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Commutative rings with zero divisor graphs of orders are 23 ,24 and25

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Abstract

For a commutative ring with identity 1≠0. We denote the set of all zero divisors of R by $Z(R)$ and $Z(R)-\{0\}=Z(R)^*$. Let $\Gamma(R)$ denote the zero-divisor graph of R. Many authors have investigated zero-divisor graphs of commutative rings. In particular, some authors gave all rings with realizable graph order less than or equal to 22, the exploration of this classification for degrees 23, 24, and 25 is still a subject of ongoing research. In this paper, we present all possible rings when zero divisor graphs order 23, 24 and 25.

Keywords: zero divisor graph, local ring, order of a graph, direct product local rings.

الحلقات التبادلية ذات البيان القاسم للصفر من الدرجة 23,24 و25

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الخلاصة

للحلقة التبادلية مع العنصر الغير صفري $1\neq 0$. نشير الى مجموعة جميع المقسومات الصفرية للحلقة R بواسطة $Z(R)$، $Z(R)-\{0\}=Z(R)^*$. تشير $\Gamma(R)$ الى الرسم البياني المقسوم على الصفر للحلقة R. قام العديد من المؤلفين بدراسة الرسوم البيانية ذات المقسوم الصفري للحلقات التبادلية. اعطى بعض المؤلفين جميع الحلقات ذات الدرجات التي تصل 22 او اقل، إلا أن استكشاف هذا التصنيف للدرجات 23 و 24 و 25 لا يزال موضوع بحث مستمر. في هذا العمل، نستعرض جميع الحلقات الممكنة عندما تكون الرسوم البيانية محققة للدرجات 23 و 24 و 25.

1. Introduction

Let R be a commutative ring with identity 1. R is said to be local if it has only one proper ideal L and it is noted by (R, L). A field with a degree S is represented by F_S. An element $a\in R$ is referred to as nilpotent if it satisfies the condition $a^t=0$ for a non- negative integer number t. The set of all nilpotent elements is called nil radical and symbolized by $N(R)$. If $N(R)=0$, then the ring is said to be. Let Z_n represent the integer ring modulo n. It follows that for each prime number p, Z_p is a field and $Z_p\cong F_p$. Every set S has its cardinality denoted by $|A|$ and $A^*=A-\{0\}$. The collection of zero divisors is represented as $Z(R)$. An element r in a ring R is said to be a unit or invertible if there is r′ satisfied $\mathrm{r}.\mathrm{r}'=1$. The set of all invertible elements is represented as $U(R)$. When R is a finite, then $R=U(R)\cup Z(R)$.

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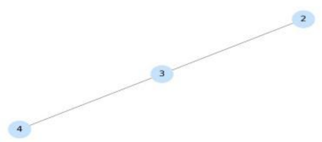
In addition, the direct product of rings $R_1, R_2, \dots, R_s$ is represented as $R_1 \times R_2 \times \dots \times R_s$ defined by $(r_1, r_2, \dots, r_s) + (r_1{}', r_2{}', \dots, r_s{}') = ((r_1 + r_1', r_2 + r_2', \dots, r_s + r_s')$ and $(r_1, r_2, \dots, r_s) \times (r_1{}', r_2{}', \dots, r_s{}') = ((r_1 \times r_1', r_2 \times r_2', \dots, r_s \times r_s')$, for every $r_t, r_t' \in R_t$ and $1 \leq t \leq s$. If R finite ring, then R is isomorphic to direct product local rings, see for example [1]. In [2], M. Behboodi and R. Beyranvand show if $|Z(R)| = p_1^{t_1}.p_2^{t_2}.\dots p_s^{t_s}$ where $p_i's$ are prime numbers and $t_i's$ positive integers for $1 \leq i \leq s$. Thus, $R \cong R_1 \times R_2 \times \dots \times R_s$ such that $|R_i| = p_i^{m_i}$, satisfied $t_i < m_i$. We denote $R[T]$ as a polynomial ring over coefficient T, which can be defined as $\{\sum_{i=0}^{\infty} a_i T^i : a_i \in R\}$, and R/I is denoted by a quotient ring or (factor ring). For more details, see [1 ‹3].

The relation between abstract algebra and graph theory is given in [4] when I. Beck associates an undirected graph whose vertex set is R and distinct $x, y \in R$ are adjacent in this graph if and only if xy = 0. In [5] D. F. Anderson and P. Livingston modified the graph introduced by Beck and called the modified graph introduced by them the zero-divisor graph of R and denoted it by $\Gamma(R)$ this graph have vertices $V(\Gamma(R)) = Z(R)^*$ satisfying distinct vertices r_1 and r_2 are adjacent if and only if $r_1 r_2 = 0$. D. F. Anderson and P. Livingston showed that this graph is linked with a diameter of less than four $\Gamma(R)$ is empty if R is an integral domain.

This field has been studied by numerous authors for instance, refer to sources [6, 7, 8]. After that, researchers gave several expanded to this. For example, Redmond [9] defined an ideal based zero divisor and D. Bennis, j. Mikram, f. Taraza in [10] gave an extended zero divisor graph. Also, E. Mehdi-Nezhad and A. M. Rahimi defined k-zero-divisor hypergraphs [11]. Later, many authors defined the idea of relating ring theory to graph theory and have given rise to the evolution of many new relationships between these two important mathematical theories, for example [12, 13, 14].
Every ring has a zero-divisor graph. However, the converse is not true in general; for example, any graph with a diameter greater than or equal to 4 is not a realized ring. In addition, if $R_1 \cong R_2$, where R_1 and R_2 are two rings, then $\Gamma(R_1) \cong \Gamma(R_2)$, while there are non-isomorphic rings R_1 and R_2 satisfied $\Gamma(R_1) \cong \Gamma(R_2)$. For example $Z_6 \ncong Z_8$ but $(Z_6) \cong \Gamma(Z_8)$, see Figures (a) and (b) as follows.



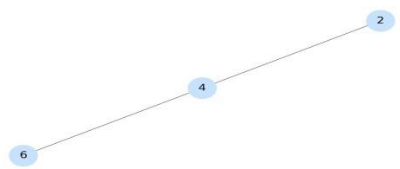


Figure -a $\Gamma(Z_6)$ **Figure -b** $\Gamma(Z_8)$

The authors in [5], investigated graphs whose order is less than or equal. Subsequently, as shown in [15] several writers classified the graphs with order less than or equal to 14. Some authors have categorized graphs with an order of 22 or less, see [16]. In this study, we categorize graphs as realizable rings with an order of 23, 24 and 25 vertices.

2. Rings where $|Z(R)^*| = 23$

In this section, we investigate commutative rings where $|Z(R)^*| = 23$. First, we give some basic lemmas that are important in our current work.

Lemma 2.1: [17] For every finite local ring (R_1, L_1) and (R_2, L_2), $|Z(R_1 \times R_2)^*| = |R_1|.|L_2| + |R_2|.|L_1| - |L_1|.|L_2| - 1$.

Corollary 2.2: [17] Assume that F_{s_1} , F_{s_2} are fields, then $\left|Z\left(F_{s_1} \times F_{s_2}\right)^*\right| = S_1 + S_2 - 2$.

Lemma 2.3: [16] For every finite local rings (R_1, L_1) , (R_2, L_2) and (R_3, L_3), $|Z(R_1 \times R_2 \times R_3)^*| = |R_1|.|R_2|.|L_3| + |Z(R_1 \times R_2)|.(|R_3| - |L_3|) - 1$ whenever $|Z(R_1 \times R_2)| = |R_1|.|L_2| + |R_2|.|L_1| - |L_1|.|L_2|$.

Corollary 2.4: [16]Let F_{s_1} , F_{s_2} and F_{s_3} are finite fields, then $\left|Z\left(F_{s_1} \times F_{s_2} \times F_{s_3}\right)^*\right| = F_{s_1}.F_{s_2} + F_{s_1}.F_{s_3} + F_{s_2}.F_{s_3} - F_{s_1} - F_{s_2} - F_{s_3}$.

Lemma 2.5: [16] Assume that R is a ring isomorphic to direct product three local rings R_1, R_2 and R_3, then

i. $|Z(R)^*| \geq 27$ whenever R_{i_1} and R_{i_2} not field for $1 \leq i_1, i_2 \leq 3$ or $|R_1|, |R_2| \geq 3$ and R_3 not field.

ii. $|Z(R)^*| \geq 26$, whenever R_i not field satisfying $|R_i| \neq 4$ and $i \in \{1,2,3\}$.

Lemma 2.6: [16]Let R isomorphic direct product four local rings R_1, R_2, R_3 and R_4, then

1- $|Z(R)^*| \geq 30$, as $|R_i| \geq 4$ for some $1 \leq i \leq 4$.

2- $|Z(R)^*| \geq 34$, as $|R_{i_1}|$ and $|R_{i_2}| \geq 3$ for some $i_1, i_2 \in \{1,2,3,4\}$.

Remark2.7: If we assume that $R \cong R_1 \times R_2 \times R_3 \times R_4$ and $|Z(R)| = 23, 24$ or 25, then Lemma 2.6 leads to a contradiction. As well if $R \cong R_1 \times R_2 \times R_3 \times R_4 \times R_5$, then $|Z(R)| \geq 30$.

It is well known that a ring R is finite if and only if $Z(R)$ is finite [5]. Additionally, if R is a finite ring, then it is an isomorphic direct product of local rings. Therefore, if $|Z(R)^*| = 23$, then $|Z(R)| = 24$. Consequently, $R \cong R_1 \times R_2 \times \ldots \times R_s$, where each R_t is a local and $t \in \{1,2, \ldots, s\}$. Since there are no local rings with $|Z(R)| = 24$, then $t \in \{2,3, \ldots, s\}$, by Remarked 2.7, we get $2 \leq s \leq 3$.

Theorem2.8: Let R be isomorphic to a direct product of local rings $R_1$and R_2, and $|Z(R)^*| = 23$, then $R \cong Z_4 \times Z_8$, $Z_4 \times Z_2\,[T_1])/(T_1{}^3)$, $Z_4 \times Z_2\,[T_1, T_2]/(T_1{}^2, T_1T_2, T_2{}^2)$, $Z_4 \times (Z_4\,[T_1])/(2T_1, T_1{}^2\,)$, $Z_2[T_1]/(T_1{}^2) \times Z_8$, $Z_2[T_1]/(T_1{}^2) \times (Z_2\,[T_1])/(T_1{}^3)$, $Z_2[T_1]/(T_1{}^2), Z_2[X]/(T_1{}^2) \times Z_2\,[T_1, T_2]/(T_1{}^2, T_1T_2, T_1{}^2)$, $Z_2[T_1]/(T_1{}^2) \times (Z_4\,[T_1])/(2T_1, T_1{}^2\,)$, $Z_4 \times Z_9$, $(Z_2[T_1])/(T_1{}^2) \times Z_9$, $Z_4 \times (Z_3\,[T_1])/(T_1{}^2)$ or $(Z_2[T_1])/(T_1{}^2) \times (Z_3\,[T_1])/(T_1{}^2)$

Proof: If we assume that R_1 and R_2 are local but not fields, then $|R_1|, |R_2| \geq 4$, and $|L_1|$, $|L_2| \geq 2$. We claim that either $|R_1|$ or $|R_2|$ equal 4. If not, then $|R_i| \geq 8$ and we get $|L_i| \geq 3$, for $i = 1, 2$

So, Lemma 2.1 leads to $|Z(R_1 \times R_2)^*| \geq |R_1||L_2| + |R_2||L_1| - |L_1| \times |L_2| - 1 \geq 8.3 + 8.3 - 3.3 - 1 = 38$ which is a contradiction. Therefore, $|R_1| = 4$ and by [18], $R_1 \cong Z_4$ or $Z_2\,[T_1]/(T_1{}^2)$ and $|L_1| = 2$. Whence $23 = |Z(R_1 \times R_2)^*| = |R_1||L_2| + |R_2||L_1| - |L_1| \times |L_2| - 1 = 4|L_2| + 2|R_2| - 2|L_2| - 1$

Which implies that

$|L_2| + |R_2| = 12$ (1)

Now to solve eq. (1), where $|R_2| \geq 8$.

If $|R_2| = 8$, then by [18], $R_2 \cong Z_8, Z_2\,[T_1]/({T_1}^3)$, $Z_2\,[T_1,T_2]/({T_1}^2, T_1T_2, {T_2}^2)$ or $Z_4\,[T_1]/(2T_1, {T_1}^2)$ and $|L_2| = 4$. Therefore
$R \cong Z_4 \times Z_8, Z_4 \times Z_2\,[T_1]/({T_1}^3), Z_4 \times Z_2\,[T_1,T_2]/({T_1}^2, T_1T_2, {T_2}^2), Z_4 \times Z_4\,[T_1]/(2T_1, {T_1}^2), Z_2[T_1]/({T_1}^2) \times Z_8, Z_2[T_1]/({T_1}^2) \times Z_2\,[T_1]/({T_1}^3), Z_2[T_1]/({T_1}^2), Z_2[T_1]/({T_1}^2) \times Z_2\,[T_1,T_2]/({T_1}^2, T_1T_2, {T_2}^2)$ or $Z_2[T_1]/({T_1}^2) \times Z_4\,[T_1]/(2T_1, {T_1}^2)$.
If $|R_2| = 9$, then by [18], $R_2 \cong Z_9$ or $Z_3\,[T_1]/({T_1}^2)$ and $|L_2| = 3$. So that $R_2 \cong Z_9$ *or* $Z_3[T_1]/({T_1}^2)$. Therefore $R \cong Z_4 \times Z_9$, $(Z_2[T_1])/({T_1}^2) \times Z_9$, $Z_4 \times Z_3\,[T_1]/({T_1}^2)$ or $Z_2[T_1]/({T_1}^2) \times Z_3\,[T_1]/({T_1}^2)$. If $|L_2| \geq 5$, then $|R_2| \leq 7$ that is a contradicts.
If R_1 local not field and R_2 field, then $|L_2| = 1$. Consequently, $|R_1| + |R_2|.|L_1| - |L_1| - 1 = 23$ or

$$|R_1| + |R_2|.|L_1| - |L_1| = 24 \qquad (2)$$

The only solution of eq. (2) is $|R_1| = 4$, $|R_2| = 11$. Therefore $|L_1| = 2$ and we have $|R_1| \cong Z_4$ *or* $Z_2\,[T_1]/({T_1}^2)$ and $R_2 \cong Z_{11}$. So that
$R \cong Z_4 \times Z_{11}$ or $Z_2[T_1]/({T_1}^2)$ .Finally, if R_1 and R_2 are fields of orders S_1 and S_2 respectively, then $|Z(R)^*| = S_1 + S_2 - 2 = 23$ Corollary 2.2, implies that $S_1 + S_2 = 25$. For that reason, $S_1 = 2$, $S_2 = 23$, $S_1 = 8$, $S_2 = 17$ or $S_1 = 9, S_2 = 16$. So $R \cong Z_2 \times Z_{23}, F_8 \times Z_{17}$ or $F_9 \times F_{16}$. ■

Theorem2.9: If R is isomorphic to a direct product of three local rings R_1, R_2 and R_3 and $|Z(R)^*| = 23$, then $R \cong Z_3 \times Z_3 \times F_4$.
Proof: Let R be a ring isomorphic to direct product of three local rings R_1, R_2 and R_3, then there are three cases:
Case (1): Let R_i be a field of order S_i for $i = 1,2$, or 3. Then by Corollary 2.4

$$S_1.S_2 + S_1.S_3 + S_2.S_3 - S_1 - S_2 - S_3 = 23 \qquad (3)$$

Without loss of generality, let $S_1 = 2$, then eq. (3) gives $2S_2 + 2S_3 + S_2.S_3 - 2 - S_2 - S_3 = 23$, or $S_2 = \frac{25 - S_3}{(1 + S_3)}$. This leads to an inconsistency. So, $S_i \geq 3$ for all $1 \leq i \leq 3$. If $S_1, S_2 \geq 3$ and $S_3 \geq 5$, then we have $|Z(R)^*| \geq 3 \times 3 + 3 \times 5 + 3 \times 5 - 3 - 3 - 5 = 28$. Which is a contradiction. Thus, the only solution for eq. (3) $S_1 = S_2 = 3$ and $S_3 = 4$. Therefor $R \cong Z_3 \times Z_3 \times F_4$.
Case(2): If R_1 ,R_2 are fields of order S_1 and S_2 respectively and R_3 is not field. If we assume $S_1, S_2 \geq 3$, then $|Z(R_1 \times R_2 \times R_3)^*| \geq 3.3.2 + (3 + 3 - 1).(4 - 2) - 1 \geq 27$ which is a contradiction. So that either $S_1 = S_2 = 2$ or $S_1 = 2$ and $S_2 = 3$
If $S_1 = S_2 = 2$. Then $|Z(R_1 \times R_2)| = S_1 + S_2 - 1 = 2 + 2 - 1 = 3$ and $|Z(R_1 \times R_2 \times R_3)^*| = 4|L_3| + 3(|R_3| - |L_3|) - 1 = 23$ or
$3|R_3| + |L_3| = 24$ or $|R_3| = 8 - \frac{|L_3|}{3}$. Since $|R_3| = p^s$ and $|L_3| = p^t$, where p prime number and s, t positive integer with $t \leq s$. There's an issue here. Likewise, if $S_1 = 2$ and $S_2 = 3$ we get a contradiction.
Case (3): If R_i is not field for all $i = 1,2$, or 3. By Lemma 2.5 we get a contradiction. Therefore for all cases $R \cong Z_3 \times Z_3 \times F_4$. ■
We denote that K_n is a complete graph of order n and $K_{n,m}$ is a complete bipartite graph order $n + m$ [19]. From the results above, we can get all graphs order 23 realized rings R.

Theorem 2.10: Assume that R is a ring such that $|\, Z(R)^* \,| = 23$, then the given graph can be represented as $\Gamma(R)$ in the following table.

Table 1: Ring with $\Gamma(R) = 23$

Ring type	Graph
$Z_4 \times Z_9$, $Z_2[T_1]/(T_1{}^2) \times Z_9$, $Z_4 \times Z_3\,[T_1]/(T_1{}^2)$ or $Z_2[T_1]/(T_1{}^2) \times Z_3\,[T_1]/(T_1{}^2)$	Fig (1)
$Z_4 \times Z_8$, $Z_4 \times Z_2\,[T_1]/(T_1{}^3), Z_4 \times Z_2\,[T_1,T_2]/(T_1{}^2, T_1T_2, T_2{}^2), Z_4 \times Z_4\,[T_1]/(2T_1, T_1{}^2), Z_2[T_1]/(T_1{}^2) \times Z_8$, $Z_2[T_1]/(T_1{}^2) \times Z_2\,[T_1]/(T_1{}^3), Z_2[T_1]/(T_1{}^2), Z_2[T_1]/(T_1{}^2) \times Z_2\,[T_1,T_2]/(T_1{}^2, T_1T_2, T_2{}^2)$ or $Z_2[T_1]/(T_1{}^2) \times Z_4\,[T_1]/(2T_1, T_1{}^2)$	Fig (2)
$Z_2 \times Z_{23}$	$K_{1,22}$
$F_8 \times Z_{17}$	$K_{7,16}$
$F_9 \times F_{16}$	$K_{8,15}$
$Z_3 \times Z_3 \times F_4$	Fig (3)

Proof: Directly by Theorems 2.8 and 2.9, we have
$R \cong Z_4 \times Z_8$, $Z_4 \times Z_2\,[T_1])/(T_1{}^3), Z_4 \times Z_2\,[T_1,T_2]/(T_1{}^2, T_1T_2, T_2{}^2), Z_4 \times\ (Z_4\,[T_1])/(2T_1, T_1{}^2), Z_2[T_1]/(T_1{}^2) \times Z_8$, $Z_2[T_1]/(T_1{}^2) \times (Z_2\,[T_1])/(T_1{}^3), Z_2[T_1]/(T_1{}^2), Z_2[T_1]/(T_1{}^2) \times Z_2\,[T_1,T_2]/(T_1{}^2, T_1T_2, T_2{}^2)$, $Z_2[T_1]/(T_1{}^2) \times\ (Z_4\,[T_1])/(2T_1, T_1{}^2), Z_4 \times Z_9$, $(Z_2[T_1])/(T_1{}^2) \times Z_9$, $Z_4 \times (Z_3\,[T_1])/(T_1{}^2)$ or $(Z_2[T_1])/(T_1{}^2) \times (Z_3\,[T_1])/(T_1{}^2)$ and $R \cong R \cong Z_3 \times Z_3 \times F_4$ **.**

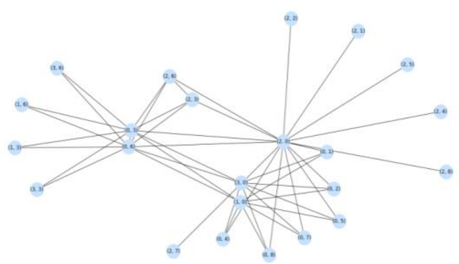


Figure 1: $\Gamma(Z_4 \times Z_9)$

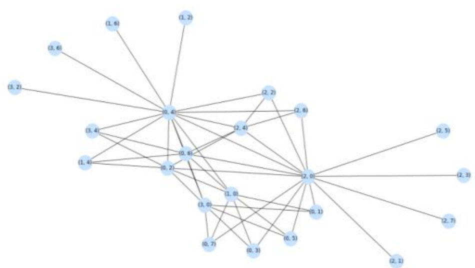


Figure 2: $\Gamma(Z_4 \times Z_8)$

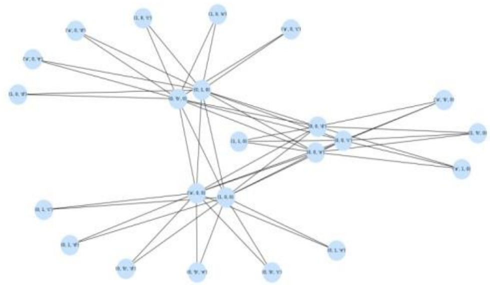


Figure 3: $\Gamma(Z_3 \times Z_3 \times F_4)$

3. Rings where $|Z(R)^*| = 24$

In this section, we investigate all possible rings to realize zero divisor graph order 24. Firstly, when R local ring, since $|Z(R)| = 25 = p^2$ then by [18, 20].
$R \cong Z_5[T_1]/(T_1{}^3)$, $Z_5[T_1, T_2]/(T_1{}^2, T_1T_2, T_1{}^2)$, $Z_{5^2}[T_1]/(5T_1, T_1{}^2)$, $Z_{5^2}[T_1]/(5T_1, T_1{}^2 - 5)$, $Z_{5^2}[T_1]/(5T_1, T_1{}^2 - 10)$, Z_{5^3} , $F_{5^2}[T_1]/(T_1{}^2)$. Additionally, from Remark 2.7, if $|Z(R)^*| = 24$, then $R \cong R_1 \times R_2 \times \ldots \times R_s$, where $s = 1, 2$ or 3. Therefore, it is enough to study whenever R direct product two or three local rings.

Theorem 3.1: If $R \cong R_1 \times R_2$, where R_i is a local ring for $i = 1,2$ and $|Z(R)^*| = 24$, then $R \cong Z_3 \times Z_{23}, Z_7 \times Z_{19}, F_9 \times Z_{17}$ or $Z_{13} \times Z_{13}$.

Proof: We claim that R_1 and R_2 are fields. If not, then there are two cases:
Case (1): When R_1 and R_2 not fields, then $|R_1|, |R_2| \geq 4$. If $|R_1|, |R_2| \geq 8$, then by Lemma2.1, $|Z(R_1 \times R_2)| \geq 38$, that is a contradiction. Similarly, if $|R_1| = |R_2| = 4$, then $|Z(R_1 \times R_2)| = 11 \neq 24$. Therefore $|R_1| = 4, |L_1| = 2$ and $|R_2| \geq 8$. So that

$$|Z(R_1 \times R_2)| = 4.|L_2| + |R_2|.2 - 2.|L_2| - 1 = 24$$

or

$$2|L_2| + 2|R_2| = 25$$

But $|L_2|$ and $|R_2|$ are integers. This leads to an equation that cannot be solved.

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Case (2): If R_1local not field and R_2 field by Lemma 2.1

$$|Z(R_1 \times R_2)| = |R_1|.|L_2| + |R_2|.|L_1| - |L_1|.|L_2| - 1 = 24$$

$$\Longrightarrow |R_1| + |L_1|(|R_2| - 1) = 25 \qquad (4)$$

Since R_1 is local but not field, then $|R_1| = p^s$ and $|L_1| = p^t$, where $1 \leq t < s$. Therefore, eq. (4) leads to $p^s + p^t(p^s - 1) = 25$ or $p^s + p^{t+s} - p^t = 5^2$
We get no solution to this equation where p is a prime number and positive integers s and t satisfy $1 \leq t < s$.
Therefore R_1 and R_2 must be fields, so $Z(R_1 \times R_2) = S_1 + S_2 = 26$, where S_i the order of the field R_i , $i = 1$ *or* 2.
It entails $S_1 = 3, S_2 = 23$, $S_1 = 7, S_2 = 19, S_1 = 9, S_2 = 17$ or $S_1 = S_2 = 13$. Therefore $R \cong Z_3 \times Z_{23}$,
$Z_7 \times Z_{19}$, $F_9 \times Z_{17}$ or $Z_{13} \times Z_{13}$. ■

Theorem 3.2: Let $R \cong R_1 \times R_2 \times R_3$, where R_i local ring for $i = 1,2$ or 3 and $|Z(R)^*| = 24$, then $R \cong Z_2 \times Z_2 \times F_8$.

Proof: Firstly, if R_i is a field of order S_i for $i = 1,2,3$. Then by Corollary 2.4

$$|Z(R_1 \times R_2 \times R_3)| = S_1.S_2 + S_1.S_3 + S_2.S_3 - S_1 - S_2 - S_3 = 24 \qquad (5)$$

Now, to solve eq. (5)
If $S_i = 2$, say $S_1 = 2$, then $2S_2 + 2S_3 + S_2.S_3 - 2 - S_2 - S_3 = 24$ or

$$S_2 = \frac{26 - S_3}{1 + S_3} \qquad (6)$$

So that $S_2 = 2$ and $S_3 = 8$, which implies $R \cong Z_2 \times Z_2 \times F_8$
Additionally, if $S_i = 3$, then we have a contradiction. Therefore $S_i \geq 4$ for $i = 1,2,3$, with substituting in eq. (5) we get $|Z(R_1 \times R_2 \times R_3)| \geq 36$ which contradicts the fact $|Z(R_1 \times R_2 \times R_3)| = 24$. So the only solution of eq. (5) is $R \cong Z_2 \times Z_2 \times F_8$
Secondly, if R_1 , R_2 are fields order S_1, S_2 respectively and R_3 not field. As S_1 and $S_2 \geq 3$, then by Lemma 2.5 we get a contradiction. So that either $S_1 = S_2 = 2$ or $S_1 = 2$ and $S_2 = 3$. If $S_1 = S_2 = 2$, then $|Z(R_1 \times R_2 \times R_3)^*| = S_1.S_2.|L_3| + |Z(R_1 \times R_2)|(|R_3| - |L_3|) - 1$, where $|Z(R_1 \times R_2)| = S_1 + S_2 - 1 = 2 + 2 - 1 = 3$. This implies that $4|L_3| + 3(|R_3| - |L_3|) - 1 = 24$ or $|R_3| = \frac{25-|L_3|}{3}$
Since $L_3 = p^t$ and $R_3 = p^s$, where p prime number and t, s positive integer with $1 \leq t < s$, we get a contradiction. Finally, if $|R_i|$ not field for some $1 \leq i_1 , i_2 \leq 3$, then by Lemma 2.5 (i) we get $|Z(R)^*| \geq 27$ a contradiction. ■

Theorem 3.3: Let R be a ring such that $Z(R)^* \,| = 24$, then the given $\Gamma(R)$ represents the graph or the opposite for the following table:

Table 2: Ring with $\Gamma(\mathrm{R}) = 24$

Ring type	Graph
$Z_5[T_1]/(T_1^{\ 3})$, Z_{5^3}	Fig (4)
$Z_5[T_1, T_2]/(T_1^{\ 2}, T_1T_2 , T_2^{\ 2})$ $F_{5^2}[T_1]/(T_1^{\ 2})$ $Z_{5^2}[T_1]/(5T_1 , T_1^{\ 2})$	K_{24}
$Z_{5^2}[T_1]/(5T_1 , T_1^{\ 2} - 5)$ $Z_{5^2}[T_1]/(5T_1 , T_1^{\ 2} - 10)$	Fig (5)
$Z_2 \times Z_2 \times F_8$	Fig (6)
$Z_3 \times Z_{23}$	$K_{2,22}$
$Z_7 \times Z_{19}$	$K_{6,18}$
$F_9 \times Z_{17}$	$K_{8,16}$
$Z_{13} \times Z_{13}$	$K_{12,12}$

Proof : Let R be a local ring, then $R \cong Z_5[T_1]/(T_1^{\ 3})$, $Z_5[T_1, T_2]/(T_1^{\ 2}, T_1T_2 , T_2^{\ 2})$, $Z_{5^2}[T_1]/(5T_1 , T_1^{\ 2})$, $Z_{5^2}[T_1]/(5T_1 , T_1^{\ 2} - 5)$, $Z_{5^2}[T_1]/(5T_1 , T_1^{\ 2} - 10)$, Z_{5^3} , $F_{5^2}[T_1]/(T_1^{\ 2})$. If R is not local, then applying Theorems 3.1 and 3.2, then we have:
$R \cong Z_3 \times Z_{23}, Z_7 \times Z_{19}, F_9 \times Z_{17}$, $Z_{13} \times Z_{13}$ *or* $Z_2 \times Z_2 \times F_8$. ■

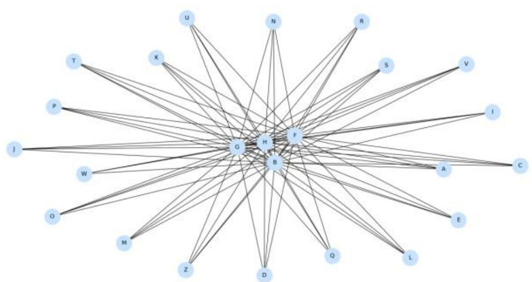

Figure 4: $\Gamma(Z_5[T_1]/(T_1^{\ 2}))$ *or* $\Gamma(Z_{5^3})$

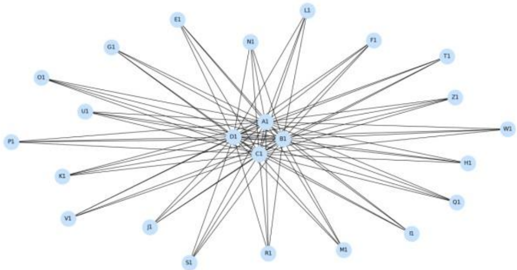

Figure 5: $\Gamma(Z_{5^2}[T_1]/(5T_1, T_1{}^2 - 5)$or $\Gamma(Z_{5^2}[T_1]/(5T_1, T_1{}^2 - 10))$

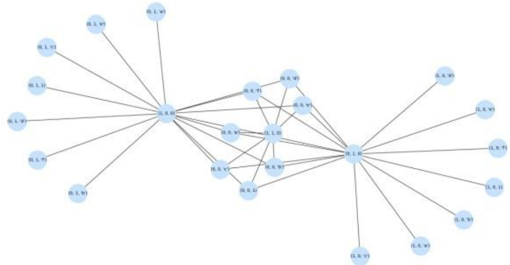

Figure 6: $\Gamma(Z_2 \times Z_2 \times F_8)$

4. Rings where $|\boldsymbol{Z}(\boldsymbol{R})^*| = \mathbf{25}$

Since $|Z(R)^*| = 25$, then $|Z(R)| = 26$ so that R not local. Therefore $R \cong R_1 \times R_2 \times \dots \times R_s$,where $2 \leq s \leq 3$ and R_i local. First, we begin when R direct product two local rings.

Theorem 4.1: Let R isomorphic direct product two local rings $R_1$and R_2, and $|Z(R)^*| = 25$, then $R \cong Z_2 \times F_{25}, F_4 \times Z_{23}, F_8 \times Z_{19}$ or $Z_{11} \times F_{16}$

Proof: There are three cases independent on R_1 and R_2.
Case (1): If R_1 and R_2 are not fields, in a similar way to the proof of Case 1 of Theorem3.1 we obtain a contradiction.
Case (2): If R_1 local not field and R_2 is a field order S_2 , then by Lemma 2.1

$$|Z(R_1 \times R_2)| = |R_1| + S_2.|L_1| - |L_1| - 1 = 25$$

or $|R_1| + |L_1|(S_2 - 1) = 26$ we get a contradiction
Case (3): If R_1, R_2 are fields order S_1 and S_2 respectively, then $|Z(R_1 \times R_2)| = S_1 + S_2 - 2 = 25$ or $S_1 + S_2 = 27$
Therefore $S_1 = 2,\ S_2 = 25,\ S_1 = 4, S_2 = 23,\ S_1 = 8,\ S_2 = 19$ or $S_1 = 11,\ S_2 = 16$. Which implies that $R \cong Z_2 \times F_{25}, F_4 \times Z_{23}, F_8 \times Z_{19}$ or $Z_{11} \times F_{16}$. ■

Theorem4.2: If $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2 and R_3 are local rings and $|Z(R)^*| = 25$, then

$$R \cong F_2 \times F_4 \times Z_4 \ or\ F_2 \times F_4 \times Z_2[T_1]/(T_1{}^2)$$

Proof: We will discuss three cases.
If R_1 , R_2 and R_3 are fields order S_1, S_2 and S_3 respectively, then

$$|Z(R_1 \times R_2 \times R_3)^*| = S_1.S_2 + S_1.S_3 + S_2.S_3 - S_1 - S_2 - S_3 = 25 \quad (7)$$

If we take $S_1 = 2$, then eq. (7) leads

$$|Z(R_1 \times R_2 \times R_3)^*| = 2S_2 + 2S_3 + S_2.S_3 - 2 - S_2 - S_3 = 25$$

or $S_2 = \frac{27-S_3}{1+S_3}$ this equation has no solution when R_2 and R_3 equal a prime number with a pawer integer number. So that $S_i \geq 3$ for all $1 \leq i \leq 3$. We note that no solution in this equation. If R_1, R_2 are fields of order S_1, S_2 respectively and R_3 not field, then by the same way prove theorem 3.2 we have the only solution is $S_1 = 2, S_2 = |R_3| = 4$ and $|L_3| = 2$

$$\therefore \quad R \cong F_2 \times F_4 \times Z_4 \ \ or\ F_2 \times F_4 \times Z_2[T_1]/({T_1}^2)$$

Finally, if R_i not field for some $1 \le i_1, i_2 \le 3$, then by Lemma 2.5
$|Z(R)^*| \ge 27$ that is a contradiction. ■

Theorem 4.3: Let R be a ring such that $|Z(R)^*| = 25$, then the given graph can be represented as $\Gamma(R)$ for the following table:

Table 3: Ring with $\Gamma(R) = 25$

Ring type	Graph
$Z_2 \times F_{25}$	$K_{1,24}$
$F_4 \times Z_{23}$	$K_{3,22}$
$F_8 \times Z_{19}$	$K_{7,18}$
$Z_{11} \times F_{16}$	$K_{10,15}$
$F_2 \times F_4 \times Z_4\ or\ F_2 \times F_4 \times Z_2[T_1]/({T_1}^2)$	Fig (7)

Proof: By applying Theorems 4.1 and 4.2. ■

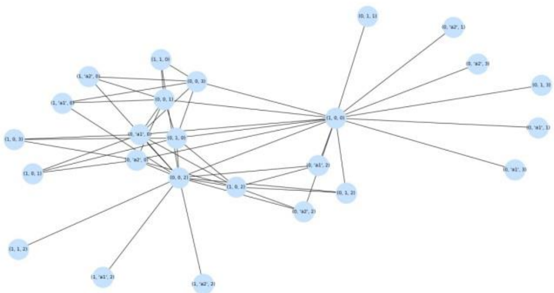

Figure 7: $\Gamma(F_2 \times F_4 \times Z_4)\ or\ \ \Gamma(F_2 \times F_4 \times Z_2[T_1]/({T_1}^2))$

Conclusion

First, when $|\Gamma(R)| = 23$, then the only six graphs (Fig. (1), Fig. (2), $K_{1,22}$, $K_{7,16}$, $K_{8,15}$ and Fig. (3)) realized a ring R. Second, when $|\Gamma(R)| = 24$, then only eight graphs (Fig. (4), Fig .(5), Fig. (6), k_{24}, $K_{2,22}$, $K_{6,18}$, $K_{8,16}$ and $K_{12,12}$) realized a ring. Finally, when $|\Gamma(R)| = 25$, then only five graphs ($K_{1,24}, K_{3,22}, K_{7,18}, K_{10,15}$ and Fig. (7)) realized a ring. In addition, we note that if $|\Gamma(R)| = 23$, then R is not necessarily a reduced ring, while if $|\Gamma(R)| = 24$, then R is either a reduced or local ring. Additional if $|\Gamma(R)| = 25$, then all rings are reduced.

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