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Bounds of sixth-order Toeplitz determinant for the families of function associated with bounded turnings

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Abstract
This research focused on studying the Toeplitz determinants for classes of functions with bounded turnings. Before that, some previous studies of Hankel's determinants have been discussed, due to their strong association with Toplitz determinants. They share most of the mathematical features. For example, if the transpose of both matrices is taken, then the original matrix is obtained. On the other hand, for the set BT of univalent functions with bounded turnings, estimates of the sixth-order Toeplitz determinants have been provided in the unit disc. Besides, the sixth Toeplitz determinant for the BT^2 and BT^4 was found, which subfamilies of the BT family.

Keywords: Toeplitz determinants, Hankel determinants, analytic functions, Bounded Turnings function.

حدود محددات توبليتز من الدرجة السادسة لعائلات الوظائف المرتبطة بالمنعطفات المحدودة

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الخلاصة
ركز هذا البحث على دراسة محددات توبليتز لفئات الدوال ذات الدورانات المحدودة. قبل ذلك، تمت مناقشة بعض الدراسات السابقة لمحددات هانكل، وذلك بسبب ارتباطها القوي بمحددات توبليتز. فهي تشترك في معظم السمات الرياضية. على سبيل المثال، إذا تم أخذ النقل لكلا المصفوفتين، فسيتم الحصول على المصفوفة الأصلية. من ناحية أخرى، بالنسبة لمجموعة BT للدوال أحادية التكافؤ ذات الدورانات المحدودة، تم توفير تقديرات لمحددات توبليتز من الدرجة السادسة في قرص الوحدة. بالإضافة إلى ذلك، تم العثور على محدد توبليتز السادس ل BT^2 و BT^4، والتي تعد عائلات فرعية من عائلة BT.

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1. Introduction

Let $\mathcal{A}$ represent the class of function of the following form:

$$h(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad (1)$$

which in the open unit disk $Ɗ = \{z: z \in ȼ: |z| < 1\}$ is analytic. Moreover, the class of all functions in $\mathcal{A}$ that are univalent in Ɗ will be indicated by ß. Furthermore, suppose that S^* and C^* stand for the classes of starlike and convex functions, respectively, which are defined as follows see [1]

$$S^* = \left\{h \in ß: Re\left[\frac{zh'(z)}{h(z)}\right] > 0, \qquad (z \in Ɗ)\right\},$$

$$C^* = \left\{h \in ß: Re\left[1 + \frac{zh''(z)}{h'(z)}\right] > 0, \qquad (z \in Ɗ)\right\},$$

when a starlike function g ∈ S^* exist and such a function h ∈ $\mathcal{A}$ is close-to-convex

$$\text{Re}\left[\frac{\text{zh}'(z)}{\text{g}(z)}\right] > 0, \ (z \in Ɗ).$$

Assume Q stands for the class of holomorphic functions q of the form:

$$q(z) = 1 + \sum_{k=1}^{\infty} e_k z^k. \quad (2)$$

Satisfies the condition $\{Re\, q(z) > 0, \ (z \in Ɗ)\}$. Suppose that the class of functions h in $\mathcal{A}$ fulfilling $\{Re\, h'(z) > 0, \ (z \in Ɗ)\}$ is indicated by BT. Functions in BT are close-to-convex, making them univalent, and this may be easily verified. Sometimes, functions in BT are referred to as functions of bounded turnings. Furthermore, assume that $\text{n} \in \text{N}: \text{n} = \{1,2, \cdots\}$. Analytic function h in Ɗ is n-fold symmetric, if

$$h\left(e^{\frac{2\pi i}{n}} * z\right) = e^{\frac{2\pi i}{n}} h(z), \qquad (z \in Ɗ).$$

The Taylor series of the set of n-fold univalent functions is represented by $ß^n$

$$h(z) = z + \sum_{j=1}^{\infty} b_{nj+1} z^{nj+1}, \qquad (z \in Ɗ). \quad (3)$$

The set of n-fold symmetric functions with bounded turnings is called the sub-family BT^n of $ß^n$. The family BT^n includes an analytic function h given by (1) if and only if

$$\text{h}'(z) = q(z),$$

where $q \in Q^n$ such that Q^n is the set defined by

$$Q^n = \{q \in Q: q(z) = 1 + \sum_{j=1}^{\infty} e_{nj} z^{nj}, (z \in Ɗ). \quad (4)$$

Hankel determinants were proposed by C. Pommmerenke [2, 3], who also defined them as follows for univalent function h ∈ ß:

$$H_p(k) = \begin{bmatrix} b_k & b_{k+1} & \cdots & b_{k+p-1} \\ b_{k+1} & b_{k+2} & \cdots & b_{k+p-2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k+p-1} & b_{k+p-2} & \cdots & b_{k+2(p-1)} \end{bmatrix}$$

One of the most researched issues in the theory of analytic functions is determining the upper bound of $|H_p(k)|$. For fixed values of p and k, several studies discovered the aforementioned bound for various subclasses of univalent functions [4, 5, 6]. Janteng, Halim, and Darus [7, 8] looked at the sharp boundaries of $|H_2(2)|$ for the subclasses S^*, C^*, and BT of the set ß. In this way, they demonstrated the bounds as follows:

$$|H_2(2)| \leq \begin{cases} 1, & for\ h \in S^* \\ 0.125, & for\ h \in C^* \\ 0.\overline{4}, & for\ h \in BT \end{cases}$$

Krishna and RamReddy [9] were able to acquire an accurate estimate of $|H_2(2)|$ for the family of Bazilevic functions. Thomas' conjecture [10] states that for subclasses of ß, if h ∈ ß, then $|H_p(2)| \leq 1$. However, Li and Srivastava's work [11] shows that this claim is false for k ≥ 4. The estimation of $|H_3(1)|$ is far more challenging. In 2010, Babalola [12] released the first article on $H_3(1)(h)$, in which he found the upper bound of $|H_3(1)|$ for the subclasses S^*, C^*,and BT. More recently, in 2017, Zaprawa [13] enhanced Babalola's results by demonstrating

$$|H_3(1)| \leq \begin{cases} 1, & for\ h \in S^* \\ 0.091, & for\ h \in C^* \\ 0.68\overline{3}, & for\ h \in BT \end{cases}$$

Regarding the classes of functions possessing bounded turnings, Arif, I. Ullah, Raza, and Zaprawa [14] determined the upper bounds of $|H_4(1)|$ and $|H_5(1)|$. A tight relation exists between Hankel determinants and Toeplitz determinants [15]. Along the diagonal, Toeplitz matrices contain fixed entries. Applications for Toeplitz matrices maybe found in both mathematics types pure and applied in [16].

The symmetric determinant $T_p(k)$ for analytic functions h given by (1) described by Thomas and Halim [17], was introduced as follows:

$$T_p(k) = \begin{bmatrix} b_k & b_{k+1} & \cdots & b_{k+p-1} \\ b_{k+1} & b_k & \cdots & b_{k+p-2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k+p-1} & b_{k+p-2} & \cdots & b_k \end{bmatrix} \quad p, k \geq 1.$$

Several authors have been drawn to the study of an accurate upper bound of $T_p(k)$ for various subclasses of analytic functions. By Ali, Thomas, and Vasudevarao [18] the Toeplitz determinant $T_p(k)$ for class ß and subclasses of ß of univalent functions was examined, refined, and explored.

Lemma 1.1. [19, 20] For $q \in Q$ given by (2), then

$$|e_k| \leq 2, \tag{5}$$

$$|e_{k+j} - \mu e_k e_j| \leq 2,\ for\ \mu \in [0,1]. \tag{6}$$

Where $k, j \in N: N = \{1, 2, \cdots\}$.

Lemma 1.2. [14] Let $h \in BT$ and $u + v = s + t$. Then

$$|b_u b_v - b_s b_t| \leq \frac{4}{\rho},\ for\ \rho = \min\{uv, st\}. \tag{7}$$

Lemma 1.3. Assume that $h \in BT$ has a form (1), then

$$|b_k| \leq \frac{2}{k}, \qquad for\ k \geq 2.$$

Proof. Let $h \in BT$ given by (1). Then there exists a function $q \in Q$ of the form (2) so that

$$h'(z) = q(z),$$

rearranging the above equation thus using the series representations for h′(z) and q(z), we get

$$1 + 2b_2z + 3b_3z^2 + 4b_4z^3 + \cdots = 1 + e_1z + e_2z^2 + e_3z^3 + \cdots,$$

By comparing the following equations, we find

$$b_2 \le \frac{e_1}{2}, \quad b_3 \le \frac{e_2}{3}, \quad b_4 \le \frac{e_3}{4}, \quad b_5 \le \frac{e_4}{5} \ \ and \ \ b_6 \le \frac{e_5}{6}.$$

From the above results, the Lemma proof deduced.

Theorem 1.3. [18] If $h \in BT$ given by (1), then

a) $|T_2(k)| \le \frac{4}{k^2} + \frac{4}{(k+1)^2}, \ k \ge 2.$ (8)

b) $|T_3(1)| \le \frac{35}{9}.$ (9)

c) $|T_3(2)| \le \frac{7}{3}.$ (10)

Theorem 1.4. [21] If $h \in BT$ given by (1), then

a) $|T_4(1)| \le \frac{5199}{486}.$ (11)

b) $|T_5(1)| \le \frac{59378}{2025}.$ (12)

2. Bound of $T_6(1)$ for the set BT

The upper bounds of the sixth-order Toeplitz determinants have been found in this part. The Toeplitz matrix written as:

$$T_6(1) = \begin{bmatrix} 1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b_2 & 1 & b_2 & b_3 & b_4 & b_5 \\ b_3 & b_2 & 1 & b_2 & b_3 & b_4 \\ b_4 & b_3 & b_2 & 1 & b_2 & b_3 \\ b_5 & b_4 & b_3 & b_2 & 1 & b_2 \\ b_6 & b_5 & b_4 & b_3 & b_2 & 1 \end{bmatrix}$$

$T_6(1)$ can be written in the format

$$T_6(1) = T_5(1) - b_2 A + b_3 B - b_4 C + b_5 D - b_6 E, \tag{13}$$

where

$$T_5(1) = T_4(1) - b_2\delta_1 + b_3\delta_2 - b_4\delta_3 + b_5\delta_4, \tag{14}$$
$$A = b_2 T_4(1) - b_2\delta_5 + b_3\delta_6 - b_4\delta_7 + b_5\delta_8, \tag{15}$$
$$B = b_2\delta_9 - \delta_{10} + b_3\delta_{11} - b_4\delta_{12} + b_5\delta_{13}, \tag{16}$$
$$C = b_2\delta_{14} - \delta_{15} + b_2\delta_{16} - b_4\delta_{17} + b_5\delta_{18}, \tag{17}$$
$$D = b_2\delta_{19} - \delta_{20} + b_2\delta_{21} - b_3\delta_{22} + b_5\delta_{23}, \tag{18}$$
$$E = b_2\delta_{24} - \delta_{25} + b_2\delta_{26} - b_3\delta_{27} + b_4\delta_{28}. \tag{19}$$

From (13), $T_6(1)$ will be considered that be a polynomial of five successive coefficients $b_2, b_3, b_4, b_5 \ and, b_6$ of a function h in a certain class. Also, each δ in above has a certain polynomial.

Several theorems have been written and proven to support our claim as follows:

Theorem 2.1. Let $h \in BT$ given by (1). Then

$$|A| \le \frac{104353}{3888} \approx 26.840.$$

Proof. Since $h \in BT$, using triangle inequality for each one and from (9) and (11) with all the lemmas, we get

$$\begin{aligned} |T_4(1)| = |T_3(1) &- b_2[b_2(1-b_2^2) - b_2(b_3 - b_2 b_4) + b_3(b_2 b_3 - b_4)] \\ &+ b_3[b_2(b_2 - b_2 b_3) - (b_3 - b_2 b_4) + b_3(b_3^2 - b_2 b_4)] - b_4[b_2(b_2^2 - b_3) \\ &- (b_2 b_3 - b_4) + b_2(b_3^2 - b_2 b_4)| \le \frac{5199}{486}, \end{aligned}$$

$$\begin{aligned} |\delta_5| = |b_3 T_3(1) &- b_2[b_4(1-b_2^2) - b_2(b_5 - b_2 b_6) + b_3(b_2 b_5 - b_6)] \\ &+ b_3[b_4(b_2 - b_2 b_3) - (b_5 - b_2 b_6) + b_3(b_3 b_5 - b_2 b_6)] - b_4[b_4(b_2^2 - b_3) \\ &- (b_2 b_5 - b_6) + b_2(b_3 b_5 - b_2 b_6)]| \le \frac{367}{54}, \end{aligned}$$

$$
\begin{aligned}
|\delta_6| = |b_3[&b_2(1-b_2^2) - b_2(b_3 - b_2b_4) + b_3(b_2b_3 - b_4)] \\
&- [b_4(1-b_2^2) - b_2(b_5 - b_2b_6) + b_3(b_2b_5 - b_6)] \\
&+ b_3[b_4(b_3 - b_2b_4) - b_2(b_5 - b_2b_6) + b_3(b_4b_5 - b_3b_6)] \\
&- b_4[b_4(b_2b_3 - b_4) - b_2(b_2b_5 - b_6) + b_2(b_4b_5 - b_3b_6)]| \le \frac{2075}{324},
\end{aligned}
$$

$$
\begin{aligned}
|\delta_7| = |b_3[&b_2(b_2 - b_2b_3) - (b_3 - b_2b_4) + b_3(b_3^2 - b_2b_4)] \\
&- [b_4(b_2 - b_2b_3) - (b_5 - b_2b_6) + b_3(b_3b_5 - b_2b_6)] \\
&+ b_2[b_4(b_3 - b_2b_4) - b_2(b_5 - b_2b_6) + b_3(b_4b_5 - b_3b_6)] \\
&- b_4[b_4(b_3^2 - b_2b_4) - b_2(b_3b_5 - b_2b_6) + (b_4b_5 - b_3b_6)]| \le \frac{6229}{1080},
\end{aligned}
$$

$$
\begin{aligned}
|\delta_8| = |b_3[&b_2(b_2^2 - b_3) - (b_2b_3 - b_4) + b_2(b_3^2 - b_2b_4)] \\
&- [b_4(b_2^2 - b_3) - (b_2b_5 - b_6) + b_2(b_3b_5 - b_2b_6)] \\
&+ b_2[b_4(b_2b_3 - b_4) - b_2(b_2b_5 - b_6) + b_2(b_4b_5 - b_3b_6)] \\
&- b_3[b_4(b_3^2 - b_2b_4) - b_2(b_3b_5 - b_2b_6) + (b_4b_5 - b_3b_6)]| \le \frac{148}{27}.
\end{aligned}
$$

From (15), using some simple mathematical operations with the triangle inequality, the proof is complete.

Theorem 2.2. Suppose that $h \in BT$ has the form (1), then

$$|B| \le \frac{75947}{3240} \approx 23.440.$$

Proof. Let $h \in BT$ and using the following equations, (5), (6), (7), and (9) and Lemma [1.3], we find that

$$
\begin{aligned}
|\delta_9| = |b_2T_3(1) &- b_2[b_3(1-b_2^2) - b_2(b_4 - b_2b_5) + b_3(b_2b_4 - b_5)] \\
&+ b_3[b_3(b_2 - b_2b_3) - (b_4 - b_2b_5) + b_3(b_3b_4 - b_2b_5)] - b_4[b_3(b_2^2 - b_3) \\
&- (b_2b_4 - b_5) + b_2(b_3b_4 - b_2b_5)]| \le \frac{2489}{270},
\end{aligned}
$$

$$
\begin{aligned}
|\delta_{10}| = |b_3T_3(1) &- b_2[b_4(1-b_2^2) - b_2(b_5 - b_2b_6) + b_3(b_2b_5 - b_6)] \\
&+ b_3[b_4(b_2 - b_2b_3) - (b_5 - b_2b_6) + b_3(b_3b_5 - b_2b_6)] - b_4[b_4(b_2^2 - b_3) \\
&- (b_2b_5 - b_6) + b_2(b_3b_5 - b_2b_6)]| \le \frac{367}{54},
\end{aligned}
$$

$$
\begin{aligned}
|\delta_{11}| = |b_3[&b_3(1-b_2^2) - b_2(b_4 - b_2b_5) + b_3(b_2b_4 - b_5)] \\
&- b_2[b_4(1-b_2^2) - b_2(b_5 - b_2b_6) + b_3(b_2b_5 - b_6)] \\
&+ b_3[b_4(b_4 - b_2b_5) - b_3(b_5 - b_2b_6) + b_3(b_5^2 - b_4b_6)] \\
&- b_4[b_4(b_2b_4 - b_5) - b_3(b_2b_5 - b_6) + b_2(b_5^2 - b_4b_6)]| \le \frac{2903}{540},
\end{aligned}
$$

$$
\begin{aligned}
|\delta_{12}| = |b_3[&b_3(b_2 - b_2b_3) - (b_4 - b_2b_5) + b_3(b_3b_4 - b_2b_5)] \\
&- b_2[b_4(b_2 - b_2b_3) - (b_5 - b_2b_6) + b_3(b_3b_5 - b_2b_6)] \\
&+ b_2[b_4(b_4 - b_2b_5) - b_3(b_5 - b_2b_6) + b_3(b_5^2 - b_4b_6)] \\
&- b_4[b_4(b_3b_4 - b_2b_5) - b_3(b_3b_5 - b_2b_6) + (b_5^2 - b_4b_6)]| \le \frac{2557}{540},
\end{aligned}
$$

$$
\begin{aligned}
|\delta_{13}| = |b_3[&b_3(b_2^2 - b_3) - (b_2b_4 - b_5) + b_2(b_3b_4 - b_2b_5)] \\
&- b_2[b_4(b_2^2 - b_3) - (b_2b_5 - b_6) + b_2(b_3b_5 - b_2b_6)] \\
&+ b_2[b_4(b_2b_4 - b_5) - b_3(b_2b_5 - b_6) + b_2(b_5^2 - b_4b_6)] \\
&- b_3[b_4(b_3b_4 - b_2b_5) - b_3(b_3b_5 - b_2b_6) + (b_5^2 - b_4b_6)]| \le \frac{199}{54}.
\end{aligned}
$$

Finally, the stated bound was obtained using the triangle inequality again in equation (16).

Theorem 2.3. Suppose that h ∈ BT has the form (1), then

$$|C| \leq \frac{386621}{16200} \approx 23.865.$$

Proof. Assume that h ∈ BT, from Lemma 1.1-1.3 beside the triangle inequality, we obtain

$$\begin{aligned}|\delta_{14}| = |b_2[b_2(1-b_2^2) - b_2(b_3 - b_2b_4) + b_3(b_2b_3 - b_4)] \\ - [b_3(1-b_2^2) - b_2(b_4 - b_2b_5) + b_3(b_2b_4 - b_5)] \\ + b_3[b_3(b_3 - b_2b_4) - b_2(b_4 - b_2b_5) + b_3(b_4^2 - b_3b_5)] \\ - b_4[b_3(b_2b_3 - b_4) - b_2(b_2b_4 - b_5) + b_2(b_4^2 - b_3b_5)]| \leq \frac{235}{27},\end{aligned}$$

$$\begin{aligned}|\delta_{15}| = |b_3[b_2(1-b_2^2) - b_2(b_3 - b_2b_4) + b_3(b_2b_3 - b_4)] \\ - [b_4(1-b_2^2) - b_2(b_5 - b_2b_6) + b_3(b_2b_5 - b_6)] \\ + b_3[b_4(b_3 - b_2b_4) - b_2(b_5 - b_2b_6) + b_3(b_4b_5 - b_3b_6)] \\ - b_4[b_4(b_2b_3 - b_4) - b_2(b_2b_5 - b_6) + b_2(b_4b_5 - b_3b_6)]| \leq \frac{2075}{324},\end{aligned}$$

$$\begin{aligned}|\delta_{16}| = |b_3[b_3(1-b_2^2) - b_2(b_4 - b_2b_5) + b_3(b_2b_4 - b_5)] \\ - b_2[b_4(1-b_2^2) - b_2(b_5 - b_2b_6) + b_3(b_2b_5 - b_6)] \\ + b_3[b_4(b_4 - b_2b_5) - b_3(b_5 - b_2b_6) + b_3(b_5^2 - b_4b_6)] \\ - b_4[b_4(b_2b_4 - b_5) - b_3(b_2b_5 - b_6) + b_2(b_5^2 - b_4b_6)]| \leq \frac{941}{180},\end{aligned}$$

$$\begin{aligned}|\delta_{17}| = |b_3[b_3(b_3 - b_2b_4) - b_2(b_4 - b_2b_5) + b_3(b_4^2 - b_3b_5)] \\ - b_2[b_4(b_3 - b_2b_4) - b_2(b_5 - b_2b_6) + b_3(b_4b_5 - b_3b_6)] \\ + [b_4(b_4 - b_2b_5) - b_3(b_5 - b_2b_6) + b_3(b_5^2 - b_4b_6)] \\ - b_4[b_4(b_4^2 - b_3b_5) - b_3(b_4b_5 - b_3b_6) + b_2(b_5^2 - b_4b_6)]| \leq \frac{2147}{540},\end{aligned}$$

$$\begin{aligned}|\delta_{18}| = |b_3[b_3(b_2b_3 - b_4) - b_2(b_2b_4 - b_5) + b_3(b_4^2 - b_3b_5)] \\ - b_2[b_4(b_2b_3 - b_4) - b_2(b_2b_5 - b_6) + b_2(b_4b_5 - b_3b_6)] \\ + [b_4(b_2b_4 - b_5) - b_3(b_2b_5 - b_6) + b_2(b_5^2 - b_4b_6)] \\ - b_3[b_4(b_4^2 - b_3b_5) - b_3(b_4b_5 - b_3b_6) + b_2(b_5^2 - b_4b_6)]| \leq \frac{1561}{405}.\end{aligned}$$

Using the values above in (17) produces the intended outcome. Thus, the proof is complete.

Theorem 2.4. Suppose that h ∈ BT has the form (1), then

$$|D| \leq \frac{71141}{3240} \approx 21.957.$$

Proof. Let h ∈ BT, using Lemmas 1.1, 1.2, and 1.3 with triangle inequality, we find that

$$\begin{aligned}|\delta_{19}| = |b_2[b_2(b_2 - b_2b_3) - (b_3 - b_2b_4) + b_3(b_3^2 - b_2b_4)] \\ - [b_3(b_2 - b_2b_3) - (b_4 - b_2b_5) + b_3(b_3b_4 - b_2b_5)] \\ + b_2[b_3(b_3 - b_2b_4) - b_2(b_4 - b_2b_5) + b_3(b_4^2 - b_3b_5)] \\ - b_4[b_3(b_3^2 - b_2b_4) - b_2(b_3b_4 - b_2b_5) + (b_4^2 - b_3b_5)]| \leq \frac{39}{5},\end{aligned}$$

$$\begin{aligned}|\delta_{20}| = |b_3[b_2(b_2 - b_2b_3) - (b_3 - b_2b_4) + b_3(b_3^2 - b_2b_4)] \\ - [b_4(b_2 - b_2b_3) - (b_5 - b_2b_6) + b_3(b_3b_5 - b_2b_6)] \\ + b_2[b_4(b_3 - b_2b_4) - b_2(b_5 - b_2b_6) + b_3(b_4b_5 - b_3b_6)] \\ - b_4[b_4(b_3^2 - b_2b_4) - b_2(b_3b_5 - b_2b_6) + (b_4b_5 - b_3b_6)]| \leq \frac{6229}{1080},\end{aligned}$$

$$|\delta_{21}| = |b_3[b_3(b_2 - b_2b_3) - (b_4 - b_2b_5) + b_3(b_3b_4 - b_2b_5)] - b_2[b_4(b_2 - b_2b_3) - (b_5 - b_2b_6) + b_3(b_3b_5 - b_2b_6)] + b_2[b_4(b_4 - b_2b_5) - b_3(b_5 - b_2b_6) + b_3(b_5^2 - b_4b_6)] - b_4[b_4(b_3b_4 - b_2b_5) - b_3(b_3b_5 - b_2b_6) + (b_5^2 - b_4b_6)]| \leq \frac{2557}{540},$$

$$|\delta_{22}| = |b_3[b_3(b_3 - b_2b_4) - b_2(b_4 - b_2b_5) + b_3(b_4^2 - b_3b_5)] - b_2[b_4(b_3 - b_2b_4) - b_2(b_5 - b_2b_6) + b_3(b_4b_5 - b_3b_6)] + [b_4(b_4 - b_2b_5) - b_3(b_5 - b_2b_6) + b_3(b_5^2 - b_4b_6)] - b_4[b_4(b_4^2 - b_3b_5) - b_3(b_4b_5 - b_3b_6) + b_2(b_5^2 - b_4b_6)]| \leq \frac{2147}{540},$$

$$|\delta_{23}| = |b_3[b_3(b_3^2 - b_2b_4) - b_2(b_3b_4 - b_2b_5) + (b_4^2 - b_3b_5)] - b_2[b_4(b_3^2 - b_2b_4) - b_2(b_3b_5 - b_2b_6) + (b_4b_5 - b_3b_6)] + [b_4(b_3b_4 - b_2b_5) - b_3(b_3b_5 - b_2b_6) + (b_5^2 - b_4b_6)] - b_2[b_4(b_4^2 - b_3b_5) - b_3(b_4b_5 - b_3b_6) + b_2(b_5^2 - b_4b_6)]| \leq \frac{271}{108}.$$

Now, using (18) and from the results obtained above, the proof of the theorem is complete.

Theorem 2.5. Let h ∈ BT, be given by (1). Then

$$|E| \leq \frac{205003}{9720} \approx 21.091.$$

Proof. Since h ∈ BT, from (5), (6), and (7) with Lemma 1.3, we have

$$|\delta_{24}| = |b_2[b_2(b_2^2 - b_3) - (b_2b_3 - b_4) + b_2(b_3^2 - b_2b_4)] - [b_3(b_2^2 - b_3) - (b_2b_4 - b_5) + b_2(b_3b_4 - b_2b_5)] + b_2[b_3(b_2b_3 - b_4) - b_2(b_2b_4 - b_5) + b_2(b_4^2 - b_3b_5)] - b_3[b_3(b_3^2 - b_2b_4) - b_2(b_3b_4 - b_2b_5) + (b_4^2 - b_3b_5)]| \leq \frac{659}{90},$$

$$|\delta_{25}| = |b_3[b_2(b_2^2 - b_3) - (b_2b_3 - b_4) + b_2(b_3^2 - b_2b_4)] - [b_4(b_2^2 - b_3) - (b_2b_5 - b_6) + b_2(b_3b_5 - b_2b_6)] + b_2[b_4(b_2b_3 - b_4) - b_2(b_2b_5 - b_6) + b_2(b_4b_5 - b_3b_6)] - b_3[b_4(b_3^2 - b_2b_4) - b_2(b_3b_5 - b_2b_6) + (b_4b_5 - b_3b_6)]| \leq \frac{148}{27},$$

$$|\delta_{26}| = |b_3[b_3(b_2^2 - b_3) - (b_2b_4 - b_5) + b_2(b_3b_4 - b_2b_5)] - b_2[b_4(b_2^2 - b_3) - (b_2b_5 - b_6) + b_2(b_3b_5 - b_2b_6)] + b_2[b_4(b_2b_4 - b_5) - b_3(b_2b_5 - b_6) + b_2(b_5^2 - b_4b_6)] - b_3[b_4(b_3b_4 - b_2b_5) - b_3(b_3b_5 - b_2b_6) + (b_5^2 - b_4b_6)]| \leq \frac{241}{54},$$

$$|\delta_{27}| = |b_3[b_3(b_2b_3 - b_4) - b_2(b_2b_4 - b_5) + b_2(b_4^2 - b_3b_5)] - b_2[b_4(b_2b_3 - b_4) - b_2(b_2b_5 - b_6) + b_2(b_4b_5 - b_3b_6)] + [b_4(b_2b_4 - b_5) - b_3(b_2b_5 - b_6) + b_2(b_5^2 - b_4b_6)] - b_3[b_4(b_4^2 - b_3b_5) - b_3(b_4b_5 - b_3b_6) + b_2(b_5^2 - b_4b_6)]| \leq \frac{1561}{405},$$

$$|\delta_{28}| = |b_3[b_3(b_3^2 - b_2b_4) - b_2(b_3b_4 - b_2b_5) + (b_4^2 - b_3b_5)] - b_2[b_4(b_3^2 - b_2b_4) - b_2(b_3b_5 - b_2b_6) + (b_4b_5 - b_3b_6)] + [b_4(b_3b_4 - b_2b_5) - b_3(b_3b_5 - b_2b_6) + (b_5^2 - b_4b_6)] - b_2[b_4(b_4^2 - b_3b_5) - b_3(b_4b_5 - b_3b_6) + b_2(b_5^2 - b_4b_6)]| \leq \frac{271}{108}.$$

Consequently, the proof has been found through using the above values in equation (19).

Theorem 2.6. Let h ∈ BT, be given by (1). Then

$$|T_6(1)| \leq \frac{14512211}{145800} \approx 99.535.$$

Proof. Assume that h ∈ BT, and from (13), then

$$|T_6(1)| \leq |T_5(1)| + |b_2||A| + |b_3||B| + |b_4||C| + |b_5||D| + |b_6||E|.$$

Now, using the bounds obtained in the Theorems {2.1-2.5}. Also, from Lemma 1.3 and (12), we have

$$|T_6(1)| \leq \frac{59378}{2025} + \frac{104353}{3888} + \left(\frac{2}{3} * \frac{75947}{3240}\right) + \left(\frac{1}{2} * \frac{386621}{16200}\right) + \left(\frac{2}{5} * \frac{71141}{3240}\right) + \left(\frac{1}{3} * \frac{205003}{9720}\right).$$

Thus, the proof is complete.

3. $T_6(1)$ bounds for BT^2 and BT^4

In this part, $T_6(1)$ bounds were obtained for subclasses BT^2 and BT^4.

Theorem 3.1. Suppose that $h \in BT^2$, has the form (3), then

$$|T_6(1)| \leq \frac{292681}{50625} \approx 5.781.$$

Proof. Let $h \in BT^2$. Then there exists a function q ∈ Q^2 as follows:

$$h'(z) = q(z),$$

by equating the coefficients of the equations

$$b_2 \leq 0, \quad b_3 \leq \frac{e_2}{3}, \quad b_4 \leq 0, \quad b_5 \leq \frac{e_4}{5} \text{ and } b_6 \leq 0. \tag{20}$$

Through a simple calculation, $T_6(1)$ may be expressed as:

$$T_6(1) = 1 - 4b_3^2 + 4b_3^2 b_5 - 2b_5^2 + 4b_3^4 - 8b_3^4 b_5 + 4b_3^2 b_5^2 + 4b_3^4 b_5^2 - 4b_3^2 b_5^3 + b_5^4.$$

From (20) and using triangle inequality, we obtain

$$T_6(1) \leq 1 + \frac{4}{9}|e_2|^2 + \frac{4}{45}|e_2|^2|e_4| + \frac{2}{25}|e_4|^2 + \frac{4}{81}|e_2|^4 + \frac{8}{405}|e_2|^4|e_4| + \frac{4}{225}|e_2|^2|e_4|^2 + \frac{4}{2025}|e_2|^4|e_4|^2 + \frac{4}{1125}|e_2|^2|e_4|^3 + \frac{1}{625}|e_4|^4.$$

It is evident from (5) that

$$T_6(1) \leq 1 + \frac{16}{9} + \frac{32}{45} + \frac{8}{25} + \frac{64}{81} + \frac{256}{405} + \frac{64}{225} + \frac{256}{2025} + \frac{128}{1125} + \frac{16}{625}.$$

From a simple calculation, the proof did deduce.

Theorem 3.2. Assume that h ∈ BT^4, submitted by (3), then

$$|T_6(1)| \leq \frac{841}{625} \approx 1.346.$$

Proof. Let h ∈ BT^4, using the same steps as the previous Theorem, the following is concluded

$$b_2 = b_3 = b_4 = b_6 = 0 \text{ and } b_5 \leq \frac{e_4}{5}. \tag{21}$$

It's easy to find the following

$$T_6(1) = 1 - 2b_5^2 + b_5^4.$$

Therefore, the required proof can be obtained using triangle inequality and from (5).

4. Conclusions

The summary presented in this work continues previous studies of the Toeplitz selector with BT function. A sixth-order Toeplitz determinant has been obtain for BT, BT^2, and BT^4. It concluded that BT $> BT^2 > BT^4$, meaning that as the force increases, the results decrease.

Researchers later can study functions other than BT or evens find forces other than those mentioned previously.

5. Disclosure and conflict of interest

"Conflict of Interest: The authors declare that they have no conflicts of interest."

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