*, vol. 33, no. 11, pp. 1-40, 2023.

[32] S. Chen, F. Chen, V. Srivastava, and R. D. Parshad, "Dynamical analysis of a Lotka-Volterra Competition Model with both Allee and Fear Effect," *International Journal of Biomathematics*, 2023. DOI:10.1142/S1793524523500778.

[33] S. Mondal, A. Maiti, and G. P. Samanta, "Effects of Fear and Additional Food in a Delayed Predator–Prey Model," Biophysical Reviews and Letters, vol. 13, no. 04, pp. 157–177, 2018.

[34] A. Das and G. P. Samanta, "Modeling the fear effect on a stochastic prey–predator system with additional food for the predator," *Journal of Physics A: Mathematical and Theoretical*, vol. 51, no. 46, pp.1-38, 2018.

[35] S. Mondal and G. P. Samanta, "Dynamics of an additional food provided predator-prey system with prey refuge dependent on both species and constant harvest in predator," *Physica A: Statistical Mechanics and Its Applications*, vol. 534, no. 122301, pp.1-21, 2019.

[36] D. Prakash and D. K. K. Bhanu, "Stochastic optimal and time-optimal control studies for additional food provided prey-predator systems involving Holling type III functional response," *Computational and Mathematical Biophysics*, vol. 11, no.1, pp.114-145, 2023.

[37] P. Panja, S. Poria, and S. K. Mondal, "Analysis of a harvested tritrophic food chain model in the presence of additional food for top predator," *International Journal of Biomathematics*, vol. 11, no. 04, pp. 1-29, 2018.

[38] P. Panja, S. Jana, and S. Kumar Mondal, "Effects of additional food on the dynamics of a three species food chain model incorporating refuge and harvesting," *International Jouurnal of Nonlinear Sciences and Numerical Simulation*, vol. 20, no. 7–8, pp. 787–801, 2019.

[39] K. S. Chaudhuri and S. S. Ray, "On the combined harvesting of a prey-predator system," *Journal of Biological Systems*, vol. 04, no. 03, pp. 373–389, 1996.

[40] S. Gakkhar and B. Singh, "The dynamics of a food web consisting of two preys and a harvesting predator," *Chaos Solitons and Fractals*, vol. 34, no. 4, pp. 1346–1356, 2007.

[41] R. P. Gupta, P. Chandra and M. Banerjee, "Dynamical complexity of a prey-predator model with nonlinear predator harvesting," *Discrete and Continuous Dynamical System B*, vol. 20, no. 2, pp. 423–443, 2015.

[42] H. Yang and J. Jia, "Harvesting of a predator–prey model with reserve area for prey and in the presence of toxicity," *Journal of Applied Mathematics and Computing*, vol. 53, no. 1, pp. 693–708, 2017.

[43] N. Mohdeb, "A dynamic analysis of a prey-predator population model with a nonlinear harvesting rate," *Arab Journal of Mathematical Sciences*, vol.29, no.2, pp.1-14, 2023.

[44] C. García, "Bifurcations on a discontinuous Leslie-Grower model with harvesting and alternative food for predators and Holling II functional response," *Communications in Nonlinear Science and Numerical Simulation*, vol. 116, no.106800, pp.1-25, 2023.

[45] A. Chatterjee and S. Pal, “A predator-prey model for the optimal control of fish harvesting through the imposition of a tax,” *International Journal for the Optimal Control Theories & Applications. (IJOCTA)*, vol. 13, no. 1, pp. 68–80, 2023.