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Parahyponormality Modeling of Operator Inequality Dominated by Norm Mapping

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Abstract
In this article, new class of operators named $(\eta,*)$-Parahyponormal operator is imposed. It is extended Paranormality, means it includes both paranormal and hyponormal operators. This concept is described analytically based on the Inequality Theory (IT). Furthermore, a new characteristic of $(\eta,*)$-Parahyponormality is investigated and provided. Several analytical merits, such as adjoint, invertible, null set, power again, and scalar multiplication, are highlighted. Besides, some algebraic merits of this considered operator are discussed including addition, multiplication and tensor product. Some illustrative examples are also provided.

Keywords: Hilbert space, Hyponormal operator, Paranormal, Parahyponormal operator.

نمذجة نواه الفوق السوية لمؤثر المتراجحات تحت تأثير دالة النورم

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الخلاصة
في هذه الدراسة تم تقديم صنف جديد من المؤثرات تسمى مؤثر نواة فوق السوية– $(\eta,*)$، هذا المؤثر توسيع لخاصية فوق السوية والتي تضمن مؤثرات نواه السوية وفوق السوية، تم وصف هذا المفهوم تحليليا على اساس نظرية المتراجحات علاوة على ذلك تمييزات جديدة لنواة فوق السوية– $(\eta,*)$ اكتشفت وبرهنت، كما تم تسليط الضوء على العديد من الخواص التحليلية، مثل الترافق، والانعكاس، مجموعة النواة ، وكذلك الرفع، والضرب العددي بالإضافة الى ذلك تم مناقشة بعض الخواص الجبرية لهذا المؤثر المقدم والتي تتضمن الجمع، الضرب، والضرب التنسوري وبعض الامثلة التوضيحية قدمت ايضا.

1. Introduction

Functional analysis (FA) is a significant discipline in mathematics and is employed in various realms, such as differential equations and fuzzy operator theory, see ([1-5]). The most distinguished step in the evolution of the area of Hilbert space was the systematic realization of Operator Theory (OT). The Inequality Theory (IT) has a dynamic role in the characterization of generalized operators, due to the thoughtful analysis of the algebraic

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merits of their image domains is of vibrant importance. Therefore, this theme has become captivating for numerous researchers, who have studied and analyzed many merits and highlighted various new characteristics. For their effective contributions [6-16]. In this regard, interest in normal operator theory has led to numerous attempts to generalize by reducing the requirement of commutativity. Interesting operators comprise unitary operators, the Hermitian operators, skew-Hermitian operators, positive operators, and others [17-24]. Hyponormality is also an interesting idea, generalizing normal operators. This is due to Halmos [25] in 1950. Following this, in 1957, Putnam [26] investigated some merits of the considered operator in terms of hyponormal. Afterwards, several scientists investigated and presented sorts of hyponormal operators in different formulations, due to their applicability in the realm of IT. The study of quasi-hyponormal class of operators was initiated by Shah and Sheth [27] in 1975. It is a generalized class of hyponormal operators and played a leading role in the advancement of the OT in the sense of FA. For the general theory of quasi-hyponormal operators, see [28-32]. A marvelous class of operators, termed semi-hyponormal ones, was provided by Xia [33] in 1980, based on the hyponormality concept. Similar outcomes were acquired to those of hyponormal operators. For a general, Aluthge [34] in 1990 proposed the so-called p-hyponormal class of operators $(0 < p < 1)$. In 1997, Huruya [35] discussed interesting spectral features in the context of a p-hyponormal operator. Building upon this, the included connection of the "p-hyponormality" and the "quasi-hyponormality" is unknown. The decisive natural generalized of the p-hyponormality arises with quasi-hyponormality and is called paranormality. The paranormal operator also includes the normal operator. It stemmed in the 1960s. The term "paranormal" was presented by Istratescu [36] and is likely attributed to Furuta [37]. It was initially formulated as an interpose class between the normaloid class and hyponormal one. Subsequently, several interesting studies were conducted based on paranormal operators and were included in those works [38-44]. In 1990s, a more general operator related to Hyponormality was Parahyponormal operator. It was first formed by Kutkut [45]. In 2005, Senthilkumar and Thirugnanasambandam [46] showed interest in discussing the weighted composition Parahyponormality. In 2008, Panayappan and Radharamani [47] presented the first extended Parahyponormal operator named quasi Parahyponormal operator. They also considered the characterization of this new operator. In 2018, Manikandan and Suganya [48] investigated a basic algebraic outcome about Parahyponormality, that the product of two Parahyponormal operators is a Parahyponormal operator. In 2020, Parvatham and Senthilkumar [49] imposed a further extension Parahyponormal operator called k-quasi-Parahyponormal operator. Several merits are discussed including spectrum, joint approximate point spectrum, and non-zero points of approximate point spectrum for this operator are discussed. In 2023, Bakir [50] established the class of (M, k)-$*$-quasi-parahyponormal operators and examined various algebraic merits for this posed operator. In 2023, Mohsen [51] investigated extended operator, namely (k, m)-n-paranormal operators and studied several related merits of this operator. In 2023, Mohsen [52] posed and studied a new modified operator, namely (M, θ)-hyponormal operator. The algebraic merits of the operator are also determined and derived.

Continuing this line of research, this sequel presents a new extended Parahyponormal operator called the $(\eta,*)$-Parahyponormal operator. Several attempts are made to transfer various significant merits of hyponormality to the $(\eta,*)$-Parahyponormality. Moreover, adequate conditions for the merits of addition and multiplication are illustrated. In addition, the tensor product achieves the $(\eta,*)$-parahyponormal operator. The following are a variety of concepts related to OT acting on a Hilbert space that are desired to achieve new exciting.

Definition 1.1 [53] Let $T: H \to H$ be a bounded linear operator on a Hilbert space H, then T is called normal operator if $T^*T = TT^*$, where T^* is adjoint operator.

Definition 1.2: [54]
Let $T: H \to H$ be a bounded linear operator on a Hilbert space H, then T is called hyponormal operator if $T^*T \geq TT^*$, that is $\langle T^*Tx, x\rangle \geq \langle TT^*x, x\rangle$, for every $x \in H$.

Definition 1.3: [34]
Let $T: H \to H$ be a bounded linear operator on a Hilbert space H, then T is called p −hyponormal operator if $(T^*T)^p \geq (TT^*)^p$, where $0 < p < 1$., that is obvious in case $p = 1$, T becomes hyponormal.

Definition 1.4: [37]
Let $T: H \to H$ be a bounded linear operator in a Hilbert space H, then T is called paranormal operator if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for every $x \in H$.

Definition 1.5: [45]
Let $T: H \to H$ be a bounded linear operator on a Hilbert space H. then T is called Parahyponormal operator if $\|Tx\|^2 \leq \|TT^*\|\|x\|$ for every $x \in H$.

Definition 1.6: [55]
Let $T: H \to H$ be a bounded linear operator on a Hilbert space H, then T is called$*$ −Parahyponormal operator if $\|T^*x\|^2 \leq \|T^*T\|\|x\|$ for every $x \in H$.

Definition 1.7: [49]
Let $T: H \to H$ be a bounded linear operator on a Hilbert space H, then T is called k −qusi$*$ −Parahyponormal operator if $T^{*k}[(T^*T)^2 - 2\lambda TT^* + \lambda^2]T^k$for every $\lambda > 0$. In case $k = 1, T$ is asserted quasi$*$ −Parahyponormal operator.

Definition 1.8: [49]
Let $T: H \to H$ be a bounded linear operator on a Hilbert space H, then T is called k −qusi$*$ −Parahyponormal operator if$T^{*k}(T^*T)^2T^k - 2\lambda T^{*(k+1)}T^{k+1} + \lambda^2 T^{*k}T^k$ for every $\lambda > 0$. In case $k = 1, T$ is asserted k −quasi−Parahyponormal operator. In case $k = 1, T$ is asserted quasi−Parahyponormal operator.
Note that, Normal ⊂ hyponormal ⊂ paranormal ⊂ Parahyponormal ⊂ quasi-Parahyponormal ⊂ k −quasi-parahyponormal, [40].

2. Attribute theories: argument and analysis

This item is considered the research, as it included the new generalization of the factor under study, as well as the basic theories and issues that illustrated some algebraic properties of the topic under study.

Definition 2.1:
Let $T : H \to H$ be bounded linear operator, T is called $(\eta,*)$-Parahyponormal if $\|T^{*(\eta+1)}x\|^2 \leq \|T^*TT^{*\eta}x\|$, for every unite vector $x \in H$. Where η is natural number, in case $\eta = 1$, the operator T is asserted $(1,*)$-Parahyponormal operator.
To illustrate this concept, one can consider the bounded operator, $T = [0\ 2\ 2\ 0\ ]$, $(\eta,*)$-Parahyponormal, so proved this concept is well defined in field of functional analysis specially operator theory, but $S = [0\ \ 0\ \ -2\ \ 2\ ]$, does not has this property of operator.

The next theorem examines the equivalent conditions to get the definition of $(\eta,*)$-Parahyponormal operator via other way, in the following (H) , where $B(H)$is a space of all bounded linear operator from H into H.

Theorem 2.2:
Let $T \in B(H)$, so T is called $(\eta,*)$-Parahyponormal if and only if $T^{\eta}(T^{*}T)^{2}T^{*\eta} - 2\lambda T^{(\eta+1)}T^{*(\eta+1)} + \lambda^{2} \geq 0$, for all $\lambda > 0$.

Proof: Let $T^{\eta}(T^{*}T)^{2}T^{*\eta} - 2\lambda T^{(\eta+1)}T^{*(\eta+1)(\eta+1)} + \lambda^{2} \geq 0$, for all $\lambda > 0$. So that one obtain, $\langle (T^{\eta}(T^{*}T)^{2}T^{*\eta} - 2\lambda T^{(\eta+1)}T^{*(\eta+1)(\eta+1)} + \lambda^{2})x, x\rangle \geq 0$, thus have $\langle (T^{\eta}(T^{*}T)^{2}T^{*\eta}x, x\rangle - 2\lambda\langle T^{*(\eta+1)}T^{(\eta+1)}x, x\rangle + \lambda^{2}\langle x, x\rangle \geq 0$, this implies to, $\langle (T^{\eta}T^{*}TT^{*}TT^{*\eta}x, x\rangle - 2\lambda\langle T^{*(\eta+1)}T^{(\eta+1)}x, x\rangle + \lambda^{2}\langle x, x\rangle \geq 0$, via properties of adjoint of bounded linear operators, that leads to $\langle T^{*}TT^{*\eta}x, (T^{\eta}T^{*}T)^{*}x\rangle - 2\lambda\langle T^{*(\eta+1)}x, T^{*(\eta+1)}x\rangle + \lambda^{2}\langle x, x\rangle \geq 0$, to calculate this, one can yield $\langle T^{*}TT^{*\eta}x, T^{*}TT^{*\eta}x\, x\rangle - 2\lambda\langle T^{*(\eta+1)}x, T^{*(\eta+1)}x\rangle + \lambda^{2}\langle x, x\rangle \geq 0$, and up on definition of inner product via norm, so we get $\|T^{*}TT^{*\eta}x\|^{2} - 2\lambda\|T^{*(\eta+1)}x\|^{2} + \lambda^{2}\|x\|^{2} \geq 0$, and $\|T^{*}TT^{*\eta}x\|^{2} - 2\lambda\|T^{*(\eta+1)}x\|^{2} + \lambda^{2} \geq 0$. Finally, $\|T^{*(\eta+1)}x\|^{2} \leq \|T^{*}TT^{*\eta}x\|$, therefore; T is $(\eta,*)$-Parahyponormal operator.
Conversely, form hypothesis T is $(\eta,*)$-Parahyponormal operator, $\|T^{*(\eta+1)}x\|^{2} \leq \|T^{*}TT^{*\eta}x\|$, to get $\|T^{*}TT^{*\eta}x\|^{2} - 2\lambda\|T^{*(\eta+1)}x\|^{2} + \lambda^{2} \geq 0$, for computing this have $\|T^{*}TT^{*\eta}x\|^{2} - 2\lambda\|T^{*(\eta+1)}x\|^{2} + \lambda^{2}\|x\|^{2} \geq 0$, thus $\langle T^{*}TT^{*\eta}x, T^{*}TT^{*\eta}x\rangle - 2\lambda\langle T^{*(\eta+1)}x, T^{*(\eta+1)}x\rangle + \lambda^{2}\langle x, x\rangle \geq 0$, so $\langle (T^{*}TT^{*\eta})^{*}T^{*}TT^{*\eta}x, x\rangle - 2\lambda\langle T^{(\eta+1)}T^{*(\eta+1)}x, x\rangle + \lambda^{2}\langle x, x\rangle \geq 0$, then $\langle T^{\eta}T^{*}TT^{*}TT^{*\eta}x, x\rangle - 2\lambda\langle T^{(\eta+1)}T^{*(\eta+1)}x, x\rangle + \lambda^{2}\langle x, x\rangle \geq 0$, to obtain,
$\langle\langle T^{\eta}(T^{*}T)^{2}T^{*\eta}x, x\rangle - 2\lambda\langle T^{(\eta+1)}T^{*(\eta+1)}x, x\rangle + \lambda^{2}\langle x, x\rangle \geq 0$, hence; $\langle (T^{*\eta}(TT^{*})^{2}T^{\eta} - 2\lambda T^{(\eta+1)}T^{*(\eta+1)} + \lambda^{2})x, x\rangle \geq 0$, therefore; $T^{\eta}(T^{*}T)^{2}T^{*\eta} - 2\lambda T^{(\eta+1)}T^{*(\eta+1)} + \lambda^{2} \geq 0$.

Theorem 2.3:
Let $T \in B(H)$ be $(\eta,*)$-Parahyponormal, then the restriction on closed invariant subspace M of H, $(\eta,*)$-Parahyponormal operator.

Proof: Since M is closed invariant subspace M of H, so H, can be written as $H = M \oplus M^{\perp}$, and T is $(\eta,*)$-Parahyponormal operator, thus via Theorem 2.2 its satisfy $T^{\eta}(T^{*}T)^{2}T^{*\eta} - 2\lambda T^{(\eta+1)}T^{*(\eta+1)} + \lambda^{2} \geq 0$, for each $\lambda > 0$, let $T = (T_1 \quad T_2\, 0 \quad T_3)$, then
$(T_1 \quad T_2\, 0 \quad T_3)^{\eta}\big((T_1 \quad T_2\, 0 \quad T_3)^{*}(T_1 \quad T_2\, 0 \quad T_3)\big)^{2}(T_1 \quad T_2\, 0 \quad T_3)^{*\eta} - 2\lambda(T_1 \quad T_2\, 0 \quad T_3)(T_1 \quad T_2\, 0 \quad T_3)^{*\eta+1} + \lambda^{2} \geq 0$, thus

$$(T_1 \quad T_2\, 0 \quad T_3)^{\eta}\big((T_1{}^{*} \quad 0\, T_2{}^{*} \quad T_3{}^{*})(T_1 \quad T_2\, 0 \quad T_3)\big)^{2}(T_1{}^{*} \quad 0\, T_2{}^{*} \quad T_3{}^{*})^{\eta} - 2\lambda(T_1 \quad T_2\, 0 \quad T_3)(T_1{}^{*} \quad 0\, T_2{}^{*} \quad T_3{}^{*})^{\eta+1} + \lambda^{2} \geq 0$$

$$(T_1 \quad T_2\, 0 \quad T_3)^{\eta}\Big[\big((T_1{}^{*} \quad 0\, T_2{}^{*} \quad T_3{}^{*})(T_1 \quad T_2\, 0 \quad T_3)\big)^{2} - 2\lambda(T_1 \quad T_2\, 0 \quad T_3)(T_1{}^{*} \quad 0\, T_2{}^{*} \quad T_3{}^{*})\Big](T_1{}^{*} \quad 0\, T_2{}^{*} \quad T_3{}^{*})^{\eta} + \lambda^{2} \geq 0.$$

Via same way appeared in [49,56], get the following inequalities, where η is natural number $\big(T_1{}^{\eta}(T_1{}^{*}T_1)^{2}T_1{}^{*\eta} - 2\lambda T_1{}^{(\eta+1)}T_1{}^{*(\eta+1)} + \lambda^{2} \;\; A\; B\; C\big) \geq 0$, for some operators A, B and C, leads to $T_1{}^{\eta}(T_1{}^{*}T_1)^{2}T_1{}^{*\eta} - 2\lambda T_1{}^{(\eta+1)}T_1{}^{*(\eta+1)} + \lambda^{2} \geq 0$, but $T_1 = \frac{T}{M}$, therefore the restriction on closed invariant subspace M of H, $(\eta,*)$-Parahyponormal operator.

Theorem 2.4:
If T is $(\eta,*)$-Parahyponormal, then the null set of T is a subset of $N(T^{*})$.

Proof: Suppose x be any vector in H, such that $x \in N(T)$, and can use the assumptions appeared in this theorem, where T $(\eta,*)$-Parahyponormal, also via Definition 2.1, we obtain $\|T^{*(\eta+1)}x\|^2 \leq \|T^*TT^{*\eta}x\|$, for every unite vector $x \in H$, and since $Tx = \lambda x$, $\lambda > 0$, so we get $T^{*\eta}x = \underline{\lambda}^\eta x$, then $\| T^*T(T^{*\eta}x) \| = \| T^*T(\underline{\lambda}^\eta x) \| = \| T^*\underline{\lambda}^\eta(Tx) \| = 0$, this implies $\|T^{*(\eta+1)}x\|^2 \leq 0$, thus $\|T^{*\eta}T^*x\|^2 \leq 0$, and since $T^{*\eta} \neq 0$, then we have $T^*x = 0$, therefore; $x \in N(T^*)$, then $N(T) \subseteq N(T^*)$.

Corollary 2.5:
If T is $(\eta,*)$-Parahyponormal, then the null set of $T - o$ is subset of the set of $(T - o)^*$, for each $o \in C - \{0\}$.

Proof: Suppose x be any vector in H, such that $x \in N(T - o)$, implies that $(T - o)(x) = 0$, so $Tx - ox = 0$, we obtain $Tx = ox$, chose $S = T - o$, directly via Theorem 2.4, we have $N(S) \subseteq N(S^*)$, therefore; then $N(T - o) \subseteq N(T - o)^*$.
The next issue represents a new addition to the properties of this class of developed operator.

Proposition 2.6:
If T is $(\eta,*)$-Parahyponormal, then ET $(\eta,*)$-Parahyponormal , for each $E \in C - \{0\}$.

Proof: Since T is $(\eta,*)$-Parahyponormal operator, we can use Theorem 2.2, to get $T^\eta(T^*T)^2T^{*\eta} - 2\lambda T^{(\eta+1)}T^{*(\eta+1)} + \lambda^2 \geq 0$, and take

$$(ET)^\eta(ET(ET)^*)^2(ET)^{*\eta} - 2\lambda(ET)^{(\eta+1)}(ET)^{*(\eta+1)} + \lambda^2$$
$$= [E^\eta E^2 \overline{(E)}^2 \overline{(E)^\eta}]T^\eta(T^*T)^2T^{*\eta} - 2\lambda[E^{\eta+1}\overline{(E)}^{\eta+1}]T^{(\eta+1)}T^{*(\eta+1)} + \lambda^2$$
$$= |E|^{\eta+2}[T^\eta(T^*T)^2T^{*\eta}] - 2\lambda|E|^{\eta+1}\left[T^{(\eta+1)}T^{*(\eta+1)}\right] + \lambda^2$$
$$= |E|^{\eta+2}\left[T^\eta(T^*T)^2T^{*\eta} - 2\frac{\lambda}{|E|^{\eta+1}}T^{(\eta+1)}T^{*(\eta+1)}\right] + \lambda^2, \text{ chose } \frac{\lambda}{|E|^{\eta+1}} = \lambda_1$$
$$= |E|^{\eta+2}\left[T^\eta(T^*T)^2T^{*\eta} - 2\lambda_1 T^{(\eta+1)}T^{*(\eta+1)}\right] + \lambda^2 \geq 0.$$

Therefore, ET $(\eta,*)$-Para-hyponormal operator.
The following outcome presents and determines the adequate to attain this operator.

Proposition 2.7:
If T is $(\eta,*)$-Parahyponormal, then $[\|T^{*(\eta+1)}x\|^2]^M \leq [\|T^*TT^{*\eta}x\|]^M$, where M is a positive integer.

Proof: We can prove this by mathematical induction technique. First, we prove this statement is true when $M = 1,2,3$ because T is $(\eta,*)$-Parahyponormal operator. Second, supposing the statement appeared in this proposition by $M = \aleph$, then we have the inequality $[\|T^{*(\eta+1)}x\|^2]^\aleph \leq [\|T^*TT^{*\eta}x\|]^\aleph$.
Third, we must show this property satisfies at $M = \aleph + 1$, then

$$[\|T^{*(\eta+1)}x\|^2]^{\aleph+1} = [\|T^{*(\eta+1)}x\|^2]^\aleph\|T^*TT^{*\eta}x\|$$
$$\leq [\|T^*TT^{*\eta}x\|]^\aleph\|T^*TT^{*\eta}x\|$$
$$\leq [\|T^*TT^{*\eta}x\|]^{\aleph+1}.$$

So, we obtain $[\|T^{*(\eta+1)}x\|^2]^M \leq [\|T^*TT^{*\eta}x\|]^M$, where M is a positive integer.

Notice that the inverse and adjoint, may not attain the merit of $(\eta,*)$-Parahyponormal operator, for T is $(\eta,*)$-Parahyponormal. To highlight that, we provide these examples.

Examples 2.8:

1) The operator $T = [0.1\ 1\ -0.1\ 0\ ]$ is $(\eta,*)$-Parahyponormal operator, since $T^{\eta}(T^{*}T)^{2}T^{*\eta} - 2\lambda T^{(\eta+1)}T^{*(\eta+1)} + \lambda^{2} = [1.031 - 0.036\lambda + \lambda^{2}\ \ -0.1 + 0.1\lambda\ \ -0.1 + 0.1\lambda\ \ 0.2\lambda + \lambda^{2}\ ] \geq 0$, for each $\lambda > 0$, but notice the inverse of operator $T^{-1} = [0 - 10\ \ 1\ 1\ ]$, is not $(\eta,*)$-Parahyponormal operator, because form compute we obtain $T^{\eta}(T^{*}T)^{2}T^{*\eta} - 2\lambda T^{(\eta+1)}T^{*(\eta+1)} + \lambda^{2}[10 = 20200 - 400\lambda + \lambda^{2}\ \ -103040 - 160\lambda\ \ -103040 - 160\lambda\ \ 1048 - 164\lambda + \lambda^{2}\ ] < 0$, for some $\lambda > 0$.

2) The operator $T = [0\ 0\ 1\ 1]$, is $(\eta,*)$-Parahyponormal operator, since $T^{\eta}(T^{*}T)^{2}T^{*\eta} - 2\lambda T^{(\eta+1)}T^{*(\eta+1)} + \lambda^{2} = [\lambda^{2}\ 0\ 0\ \ 8 - 4\lambda + \lambda^{2}\ ] \geq 0$, for each $\lambda > 0$, but notice the adjoint $T^{*} = [0\ 1\ 0\ 1\ ]$, is not $(\eta,*)$-Parahyponormal, because $T^{\eta}(T^{*}T)^{2}T^{*\eta} - 2\lambda T^{(\eta+1)}T^{*(\eta+1)} + \lambda^{2} = [16 - 2\lambda + \lambda^{2}\ \ 6 - 2\lambda\ \ 6 - 2\lambda\ \ 16 - 2\lambda + \lambda^{2}\ ] < 0$, for some $\lambda > 0$.

Down theorem show the inverse and adjoint of $(\eta,*)$-Para-hyponormal operator continue to satisfy the property of $(\eta,*)$-Parahyponormal operator, according to the specific requirements

Theorem 2.9:

Let $T: H \to H$ be $(\eta,*)$-Parahyponormal operator,

1) If T is invertible normal and $\|(T^{-1})^{\eta}\| = 1$, then T^{-1} is $(\eta,*)$-Parahyponormal.

2) If T is normal and $\|(T^{*})^{(\eta)}\| = 1$, then T^{*} is $(\eta,*)$-Parahyponormal.

Proof:

1)
$$
\begin{aligned}
\|(T^{-1})^{*(\eta+1)}x\|^{2} &= \|(T^{-1})^{*(\eta+1)}(T^{-1})^{(\eta+1)}x\| \\
&= \|(T^{-1})^{*}(T^{-1})^{*\eta}(T^{-1})(T^{-1})^{(\eta)}x\| \\
&= \|(T^{-1})^{*}(TT^{*\eta})^{-1}x\| \\
&= \|(T^{-1})^{*}(T^{*\eta}\,T)^{-1}xx\| \\
&= \|(T^{-1})^{*}(T^{-1})^{*}(T^{-1})^{*(\eta)}x\| \\
&\leq \|(T^{-1})^{*}(T)^{-1}(T^{-1})^{*(\eta)}x\|\|(T^{-1})^{\eta}\| \\
&\leq \|(T^{-1})^{*}(T)^{-1}(T^{-1})^{*(\eta)}x\|.
\end{aligned}
$$

Therefore; T be invertible $(\eta,*)$-Parahyponormal operator

2)
$$
\begin{aligned}
\|(T^{*})^{*(\eta+1)}x\|^{2} &= \|(T^{*})^{*(\eta+1)}(T^{*})^{(\eta+1)}x\| \\
&= \|(T)^{(\eta+1)}(T)^{*(\eta+1)}x\| \\
&= \|T\,(T)^{\eta}T^{*}\,(T^{*})^{(\eta)}x\| \\
&= \|TT^{*}\,(T)^{\eta}(T^{*})^{(\eta)}x\| \\
&\leq \|TT^{*}\,(T)^{\eta}x\|\|(T^{*})^{(\eta)}x\| \\
&\leq \|TT^{*}\,(T)^{\eta}x\|.
\end{aligned}
$$

Therefore, T^{*} be $(\eta,*)$-Parahyponormal operator.

3- Algebraic properties and their validation

The algebraic characteristics of these kinds of operators, which is addressed in this section established the prerequisites for their realization.

Remark 3.1:

The algebraic properties of these operators, like addition and multiplication, may not be attain the merit of $(\eta,*)$-Parahyponormal operator, for T $(\eta,*)$-Parahyponormal. To highlight that, we provide these examples

Examples 3.2:
1) Let $T = [0\ 2\ 2\ 0]$, $S = [0\ 0\ 0\ 2]$, so T and S are $(\eta,*)$-Parahyponormal operators, because they satisfy the conditions appeared in Theorem 2.2, since $T^{\eta}(T^*T)^2T^{*\eta} - 2\lambda T^{(\eta+1)}T^{*(\eta+1)} + \lambda^2 = [(8-\lambda)^2\ 0\ 0\ (8-\lambda)^2] \geq 0$, for each $\lambda \geq 0$ taken and applied the same condition
$S^{\eta}(S^*S)^2S^{*\eta} - 2\lambda S^{(\eta+1)}S^{*(\eta+1)} + \lambda^2 = [\lambda^2\ 0\ 0 (8-\lambda)^2] \geq 0$, for each λ taken, but $(T+S) = [0\ \ 2\ \ 2\ \ 2]$, is not $(\eta,*)$-Parahyponormal, since it has $(T+S)^{\eta}((T+S)^*(T+S))^2(T+S)^{*\eta} - 2\lambda(T+S)^{\eta+1}(T+S)^{*\eta+1} + \lambda^2[128 - 64\lambda + \lambda^2\ 140 - 128\,\lambda\ 192 - 128\lambda\ \ 281 - 128\lambda + \lambda^2] < 0$, for some $\lambda > 0$.
2) Let $T = [0\ 2\ 2\ 0]$, $S = [-1\ 0\ 1\ 0]$, then T is a $(\eta,*)$-Parahyponormal operator according to part (1) in this example and S is a $(\eta,*)$-Parahyponormal operator, because it satisfies the conditions appeared in Theorem 2.2, since $S^{\eta}(S^*S)^2S^{*\eta} - 2\lambda S^{(\eta+1)}S^{*(\eta+1)} + \lambda^2 = [(2-\lambda)^2\ 0\ 0\ \lambda^2] \geq 0$, for each $\lambda \geq 0$, but $(T.S) = [0\ \ 0\ \ -2\ \ 2]$, through Theorem 2.2, we obtained
$(T.S)^{\eta}((T.S)^*(T.S))^2(T.S)^{*\eta} - 2\lambda(T.S)^{\eta+1}(T.S)^{*\eta+1} + \lambda^2 = [\lambda^2\ \ 128\ \ 0\ -128 - 64\lambda + \lambda^2] < 0$, for some $\lambda \geq 0$, therefore; $(T.S)$ is not $(\eta,*)$-Parahyponormal operator.

The following outcome presents and determines the adequate to attain this operator.

Theorem 3.3:
Let S,T be two$(\eta,*)$-Parahyponormal operators. If $TS^* = S^*T$ S,T are normal operators and $\|(T+S)^{*\eta}\| \leq 1$, then $(T+S)$ is $(\eta,*)$-Parahyponormal operator

Proof: $\|(T+S)^{*\eta+1}x\|^2 =< (T+S)^{*\eta+1}x, (T+S)^{*\eta+1}x >$
$=< (T+S)^*\ (T+S)^{*\eta}x, (T+S)^*(T+S)^{*\eta}x >$
$=< (T+S)(T+S)^*\ (T+S)^{*\eta}x, (T+S)^{*\eta}x >$
$=< (T+S)(T^*+S^*)(T+S)^{*\eta}x, (T+S)^{*\eta}x >$
$=< (TT^*+TS^*+ST^*+SS^*)(T+S)^{*\eta}x, (T+S)^{*\eta}x >$
$=< (T^*T+S^*T+T^*S+S^*S)(T+S)^{*\eta}x, (T+S)^{*\eta}x >$
$=< (T^*+S^*)(T+S)(T+S)^{*\eta}x, (T+S)^{*\eta}x >$
$\leq \|(T^*+S^*)(T+S)(T+S)^{*\eta}x\|\|(T+S)^{*\eta}x\|$
$\leq \|(T^*+S^*)(T+S)(T+S)^{*\eta}x\|\|(T+S)^{*\eta}\|\|x\|$
$\leq \|(T^*+S^*)(T+S)(T+S)^{*\eta}x\|\|x\|.$
Therefore, $\|(T+S)^{*\eta+1}\| \leq \|(T^*+S^*)(T+S)(T+S)^{*\eta}\|$.
Hence, buy using Definition 2.1, reach out $(T+S)$ is $(\eta,*)$-Parahyponormal operator.

Theorem 3.4:
Let S,T be two$(\eta,*)$-Parahyponormal operators. If $TS^* = S^*T$ S,T are normal operators and $\|(T.S)^{*\eta}\| \leq 1$, then $(T+S)$ is $(\eta,*)$-Parahyponormal operator

Proof: $\|(T.S)^{*\eta+1}x\|^2 =< (T.S)^{*\eta+1}x, (T.S)^{*\eta+1}x >$
$=< (T.S)^*\ (T.S)^{*\eta}x, (T.S)^*(T.S)^{*\eta}x >$
$=< (T.S)(T.S)^*\ (T.S)^{*\eta}x, (T.S)^{*\eta}x >$
$=< (T.S)(S^*T^*)(T+S)^{*\eta}x, (T+S)^{*\eta}x >$
$=< (TSS^*T^*)(T+S)^{*\eta}x, (T+S)^{*\eta}x >$
$=< (TS^*ST^*)(T.S)^{*\eta}x, (T.S)^{*\eta}x >$
$=< (S^*TT^*S)(T.S)^{*\eta}x, (T.S)^{*\eta}x >$
$=< (S^*T^*TS)(T.S)^{*\eta}x, (T.S)^{*\eta}x >$
$=< (S^*T^*)(T.S)(T.S)^{*\eta}x, (T.S)^{*\eta}x >$
$=< (T.S)^*\ (T.S)(T.S)^{*\eta}x, (T.S)^{*\eta}x >$

$$\leq \| (T.S)^*(T.S)(T.S)^{*\eta}x\| \| (T.S)^{*\eta}x\|$$
$$\leq \| (T.S)^*(T.S)(T.S)^{*\eta}x\| \| (T.S)^{*\eta}\| \| x\|.$$

Therefore; $\| (T.S)^{*\eta+1}\| \leq \| (T.S)^*(T.S)(T.S)^{*\eta}\|$

Hence, buy using Definition 2.1, reach out $(T.S)$ is $(\eta,*)$-Parahyponormal operator.

Theorem 3.5:
Let T and S be $(\eta,*)$-Parahyponormal operators, then $(T \otimes S)$ is $(\eta,*)$-Parahyponormal operator.

Proof: Depending on the basic definition of the topic tensor protected operators, it possible submitted the first $\|(T \otimes S)^{*(\eta+1)}(x_1 \otimes x_2)\|^2 = \|\left(T^{*(\eta+1)}x_1 \otimes S^{*(\eta+1)}x_2\right)\|^2$ $= \|T^{*(\eta+1)}x_1\|^2 . \|S^{*(\eta+1)}x_2\|^2$, since T and S are $(\eta,*)$-Parahyponormal operators, to obtain

$$\|(T \otimes S)^{*(\eta+1)}(x_1 \otimes x_2)\|^2 \leq \|T^*TT^{*\eta}x_1\|. \|S^*SS^{*\eta}x_2\|$$
$$= \|(T^*TT^{*\eta}x_1 \otimes S^*SS^{*\eta}x_2)\|$$
$$= \|(T^* \otimes S^*)(T \otimes S)(T^{*\eta} \otimes S^{*\eta})(x_1 \otimes x_2)\|$$
$$= \|(T \otimes S)^*(T \otimes S)(T \otimes S)^{*\eta}(x_1 \otimes x_2)\|.$$

Hence, $(T \otimes S)$ is a $(\eta,*)$-Parahyponormal operator.

Corollary 3.6:
Let T be $(\eta,*)$-Parahyponormal operators, then $(T \otimes I)$ and $(T \otimes I)$ are $(\eta,*)$-Parahyponormal operator.

4. Discussions and conclusions

It has been observed in the last few years, many scholars and specialists in the realm of functional analysis, especially the operator theory of influences their articles focused on developing extraordinary effects and knowledge on complex Hilbert spaces. Subsequently, other research works have emerged with an interest in developing and debating various generalizations and sophisticated expansions of this kind of operator. This has sparked interest in important functional analysis implementations as well as solving equations built on the types of these operators and important spaces in mathematical analysis, like Banach space, Hardy space, and Hilbert space.

This study found several significant changes to the topic, including an operator extension for the standard operator and certain transferable features from the hyponormal operator, moreover addition and multiplication do not directly meet, but might hold if certain requirements are met for these operations, however, we discovered that the tensor product and direct sum satisfy the direct on definition of the N-Parahyponormal operator.

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