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# Certain Subclasses of Univalent and Bi-Univalent Functions Involving New Operator

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**Abstract**
The authors introduce novel subclasses within the univalent and bi-univalent function classes in the open unit disk $\mathfrak{A}$. Using a newly defined operator, the study explores the properties of holomorphic bi-univalent functions in $\mathfrak{A}$, focusing on parameters such as the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$, the Fekete-Szegö functional, and the second-order Hankel determinant. These parameters are critical for understanding the geometric and holomorphic characteristics of these functions. The investigation emphasizes the significance of the proposed operator and subclasses under varying parameter conditions, contributing valuable insights to geometric function theory. Derived corollaries further refine existing results, demonstrating practical applications and expanding the theoretical framework of the field.

**Keywords:** Holomorphic functions, Bi-Univalent functions, Fibonacci numbers, Quasi-subordination, Univalent function.

# بعض الفئات الفرعية من الدوال أحادية التكافؤ وثنائية التكافؤ التي تتضمن مؤثر جديد

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**الخلاصة**
يقدم المؤلفون فئات فرعية جديدة داخل فئات الدوال أحادية التكافؤ وثنائية التكافؤ في القرص الوحدوي المفتوح $\mathfrak{A}$. باستخدام مؤثر تم تعريفه حديثًا، تستكشف الدراسة خصائص الدوال ثنائية التكافؤ المجسمة في $\mathfrak{A}$ ، مع التركيز على معلمات مثل معاملات تايلور–ماكلورين $|a_2|$ و $|a_3|$ ،، ودالة فيكيتي–سيجو، ومحدد هانكل من الدرجة الثانية. هذه المعلمات مهمة لفهم الخصائص الهندسية والشكلية لهذه الدوال. يؤكد البحث على أهمية المؤثر والفئات الفرعية المقترحة في ظل ظروف معلمات مختلفة، مما يساهم في تقديم رؤى قيمة لنظرية الدالة الهندسية. تعمل النتائج المترتبة على ذلك على تحسين النتائج الحالية بشكل أكبر، وإظهار التطبيقات العملية وتوسيع الإطار النظري للمجال.

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## 1. Introduction

The normalized class of functions $\mathcal{F}$ satisfying the conditions $\mathcal{F}(0) = 0$ and $\mathcal{F}'(0) = 1$, denoted by $\mathcal{K}$, is represented by the following Taylor expansion:

$$\mathcal{F}(\zeta) = \zeta + \sum_{j=2}^{\infty} a_j \zeta^j, \quad \zeta \in \mathfrak{U}. \tag{1}$$

Here, $\mathfrak{U} = \{\zeta \in \mathbb{C} \text{ and } |\zeta| < 1\}$ denotes the open unit disk in the complex plane, where $\mathbb{C}$ represents complex numbers, and the functions are holomorphic. Let $\mathcal{G}$ denote the magnificence of all features in $\mathcal{K}$ which might be univalent in $\mathfrak{U}$. According to the Koebe One-Sector Theorem [1], for any univalent function $\mathcal{F} \in \mathcal{G}$, the image of $\mathfrak{U}$ under $\mathcal{F}$ includes a disk of radius $\frac{1}{4}$.

Thus, every univalent characteristic $\mathcal{F}$ has an inverse function $\mathcal{F}^{-1}$ such that:

$\mathcal{F}(\mathcal{F}^{-1}(\zeta)) = \zeta$ , $(\zeta \in \mathfrak{U})$ and

$\mathcal{F}(\mathcal{F}^{-1}(w)) = w$ , $\left(|w| < \rho_0(\mathcal{F}); \rho_0(\mathcal{F}) \geq \frac{1}{4}\right)$.

The inverse function can be expressed as:

$$g(w) = \mathcal{F}^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

They are normalized as in (1).

The principles of subordination and majorization are fundamental when comparing two holomorphic functions $\mathcal{F}(\zeta)$ and $r(\zeta)$ in $\mathfrak{U}$. Function $\mathcal{F}(\zeta)$ is subordinate to $r(\zeta)$, denoted $\mathcal{F}(\zeta) \prec r(\zeta)$ , if there exists a holomorphic function $t(\zeta)$ in $\mathfrak{U}$, with $t(0) = 0$ and $|t(\zeta)| < 1$, such that $\mathcal{F}(\zeta) = r(t(\zeta))$. If $r$ is univalent in $\mathfrak{U}$, then $\mathcal{F}(\zeta) \prec r(\zeta)$ implies $\mathcal{F}(0) = r(0)$ and $\mathcal{F}(\mathfrak{U}) \subset r(\mathfrak{U})$, [1].

Similarly, $\mathcal{F}(\zeta)$ is majorized by $r(\zeta)$, denoted $\mathcal{F}(\zeta) \prec\prec r(\zeta)$, if there exists a holomorphic function $|h(\zeta)|$, $|h(\zeta)| \leq 1$, such that $\mathcal{F}(\zeta) = h(\zeta) r(\zeta)$.

For the bi-univalent class, Lewin [2] established a coefficient bound $|a_2| \leq 1.51$. Clunie and Brannan [3], conjectured a bound of $|a_2| \leq 2$ .

For the function $\mathcal{F} \in \mathcal{K}$ defined in (1) and $g \in \mathcal{K}$, they are defined in the following expressed.

$$g(\zeta) = \zeta + \sum_{j=2}^{\infty} b_j \zeta^j, \quad (\zeta \in \mathfrak{U}). \tag{3}$$

The Hadamard product, or convolution, denoted by $(\mathcal{F} * g)(\zeta)$, is defined as:

$$(\mathcal{F} * g)(\zeta) = \zeta + \sum_{j=2}^{\infty} a_j b_j \zeta^j, \quad (\zeta \in \mathfrak{U}). \tag{4}$$

The $q$ -factorial [4], is represented as $[j]_q!$ , and is calculated as:

$$[j]_q! = \begin{cases} [j]_q[j-1]_q[j-2]_q[j-3]_q \ldots [3]_q[2]_q[1]_q, & if\ j = 1,2,\ldots, \\ 1 & if\ j = 0 \end{cases},$$

where

$$[j]_q = \frac{q^j - 1}{q - 1}.$$

The $q$ -derivative operator $D_q$ is defined as:

$$D_q(\zeta) = \begin{cases} \dfrac{\mathcal{F}(q\zeta) - \mathcal{F}(\zeta)}{(q-1)\zeta} & if\ \zeta \neq 0; \\ \mathcal{F}'(0), & if\ \zeta = 0\ , \end{cases} \tag{5}$$

where $0 < q < 1$.

For further details, see [5]. The $q$-exponential function $e_{q}(\zeta)$ is defined as:

$$e_{q}(\zeta)=\sum_{j=0}^{\infty}\frac{\zeta^{j}}{[j]_{q}!}\,,\qquad(\zeta\in\mathfrak{U}).$$

The $q$-binomial series [6] is given as**:**

$$(1-\alpha)_{q}^{\rho}=\sum_{j=0}^{\rho}\binom{\rho}{j}(-1)^{j}\alpha^{j}\,,\rho\in\mathbb{N}\,,j\in\mathbb{N}_{0},$$

where, $\binom{\rho}{j}$ represents the coefficients of the $q$ -binomial and formulated as follows:

$$\binom{\rho}{j}=\frac{[\rho]_{q}!}{[j]_{q}!\,[\rho-j]_{q}!}\,,$$

it represents the coefficients of the $q$ -binomial.

The operator with $q$ -differential $\mathfrak{E}_{\alpha,\rho,d,b}^{n,\sigma,v,q}:\mathcal{K}\longrightarrow\mathcal{K}$, specifically designed for $q$-differential frameworks. For $\alpha>0,\sigma>0,v\geq 0,\ 0\leq b\leq d$, and $\zeta\in\mathfrak{U}$, the operator is defined iteratively as follows:

$$\mathfrak{E}_{\alpha,\rho,d,b}^{0,\sigma,v,q}\mathcal{F}(\zeta)=\mathcal{F}(\zeta) \tag{6}$$

$$\mathfrak{E}_{\alpha,\rho,d,b}^{1,\sigma,v,q}\mathcal{F}(\zeta)=\Big(\big(1+\sigma(M_{q}^{\rho}(\alpha)-v\big)(d-b-1)\Big)\mathcal{F}(\zeta)-\zeta\sigma\Big(\big(M_{q}^{\rho}(\alpha)-v\big)(d-b)\Big)+\sigma\big(M_{q}^{\rho}(\alpha)-v\big)\zeta a_{q}\big(\mathcal{F}(\zeta)\big)\,, \tag{7}$$

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$$\mathfrak{E}_{\alpha,\rho,d,b}^{n,\sigma,v,q}\mathcal{F}(\zeta)=\mathfrak{E}_{\alpha,\rho,d,b}^{1,\sigma,v,q}(\mathfrak{E}_{\alpha,\rho,d,b}^{n-1,\sigma,v,q}\mathcal{F}(\zeta)). \tag{8}$$

Next, using functions (1) and (8), getting

$$\mathfrak{E}_{\alpha,\rho,d,b}^{n,\sigma,v,q}\mathcal{F}(\zeta)=\zeta+\sum_{j=2}^{\infty}\Big(\big(1+\sigma\big(M_{q}^{\rho}(\alpha)-v\big)[j]_{q}+d-b-1\big)\Big)^{n}a_{j}\zeta^{j}\,,(n\in\mathbb{N}_{0}=\mathbb{N}\cup\{0\})$$

where

$M_{q}^{\rho}(\alpha)=\sum_{j=0}^{\rho}\binom{\rho}{j}(-1)^{j+1}\alpha^{j}$.

This operator incorporates the coefficients from the $q$-binomial expansion and provides a versatile tool for analyzing $q$-deformed systems across diverse physical contexts.

The introduction of $q$-deformed oscillators, including the Arik-Coon oscillator and related constructs, represents a significant step forward in quantum physics. These frameworks not only extend our understanding of fundamental principles but also provide practical tools for modeling complex physical systems under non-commutative and deformed settings. For further details, readers are referred to [7-9].

For $\mathcal{F}\in\mathcal{K}$, the generalized derivative [10] operator $\mathcal{T}_{\kappa,\mu}^{m}:\mathcal{K}\longrightarrow\mathcal{K}$, it is defined by

$$\mathcal{T}_{\kappa,\mu}^{m}\,\mathcal{F}(\zeta)=\zeta+\sum_{j=2}^{\infty}[1+\mu(j-1)]^{n}\,C(\kappa,j)a_{j}\zeta^{j}\,,\qquad(\zeta\in\mathfrak{U})$$

where $n,\kappa\in N_{0}=\{0,1,2,\dots\},\mu>0$ and $C(\kappa,j)=\binom{j+\mu-1}{\kappa}=\frac{(\kappa+1)_{j-1}}{(1)_{j-1}}$.

**Definition 1.1 :** Let $\mathcal{F}\in\sum,\zeta\in\mathfrak{U}$, and consider the parameters $\alpha>0,\sigma>0,v\geq 0$, $0\leq b\leq d\quad\kappa,n\in N_{0}=\{0,1,2,\dots\}$, and $\mu>0$ .We define a new operator $\mathcal{N}_{\sigma,v,q,d,b}^{n,\kappa,\mu,\alpha,\rho,}\mathcal{J}(\zeta)=:\sum\longrightarrow\sum$ as follows:

$$\mathcal{N}_{\sigma,v,q,d,b}^{n,\kappa,\mu,\alpha,\rho}\mathcal{F}(\zeta)=\mathfrak{E}_{\alpha,\rho,d,b}^{n,\sigma,v,q}\mathcal{F}(\zeta)*\mathcal{T}_{\kappa,\mu}^{n}\ \mathcal{F}(\zeta)$$
$$=\zeta+\sum_{j=2}^{\infty}\left(\left(1+\sigma\left(M_{q}^{\rho}(\alpha)-v\right)[j]_{q}\mu(j-1)+d-b-1\right)\right)^{n}C(\kappa,j)a_{j}\zeta^{j}.$$

The preceding definition implies that

$$\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)=\zeta+\sum_{j=2}^{\infty}\Psi_{j,\alpha}a_{j}\zeta^{j},\quad \zeta\in\mathfrak{A} \tag{9}$$

with

$$\Psi_{j,\alpha}=\left(\left(1+\sigma\left(M_{q}^{\rho}(\alpha)-v\right)[j]_{q}\mu(j-1)+d-b-1\right)\right)^{n}C(\kappa,j). \tag{10}$$

**Remark 1.2:** Special cases of the operator

The operator defined above is a specific instance of the convolution operator $\mathfrak{E}_{\alpha,\rho,d,b}^{n,\sigma,v,q}\mathcal{F}(\zeta)$, which employs the generalized differential operator $\mathcal{T}_{\kappa,\mu}^{n}$ , previously introduced by various researchers.

1. For $\kappa=2, j=2, \mu=1, \rho=1, d=b$, and the operators $\mathcal{T}_{2,1}^{n}$ and $\mathfrak{E}_{\alpha,1,d,b}^{n,\sigma,v,q}$ , we derive the operator specified by Darus and Hadi, [11].
2. For $\kappa=2, j=2, \mu=1, \rho=1, \sigma=1, v=0$, and the operators $\mathcal{T}_{2,1}^{n}$ and $\mathfrak{E}_{\alpha,1,d,b}^{n,1,1,q}$, we derive the operator specified by Opoola and Lasode, [12].
3. For $\kappa=2, j=2, \mu=1, d=b, \sigma=1, v=0$, and the operators $\mathcal{T}_{2,1}^{n}$ and $\mathfrak{E}_{\alpha,\rho,d,d}^{n,1,0,q}$ , we derive the operator specified by Hadi et al., [13] .
4. For $\kappa=2, j=2, \mu=1, \rho=1, \sigma=1, v=0, d=b$, $\mathcal{T}_{2,1}^{n}$ and $\mathfrak{E}_{\alpha,\rho,d,d}^{n,1,1,q}$, we have access to the $q$ -Al-Oboudi operator, which was first presented by Aouf et al. in their work mentioned as, [14].
5. For $\kappa=2, j=2, \mu=1, q\to 1, \rho=1, d=b$, and the operators $\mathcal{T}_{2,1}^{n}$ and $\mathfrak{E}_{\alpha,1,d,b}^{n,\sigma,v,1}$, we derive the operator specified by Ibrahim and Darus in their work, [15].
6. For $\kappa=2, j=2, \mu=1, q\to 1, v=0, d=b, \sigma=1$, and the operators $\mathcal{T}_{2,1}^{n}$ and $\mathfrak{E}_{\alpha,\rho,d,d}^{n,1,0,1}$ , we derive the operator specified by Frasin, [16].
7. For $\kappa=2, j=2, \mu=1, \rho=1, \sigma=1, v=0, d=b, \alpha=1$, $\mathcal{T}_{2,1}^{n}$ and $\mathfrak{E}_{1,1,d,d}^{n1,1,q}$ , at our disposal. Possessing possessing the $q$ -Salagean operator, as demonstrated by Govindaraj and Sivasubramanian, [17].
8. For $q\to 1, \kappa=2, j=2, \mu=1, \rho=1, \sigma=1, v=1$, $\mathcal{T}_{2,1}^{n}$ and $\mathfrak{E}_{\alpha,1,d,b}^{n,1,1,1}$, we derive the operator specified by Opoola obtained in, [18].
9. For $q\to 1, \kappa=2, j=2, \mu=1, \rho=1, \sigma=1, v=1, d=b$, $\mathcal{T}_{2,1}^{n}$ and $\mathfrak{E}_{\alpha,1,d,d}^{n,1,1,1}$, Possessing the Al-Oboudi operator that was introduced by Al-Oboudi, [19].
10. For $\kappa=2, j=2, \mu=1, q\to 1, \rho=1, \sigma=1, d=b, v=1, \alpha=1$, $\mathcal{T}_{2,1}^{n}$ and $\mathfrak{E}_{1,1,d,d}^{n,1,1,1}$ possessing the Salagean operator as described by Salagean, [20].
11. For $n=0, \kappa\in N_0$, $\mu>0$ and $\mathfrak{E}_{\alpha,\rho,d,b}^{0,\sigma,v,q}$, we derive the Ruscheweyh derivative operator, [21].

Quasi-subordination, a generalization of subordination and majorization, was introduced by Robertson, [22]. For holomorphic functions $\mathcal{F}(\zeta)$ and $g(\zeta)$, the function $\mathcal{F}(\zeta)$ is quasi-subordinate to $g(\zeta)$ denoted $\mathcal{F}(\zeta)\prec_{q} g(\zeta)$, if there exists holomorphic functions $h(\zeta)$ and $t(\zeta)$ such that:

$\mathcal{F}(\zeta)=h(\zeta)g(t(\zeta)), t(0)=0, |h(\zeta)|\le 1, |t(\zeta)|<1, \zeta\in\mathfrak{A}$.

If $h(\zeta)=1$, then $\mathcal{F}(\zeta)=g\left(t(\zeta)\right)$, implying $\mathcal{F}(\zeta)\prec g(\zeta)$ in $\mathfrak{A}$. Similarly, if $t(\zeta)=\zeta$, then $\mathcal{F}(\zeta)=h(\zeta)g(\zeta)$, indicating $F(\zeta)\prec\prec g(\zeta)$ in $\mathfrak{A}$.

Many subclasses of analytic and bi-univalent functions were introduced and studied and the non-sharp estimates of first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were offered, mention to Xu et al. [23], Srivastava et al. [24]. Recently, Srivastava et al. [25] introduced some new subclasses of analytic and bi-univalent functions to integrate the studies of former researchers. Moreover, we also mention to Goyal et al. [26] for the subclasses of analytic and bi-univalent functions associated with quasi-subordination. Fekete- Szegö functional problem was investigated in many classes of functions, refer to Al-Hawary et al. [27], Orhan et al. [28] for the classes of bi-convex and bi-starlike type functions, Panigrahi and Raina [29] for class of quasi-subordination functions and Pati and Naik [30] investigated many new subclasses of analytic and bi-univalent involving the hohlov operator.

The class of bi-univalent described by quasi-subordination was presented in 2017 by Magesh et al. [31] and the coefficient bounds were established. Ma and Minda introduced the following unified function classes [32]:

$$D(\Xi) = \left\{ \mathcal{F} \in \mathcal{K} \ : \frac{\zeta\mathcal{F}'(\zeta)}{\mathcal{F}(\zeta)} \prec \Xi(\zeta) : \zeta \in \mathfrak{U} \right\},$$

$$L(\Xi) = \left\{ \mathcal{F} \in \mathcal{K} : 1 + \frac{\zeta\mathcal{F}''(\zeta)}{\mathcal{F}'(\zeta)} \prec \Xi(\zeta) : \zeta \in \mathfrak{U} \right\},$$

where $\Xi(\zeta)$ is holomorphic, univalent, and satisfies $Re(\Xi(\zeta)) > 0$ in the unit disk $\mathfrak{U}$, with $\Xi(0) = 1, \Xi'(0) > 0$. The region $\Xi(\mathfrak{U})$ is symmetric about the real axis and starlike with respect to 1(see to [32]). In this study, we assume that

$$r(\zeta) = r_0 + r_1\zeta + r_2\zeta^2 + r_3\zeta^3 + \cdots \quad (11)$$

and

$$\Xi(\zeta) = 1 + k_1\zeta + k_2\zeta^2 + k_3\zeta^3 + \cdots \quad (12)$$

Magesh et. al. [33] introduced the $k$-Fibonacci sequence $\{F_{k,j}\}_{j=0}^{\infty}$, $k \in \mathbb{R}^+$ defined as:

$$F_{k,j+1} = kF_{k,j} + F_{k,j-1}, \ F_{k,0} = 0, \ F_{k,1} = 1, \ (j \in \mathbb{N} = 1,2,\ldots). \quad (13)$$

Its closed form is given by:

$$F_{k,j} = \frac{(k-\mathcal{L}_k)^j - \mathcal{L}_k^j}{\sqrt{k^2+4}} \quad \text{with} \quad \mathcal{L}_k = \frac{k-\sqrt{k^2+4}}{2}, \quad (14)$$

where $F_{k,j}$ respresents the $nth$ element of the k-Fibonacci sequence [34].
Ozgur and Sokół [35] demonstrated that for the function

$$\tilde{p}_k(\zeta) = \frac{1 + \mathcal{L}_k^2\zeta^2}{1 - k\mathcal{L}_k\zeta - \mathcal{L}_k^2\zeta^2} = 1 + \sum_{j=1}^{\infty} \tilde{p}_{k,j}\zeta^j, \quad (15)$$

then

$\tilde{p}_{k,j} = (F_{k,j-1} + F_{k,j+1})\,\mathcal{L}_k^j, \ j \geq 1$

such that

$$\tilde{p}_k(\zeta) = 1 + (F_{k,0} + F_{k,2})\mathcal{L}_k\zeta + (F_{k,1} + F_{k,3})\mathcal{L}_k^2\zeta^2 + \cdots$$
$$= 1 + k\mathcal{L}_k\zeta + (k^2 + 2)\mathcal{L}_k^2\zeta^2 + (k^3 + 3k)\mathcal{L}_k^3\zeta^3 + \cdots,$$

where $\mathcal{L}_k = \frac{k-\sqrt{k^2+4}}{2}$ $(\zeta \in \mathfrak{U})$.

Indeed, if $k = 1, \mathcal{L}_k = \mathcal{L}$ it is possible to write

$$F_j = \frac{\ell^j - \mathcal{L}^j}{\sqrt{5}} = \frac{(1-\mathcal{L})^j - \mathcal{L}^j}{\sqrt{5}}, \qquad j \in \mathbb{N}_0$$

such that $\ell = \frac{1+\sqrt{5}}{2} \approx 1.6180339887$ ... is called the golden ratio and

$\mathcal{L} = \frac{1-\sqrt{5}}{2} = 1 - \ell = -\frac{1}{\ell} \approx -0.6180339887 \ldots$ .

For $q > 1$ and $n \geq 1$, Noonan and Thomas [36] declared the q$^{th}$ Hankel determinant as:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q+1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Multiple writers have also examined this determinant. For instance, Noor [37] calculated the limit of the average increase of $H_q(n)$) as n approaches infinity for functions defined by (1) with boundary constraints. It is evident that the Fekete-Szegö functional corresponds to $H_2(1)$. Fekete and Szegö expanded [38] on this by providing a more comprehensive estimation of $|a_3 - \varepsilon a_2^2|$, where $\varepsilon$ is a real number for detail (see [39-44] ). In this work, we will examine the Hankel determinant where $q = 2$ and $n = 2$.

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

**Definition 1.3:** Let $\mathcal{F} \in \sum$ be a function given in (1) in the class $\sum_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\delta,\gamma,\beta,\lambda,\Xi)$ satisfying the following quasi-subordination conditions:

$$\left\{\frac{\delta\zeta\left(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'}{(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))} + \frac{\gamma\zeta^2\left(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''}{(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))} + \frac{\zeta\left(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)' + \zeta^2(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}{\zeta\left(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)' + \beta\zeta^2(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''} + (1-\lambda)\zeta\left(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''\right\} - 1 \prec_q \Xi(\zeta) - 1, \qquad (16)$$

and

$$\left\{\frac{\delta w\left(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'}{(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))} + \frac{\gamma w^2\left(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)''}{(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))} + \frac{w\left(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)' + w^2(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}{w\left(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)' + \beta w^2(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''} + (1-\lambda)w(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)'')\right\} - 1 \prec_q \Xi(w) - 1, \qquad (17)$$

where ( $0 \le \delta \le 1$ , $\gamma \ge 0$, $0 \le \beta \le 1$ and $\lambda \ge 0$ , $\zeta, w \in \mathfrak{A}$).
For specific parameter values $\alpha, \rho, d, b, n, \sigma, \vartheta, q, \delta, \gamma, \beta$ *and* $\lambda$ new and well-known classes can be derived.

**Remark 1.4:** For $\delta = 0$ a function $\mathcal{F} \in \sum$, as defined in (1), belongs to the class $\sum_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\gamma,\beta,\lambda,\Xi)$, and fulfills the following quasi−subordination conditions:

$$\left\{\frac{\gamma\zeta^2\left(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''}{(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))} + \frac{\zeta\left(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)' + \zeta^2(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}{\zeta\left(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)' + \beta\zeta^2(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''} + (1-\lambda)\zeta\left(\mathcal{N}_{\sigma,\vartheta,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''\right\} - 1 \prec_q \Xi(\zeta) - 1,$$

and

$$\left\{\frac{\gamma w^2\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)''}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}+\frac{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+w^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+\beta w^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}\right.$$
$$\left.+(1-\lambda)w(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)''\right\}-1\prec_q \Xi(w)-1,$$

that ( $\gamma\geq 0,\ 0\leq\beta\leq 1$ and $\lambda\geq 0$ , $\zeta, w\in\mathfrak{A}$).

**Remark 1.5:** For $\delta=1$, a function $\mathcal{F}\in\Sigma$, as defined in (1), belongs to the class $\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(1,\gamma,\beta,\lambda,\Xi)$, and fulfills the following quasi–subordination conditions:

$$\left\{\frac{\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}+\frac{\gamma\zeta^2\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}\right.$$
$$+\frac{\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'+\zeta^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}{\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'+\beta\zeta^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}$$
$$\left.+(1-\lambda)\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''\right\}-1\prec_q \Xi(\zeta)-1\,,$$

and

$$\left\{\frac{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}+\frac{\gamma w^2\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)''}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}\right.$$
$$+\frac{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+w^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+\beta w^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}$$
$$\left.+(1-\lambda)w(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)''\right\}-1\prec_q \Xi(w)-1,$$

where ( $\gamma\geq 0,\ 0\leq\beta\leq 1$ and $\lambda\geq 0$ , $\zeta, w\in\mathfrak{A}$).

**Remark 1.6:** For $\gamma=0$ a function $\mathcal{F}\in\sum$, as defined in (1), belongs to the class $\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\delta,\beta,\lambda,\Xi)$ and fulfills the following quasi–subordination conditions:

$$\left\{\frac{\delta\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}+\frac{\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'+\zeta^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}{\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'+\beta\zeta^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}\right.$$
$$\left.+(1-\lambda)\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''\right\}-1\prec_q \Xi(\zeta)-1\,,$$

and

$$\left\{\frac{\delta w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}+\frac{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+w^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+\beta w^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}\right.$$
$$\left.+(1-\lambda)w\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''\right\}-1\prec_q \Xi(w)-1,$$

where ( $0 \leq \delta \leq 1$ , $0 \leq \beta \leq 1$ and $\lambda \geq 0$ , $\zeta, w \in \mathfrak{A}$).

**Remark 1.7:** For $\beta = 0$ a function $\mathcal{F} \in \sum$, as defined in (1), belongs to the class $\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\delta,\gamma,\lambda,\Xi)$ and fulfills the following quasi−subordination requirements:

$$\left\{\frac{\delta\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}+\frac{\gamma\zeta^2\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}\right.$$

$$+\frac{\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'+\zeta^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}{\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'}$$

$$\left.=+(1-\lambda)\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''\right\}-1\prec_q \Xi(\zeta)-1,$$

and

$$\left\{\frac{\delta w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}+\frac{\gamma w^2\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)''}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}\right.$$

$$+\frac{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+w^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'}$$

$$\left.+(1-\lambda)w(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)''\right\}-1\prec_q \Xi(w)-1,$$

where, ( $0 \leq \delta \leq 1$ , $\gamma \geq 0$ and $\lambda \geq 0$ , $\zeta, w \in \mathfrak{A}$).

**Remark 1.8:** For $\beta = 0$ a function $\mathcal{F} \in \Sigma$, as defined in (1), belongs to the class $\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\delta,\gamma,1,\lambda,\Xi)$ and fulfills the following quasi−subordination requirements:

$$\left\{\frac{\delta\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}+\frac{\gamma\zeta^2\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}\right.$$

$$+\frac{\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'+\zeta^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}{\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'+\zeta^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}$$

$$\left.+(1-\lambda)\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''\right\}-1\prec_q \Xi(\zeta)-1,$$

and

$$\left\{\frac{\delta w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}+\frac{\gamma w^2\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)''}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}\right.$$

$$+\frac{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+w^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+w^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}$$

$$\left.+(1-\lambda)w(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''\right\}-1\prec_q \Xi(w)-1,$$

where ($0 \leq \delta \leq 1$ , $\gamma \geq 0$ and $\lambda \geq 0$ , $\zeta, w \in \mathfrak{A}$).

**Remark 1.9:** For $\lambda = 0$ a function $\mathcal{F} \in \Sigma$, as defined in (1), belongs to the class$\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\delta,\gamma,\beta,\Xi)$ and fulfills the following quasi−subordination requirements:

$$\left\{\frac{\delta\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}+\frac{\gamma\zeta^2\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}\right.$$
$$\left.+\frac{\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'+\zeta^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}{\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'+\beta\zeta^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}+\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''\right\}$$
$$-1 \prec_q \Xi(\zeta)-1\,,$$

and

$$\left\{\frac{\delta w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}+\frac{\gamma w^2\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)''}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}\right.$$
$$+\frac{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+w^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+\beta w^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}$$
$$\left.+w(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)'')\right\}-1 \prec_q \Xi(w)-1\,,$$

where ($0 \le \delta \le 1$, $\gamma \ge 0$, $0 \le \beta \le 1$, and $\zeta, w \in \mathfrak{A}$).

**Remark 1.10:** For $n = 0, \sigma > 0, v \ge 0, \alpha > 0, 0 \le b \le d$, and a function $\mathcal{F} \in \sum$, as defined in (1), belongs to the class $\sum(\delta,\gamma,\beta,\lambda,\Xi)$ and fulfills the following quasi−subordination requirements:

$$\left\{\frac{\delta\zeta\mathcal{F}'(\zeta)}{(\mathcal{F}(\zeta)}+\frac{\gamma\zeta^2\mathcal{F}''(\zeta)}{\mathcal{F}(\zeta)}+\frac{\zeta\mathcal{F}'(\zeta)+\zeta^2\mathcal{F}''(\zeta)}{\zeta\mathcal{F}'(\zeta)+\beta\zeta^2\mathcal{F}''(\zeta)}+(1-\lambda)\zeta\mathcal{F}''(\zeta)\right\}-1 \prec_q \Xi(\zeta)-1\,,$$

and

$$\left\{\frac{\delta w g'(w)}{g(w)}+\frac{\gamma w^2 g''(w)}{g(w)}+\frac{w g'(w)+w^2 g''(w)}{w g'(w)+\beta w^2 g''(w)}+(1-\lambda)w g''(w))\right\}$$
$$-1 \prec_q \Xi(w)-1\,,$$

where ( $0 \le \delta \le 1$, $\gamma \ge 0$, $0 \le \beta \le 1$ and $\lambda \ge 0$, $\zeta, w \in \mathfrak{A}$.

**Definition 1.11:** Let $\mathcal{F} \in \sum$ be a function given in (1) in the class $\mathfrak{B}\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\gamma,\beta,\lambda,\eta,\tilde{p}_k)$ satisfying the following quasi-subordination conditions:

$$\left\{(1-\gamma-2\beta)\frac{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}{\zeta}+\frac{\lambda\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}+\eta\zeta^2\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'''\right\}$$
$$-1 \prec_q \tilde{p}_k(\zeta)\,,$$

where ( $0 \le \lambda \le 1$, $\gamma \ge 0$, $0 \le \beta$, $\eta \ge 0$ and $\zeta \in \mathfrak{A}$).

**Definition 1.12:** Let $\mathcal{F} \in \sum$ be a function given in (1) in the class $\mathcal{Q}\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\gamma,\beta,\lambda,\eta,\tilde{p}_k)$ satisfying the following quasi-subordination conditions:

$$\left\{(1-\gamma-2\beta)\frac{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}{\zeta}+\frac{\lambda\zeta\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}+\eta\zeta^2\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'''\right\}$$
$$-1 \prec_q \tilde{p}_k(\zeta)\,, \qquad (18)$$

and

$$\left\{(1-\gamma-2\beta)\frac{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}{w}+\frac{\lambda w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}+\eta w^2\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'''\right\}$$

$$-1\prec_q \tilde{p}_k(w)\ , \qquad (19)$$

where ( $0 \le \lambda \le 1$ , $\gamma \ge 0$, $0 \le \beta$ , $\eta \ge 0$ and $\zeta, w \in \mathfrak{U}$).

To determine the estimates coefficients $|a_2|$ and $|a_3|$, the following Lemma must be investigated.

**Lemma 1.13:** [45] If $t \in P$, then $|t_j| \le 2$ for all $j$, where $P$ is the collection of all function $t$, holomorphic in $\mathfrak{U}$ , for which $\Re\{t(\zeta)\} > 0\}$, ($\zeta \in \mathfrak{U}$), where $t(\zeta) = 1 + t_1\zeta + t_2\zeta^2 + t_3\zeta^3 + \ldots$, ($\zeta \in \mathfrak{U}$).

**Lemma 1.14:** [46] If $t(\zeta) = 1 + t_1\zeta + t_2\zeta^2 + \cdots$, and $t(\zeta) \prec \tilde{p}_k(\zeta)$, then

$$|t_1| \le |\mathcal{L}_k|,\ \ |t_2| \le 3\mathcal{L}_k^2 \ \text{ and } |t_3| \le 4\mathcal{L}_k^3.$$

This study introduces subclasses of univalent and bi-univalent functions associated with shell-like curves related to k-Fibonacci numbers. Additionally, initial coefficient bounds $|a_2|$, $|a_3|$, the Fekete–Szegö inequality, and the second Hankel determinant have been examined for functions within these classes.

## 2. Coefficient bounds for the class $\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\boldsymbol{\delta},\boldsymbol{\gamma},\boldsymbol{\beta},\boldsymbol{\lambda},\Xi)$

**Theorem 2.1:** Let $\mathcal{F}(\zeta)$ be defined as in (1) belong to the class $\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\delta,\gamma,\beta,\lambda,\Xi)$ . Then, the following coefficient bounds hold:

$$|a_2| \le min\left\{\begin{array}{c}\dfrac{|r_0|k_1}{\left|(4+\delta+2\,\gamma-2\beta-2\lambda)\Psi_{2,\alpha}\right|}\\ \sqrt{\dfrac{|4r_0|(k_1+|(k_2-k_1)|}{|(12+2\delta+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}-(\,4+\delta+2\gamma-4\beta^2)\Psi_{2,\alpha}^2|}}\end{array}\right\}, \qquad (20)$$

$$|a_3| \le min\left\{\frac{2k_1|r_0|+|r_1|}{\left|(12+2\delta+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}\right|} - \frac{|r_0^2|k_1^2}{\left|(4+\delta+2\,\gamma-2\beta-2\lambda)^2\Psi_{2,\alpha}^2\right|}, \frac{2k_1|r_0|+|r_1|}{\left|(12+2\delta+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}\right|} + \frac{|4r_0|(k_1+|(k_2-k_1)|}{\left|(12+2\delta+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}-(\,4+\delta+2\gamma-4\beta^2)\Psi_{2,\alpha}^2\right|}\right\}, \qquad (21)$$

where

$$\Psi_{2,\alpha} = \left(\left(1+\sigma\left(M_q^\rho(\alpha)-v\right)[2]_q\mu+d-b-1\right)\right)^n C(\kappa,2)$$

and

$$\Psi_{3,\alpha} = \left(\left(1+\sigma\left(M_q^\rho(\alpha)-v\right)[3]_q 2\mu+d-b-1\right)\right)^n C(\kappa,3).$$

**Proof.**

Since $\mathcal{F} \in \sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\delta,\gamma,\beta,\lambda,\Xi)$ and $g = \mathcal{F}^{-1}$, there exists holomorphic functions $u, v \in \mathcal{K}$ such that $u, v \in k$ such that $u : \mathfrak{U} \to \mathfrak{U}$ , $v : \mathfrak{U} \to \mathfrak{U}$ with $v(0) = u(0) = 0$, satisfying the following conditions:

$$\left\{\frac{\delta\zeta\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'}{(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}+\frac{\gamma\zeta^2\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''}{(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}+\frac{\zeta\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'+\zeta^2(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}{\zeta\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'+\beta\zeta^2(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}+(1-\lambda)\zeta\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''\right\}-1\prec_q r(\zeta)(\Xi(v(\zeta))-1), \quad (22)$$

and

$$\left\{\frac{\delta w\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'}{(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}+\frac{\gamma w^2\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)''}{(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}+\frac{w\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+w^2(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}{w\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+\beta w^2(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}+(1-\lambda)w(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)'')\right\}-1\prec_q r(\mathrm{w})(\Xi(u(w))-1). \quad (23)$$

Define the functions $\tau$ and $t$ as follows:

$$\tau(\zeta)=\frac{1+v(\zeta)}{1-v(\zeta)}=1+\tau_1(\zeta)+\tau_2\zeta^2+\tau_3\zeta^3+\cdots. \quad (24)$$

$$t(w)=\frac{1+u(w)}{1-u(w)}=1+t_1(w)+t_2w^2+t_3w^3+\cdots\ . \quad (25)$$

Equivalently,

$$v(\zeta)=\frac{\tau(\zeta)-1}{\tau(\zeta)+1}=\frac{1}{2}\left(\tau_1\zeta+\left(\tau_2-\frac{\tau_1^2}{2}\right)\zeta^2+\cdots\right). \quad (26)$$

$$u(w)=\frac{t(w)+1}{t(w)+1}=\frac{1}{2}\left(t_1\mathrm{w}+\left(t_2-\frac{t_1^2}{2}\right)w^2+\cdots\right). \quad (27)$$

The functions $\tau(\zeta)$ and $t(w)$ are holomorphic in $\mathfrak{A}$ with $\tau(\zeta)=t\ (w)>1$. Since both have a positive real part in the unit disk $\mathfrak{A}$, it follows that $v_i(\zeta)\le 2$ and $u_i(w)\le 2$ for $i=1,2,\ldots$

Substituting (26), (27) into (22) and (23), we obtain:

$$\left\{\frac{\delta\zeta\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'}{(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}+\frac{\gamma\zeta^2\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''}{(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}+\frac{\zeta\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'+\zeta^2(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}{\zeta\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'+\beta\zeta^2(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))''}+(1-\lambda)\zeta\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)''\right\}-1\prec_q r(\zeta)\left(\Xi\left(\frac{\tau(\zeta)-1}{\tau(\zeta)+1}\right)-1\right), \quad (28)$$

and

$$\left\{\frac{\delta w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}+\frac{\gamma w^2\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)''}{(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}+\frac{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+w^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}{w\left(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'+\beta w^2(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))''}+\right.$$

$$\left.(1-\lambda)w(\mathcal{N}_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)'')\right\}-1\prec_q r(w)(\Xi\left(\frac{t(w)+1}{t(w)+1}\right)-1). \quad (29)$$

By using (26) and (27) in conjunction with (11) and (12), we find:

$$r(\zeta)(\Xi\left(\frac{\tau(\zeta)-1}{\tau(\zeta)+1}\right)-1) = \frac{1}{2}r_0\tau_1k_1\zeta+\left(\frac{1}{2}r_1\tau_1k_1+\frac{1}{2}r_0k_1\left(\tau_2-\frac{\tau_1^2}{2}\right)+\frac{1}{4}r_0k_2\tau_1^2\right)\zeta^2 + \cdots, \quad (30)$$

$$r(w)(\Xi\left(\frac{t(w)-1}{t(w)+1}\right)-1) = \frac{1}{2}r_0t_1k_1w+\left(\frac{1}{2}r_1t_1k_1\frac{1}{2}r_0k_1\left(t_2-\frac{t_1^2}{2}\right)+\frac{1}{4}r_0k_2t_1^2\right)w^2 + \cdots . \quad (31)$$

From (28), (29), (30) and (31), it follows that:

$$(4+\delta+2\gamma-2\beta-2\lambda)\Psi_{2,\alpha}a_2=\frac{1}{2}r_0k_1\tau_1 \quad , \quad (32)$$

$$(12+2\delta+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}a_3-(4+\delta+2\gamma-4\beta^2)\Psi_{2,\alpha}^2a_2^2 = \frac{1}{2}r_1\tau_1k_1+\frac{1}{2}r_0k_1\left(\tau_2-\frac{\tau_1^2}{2}\right) + \frac{1}{4}r_0k_2\tau_1^2, \quad (33)$$

$$-(4+\delta+2\gamma-2\beta-2\lambda)\Psi_{2,\alpha}a_2=\frac{1}{2}r_0k_1t_1 \quad , \quad (34)$$

and

$$-(12+2\delta+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}a_3-(4+\delta+2\gamma-4\beta^2)\Psi_{2,\alpha}^2a_2^2+(24+4\delta+12\gamma-12\beta-12\lambda)\Psi_{3,\alpha}a_2^2 = \frac{1}{2}r_1t_1k_1+\frac{1}{2}r_0k_1\left(t_2-\frac{t_1^2}{2}\right)+\frac{1}{4}r_0k_2t_1^2, \quad (35)$$

From (32) and (34), we deduce:

$$a_2=\frac{r_0k_1\tau_1}{(4+\delta+2\gamma-2\beta-2\lambda)\Psi_{2,\alpha}} = -\frac{r_0k_1\tau_1}{(4+\delta+2\gamma-2\beta-2\lambda)\Psi_{2,\alpha}}, \quad (36)$$

which implies:

$$\tau=-t \quad (37)$$

and

$$(4+\delta+2\gamma-2\beta-2\lambda)^2\Psi_{2,\alpha}^2a_2^2=\frac{1}{4}r_0^2k_1^2(\tau_1^2+t_1^2). \quad (38)$$

Adding (33) and (35), we derive:

$$\left[2(12+2\delta+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}-2(4+\delta+2\gamma-4\beta^2)\Psi_{2,\alpha}^2\right]a_2^2 = 2r_0k_1(\tau_2+t_2)+r_0(k_2 - k_1(\tau_1^2+t_1^2). \quad (39)$$

By applying Lemma 1.13 to the coefficients $\tau_1\tau_2, t_1$ and $t_2$, it follows from (37) and (38) that:

$$|a_2| \leq \frac{|r_0|k_1}{|(4+\delta+2\gamma-2\beta-2\lambda)\Psi_{2,\alpha}|}, \tag{40}$$

and

$$|a_2| \leq \sqrt{\frac{|4r_0|(k_1+|(k_2-k_1)|}{|(12+2\delta+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}-(4+\delta+2\gamma-4\beta^2)\Psi_{2,\alpha}^2|}}. \tag{41}$$

This provides the coefficient bound for $|a_2|$ as stated in (20).

Subtracting (35) from (33), we obtain:

$$2(12+2\delta-6\beta-6\lambda)(a_3+a_2^2)\Psi_{3,\alpha} = 2r_0k_1\tau_1 + r_0k_1(\tau_2-t_2). \tag{42}$$

By substituting (38) and (39) into (42) and using Lemma 1.13, we get:

$$|a_3| \leq \frac{2k_1|r_0|+|r_1|}{|(12+2\delta+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}|} - \frac{|r_0^2|k_1^2}{|(4+\delta+2\gamma-2\beta-2\lambda)^2\Psi_{2,\alpha}^2|}, \tag{43}$$

$$|a_3| \leq \frac{2k_1|r_0|+|r_1|}{|(12+2\delta+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}|} + \frac{|4r_0|(k_1+|(k_2-k_1)|}{|(12+2\delta+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}-(4+\delta+2\gamma-4\beta^2)\Psi_{2,\alpha}^2|}. \tag{44}$$

Equations (43) and (44) provide the coefficient bound as stated in (21), completing the proof.

## 3. Coefficient bounds and Fekete-Szego inequality for the function class $\mathcal{Q}\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\gamma,\beta,\lambda,\eta,\tilde{p}_k)$

**Theorem 3.1:** Let $\mathcal{F}(\zeta)$ be defined as in (1) belong to the class $\mathcal{Q}\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\gamma,\beta,\lambda,\eta,\tilde{p}_k)$. Then the following coefficient bounds hold:

$$|a_2| \leq \frac{2k|k||\mathcal{L}_k|}{\sqrt{|k^2\mathcal{L}_k[2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}-\lambda\Psi_{2,\alpha}^2]+2N_K(1+\lambda-\gamma-2\beta)^2\Psi_{2,\alpha}^2|}}, \tag{45}$$

$$|a_3| \leq \frac{|k||\mathcal{L}_k|}{2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}} + \frac{k^3\mathcal{L}_k^2}{|k^2\mathcal{L}_k[2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}-\lambda\Psi_2^2]+2N_K(1+\lambda-\gamma-2\beta)^2\Psi_{2,\alpha}^2|}, \tag{46}$$

and for $\varepsilon \in \mathbb{R}$

$$|a_3-\varepsilon a_2^2| \leq \begin{cases} \dfrac{|k||\mathcal{L}_k|}{(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}}, & 0 \leq \Im(\varepsilon) \leq \dfrac{|k||\mathcal{L}_k|}{2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}} \\ 2|\Im(\varepsilon)||k||\mathcal{L}_k|, & \Im(\varepsilon) \geq \dfrac{|k||\mathcal{L}_k|}{2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}}. \end{cases} \tag{47}$$

where $N_K = k-(k^2+2)\mathcal{L}_k$,

$$\Im(\xi) = \frac{(1-\varepsilon)k^3\mathcal{L}_k^2}{2k^2\mathcal{L}_k[2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}-\lambda\Psi_{2,\alpha}^2]+2N_K(1+\lambda-\gamma-2\beta)^2\Psi_{2,\alpha}^2}$$

**Proof.**

Assume that $\mathcal{F}(\zeta) \in \mathcal{Q}\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\gamma,\beta,\lambda,\eta,\tilde{p}_k)$. Then, from (18) and (19), we have:

$$\left\{(1-\gamma-2\beta)\frac{(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}{\zeta}+\frac{\lambda\zeta\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'}{(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}+\eta\zeta^2\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'''\right\}$$
$$-1\prec_q \tilde{p}_k(\zeta), \qquad (48)$$

and

$$\left\{(1-\gamma-2\beta)\frac{(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}{w}+\frac{\lambda w\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'}{(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w))}+\eta w^2\left(\mathcal{N}_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}g(w)\right)'''\right\}$$
$$-1\prec_q \tilde{p}_k(w), \qquad (49)$$

where $g(w)=\mathcal{F}^{-1}(\zeta)$ as defined in Equation (2), and $\tilde{p}_k$ is given by (15).

Let $\upsilon(\zeta)=1+\upsilon_1\zeta+\upsilon_2\zeta^2+\cdots$, and assume $\upsilon\prec\tilde{p}_k$ .Then, there exists a holomorphic function $u(\zeta)$ such that $|u(\zeta)|<1$ in $\mathfrak{U}$ and $\upsilon(\zeta)=\tilde{p}_k(u(\zeta))$. Therefore, the function

$$\mathcal{Y}(\zeta)=\frac{1+u(\zeta)}{1-u(\zeta)}=1+u_1\zeta+u_2\zeta^2+\cdots,$$

belongs to the class $P$. Consequently,

$$u(\zeta)=\frac{\mathcal{Y}(\zeta)-1}{\mathcal{Y}(\zeta)+1}=\frac{u_1}{2}\zeta+\left(u_2-\frac{u_1^2}{2}\right)\frac{\zeta^2}{2}+\left(u_3-u_1u_2+\frac{u_1^3}{4}\right)\frac{\zeta^3}{2}+\cdots,$$

and

$$\begin{aligned}\tilde{p}_k(u(\zeta))&=1+\tilde{p}_{k,1}\left(\frac{u_1\zeta}{2}+\left(u_2-\frac{u_1^2}{2}\right)\frac{\zeta^2}{2}+\left(u_3-u_1u_2+\frac{u_1^3}{4}\right)\frac{\zeta^3}{2}+\cdots\right)\\&+\tilde{p}_{k,2}\left(\frac{u_1\zeta}{2}+\left(u_2-\frac{u_1^2}{2}\right)\frac{\zeta^2}{2}+\left(u_3-u_1u_2+\frac{u_1^3}{4}\right)\frac{\zeta^3}{2}+\cdots\right)^2\\&+\tilde{p}_{k,3}\left(\frac{u_1\zeta}{2}+\left(u_2-\frac{u_1^2}{2}\right)\frac{\zeta^2}{2}+\left(u_3-u_1u_2+\frac{u_1^3}{4}\right)\frac{\zeta^3}{2}+\cdots\right)^3+\cdots\\&=1+\frac{\tilde{p}_{k,1}u_1\zeta}{2}+\left(\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)\tilde{p}_{k,1}+\frac{u_1^2}{4}\tilde{p}_{k,2}\right)\zeta^2\\&+\left(\frac{1}{2}\left(u_3-u_1u_2+\frac{u_1^3}{4}\right)\tilde{p}_{k,1}+\frac{1}{2}u_1\left(u_2-\frac{u_1^2}{2}\right)\tilde{p}_{k,2}+\frac{u_1^3}{8}\tilde{p}_{k,3}\right)\zeta^3\\&+\cdots. \quad (50)\end{aligned}$$

Similarly, there exists a holomorphic function $q(w)$ such that $|q(w)|<1$ in $\mathfrak{U}$ and $\upsilon(w)=\tilde{p}_k(q(w))$. Then, the function

$$\mathcal{X}(w)=\frac{1+q(w)}{1-q(w)}=1+q_1w+q_2w^2+\cdots,$$

belongs to the class $P$. Thus,

$$\begin{aligned}\tilde{p}_k(q(w))&=1+\frac{\tilde{p}_{k,1}q_1w}{2}+\left(\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)\tilde{p}_{k,1}+\frac{q_1^2}{4}\tilde{p}_{k,2}\right)w^2\\&+\left(\frac{1}{2}\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)\tilde{p}_{k,1}+\frac{1}{2}q_1\left(q_2-\frac{q_1^2}{2}\right)\tilde{p}_{k,2}+\frac{q_1^3}{8}\tilde{p}_{k,3}\right)w^3\\&+\cdots. \quad (51)\end{aligned}$$

Using (48), (49), (50), and (51), we get:

$$(1+\lambda-\gamma-2\beta)\Psi_2a_2=\frac{u_1k\mathcal{L}_k}{2}, \qquad (52)$$

$$(1+2\lambda+6\eta-\gamma-2\beta)\Psi_3a_3-\lambda\Psi_2^2a_2^2=\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)k\mathcal{L}_k+\frac{u_1^2}{4}(k^2+2)\mathcal{L}_k^2, \qquad (53)$$

and

$$-(1+\lambda-\gamma-2\beta)\Psi_{2,\alpha}a_2=\frac{q_1 k\mathcal{L}_k}{2}, \tag{54}$$

$$\begin{aligned}-(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}a_3+2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}a_2^2-\lambda\Psi_{2,\alpha}^2a_2^2\\ =\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)k\mathcal{L}_k\\ +\frac{q_1^2}{4}(k^2+2)\mathcal{L}_k^2,\end{aligned} \tag{55}$$

From (52) and (54), it follows that

$$u_1=-q_1, \tag{56}$$

and

$$2(1+\lambda-\gamma-2\beta)^2\Psi_{2v}^2a_2^2=\frac{(u_1^2+q_1^2)k^2\mathcal{L}_k^2}{4}. \tag{57}$$

Adding (53) and (55) gives:

$$\begin{aligned}\left[2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}-\lambda\Psi_2^2\right]a_2^2\\ =\frac{1}{2}(u_2+q_2)k\mathcal{L}_k\\ -\frac{1}{4}(k\mathcal{L}_k-(k^2+2)\mathcal{L}_k^2)(u_1^2+q_1^2).\end{aligned} \tag{58}$$

Substituting (57) into (58) yields:

$$a_2^2=\frac{(u_2+q_2)k^3\mathcal{L}_k^2}{2k^2\mathcal{L}_k\left[2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}-\lambda\Psi_2^2\right]+2N_K(1+\lambda-\gamma-2\beta)^2\Psi_{2,\alpha}^2}, \tag{59}$$

where $N_K=k-(k^2+2)\mathcal{L}_k$. Using Lemma 1.13, we obtain (45).

Now, subtraction (55) from (53) and applying (55), we find:

$$a_3=a_2^2+\frac{(u_2-q_2)k\mathcal{L}_k}{2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}}, \tag{60}$$

By substituting (59) into Lemma 1.13, (60) reduces to (45). From (59) and (60), for $\varepsilon\in\mathbb{R}$ , we have:

$$a_3-\varepsilon a_2^2=\frac{(u_2-q_2)k\mathcal{L}_k}{2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}}+(1-\varepsilon)\frac{(u_2+q_2)k^3\mathcal{L}_k^2}{2k^2\mathcal{L}_k\left[2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}-\lambda\Psi_2^2\right]+2N_K(1+\lambda-\gamma-2\beta)^2\Psi_{2,\alpha}^2},$$

which can be expressed as

$$\begin{aligned}a_3-\varepsilon a_2^2=\left[\mathfrak{J}(\xi)+\frac{k\mathcal{L}_k}{2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}}\right]u_2\\ +\left[\mathfrak{J}(\varepsilon)-\frac{k\mathcal{L}_k}{2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}}\right]q_2,\end{aligned}$$

where

$$\mathfrak{J}(\xi)=\frac{(1-\varepsilon)k^3\mathcal{L}_k^2}{2k^2\mathcal{L}_k\left[2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_3-\lambda\Psi_{2,\alpha}^2\right]+2N_K(1+\lambda-\gamma-2\beta)^2\Psi_{2,\alpha}^2}.$$

Taking the modulus, we get:

$$|a_3-\varepsilon a_2^2|\leq\begin{cases}\dfrac{|k||\mathcal{L}_k|}{(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}}, 0\leq\mathfrak{J}(\varepsilon)\leq\dfrac{|k||\mathcal{L}_k|}{2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}}\\ 2|\mathfrak{J}(\varepsilon)||k||\mathcal{L}_k|, \mathfrak{J}(\varepsilon)\geq\dfrac{|k||\mathcal{L}_k|}{2(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}}.\end{cases}$$

## 4. Second Hankel determinant for the function class $\mathfrak{B}\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\gamma,\beta,\lambda,\eta,\tilde{p}_k)$

**Theorem 4.1:** Let $\mathcal{F}(\zeta)\in\mathfrak{B}\sum_{\sigma,v,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\gamma,\beta,\lambda,\eta,\tilde{p}_k)$ . Then

$$|a_2a_4 - a_3^2| \leq \frac{[4(1+2\lambda+6\eta-\gamma-2\beta)^2\Psi_{3,\alpha}^2 + 9(1+\lambda-\gamma-2\beta)(1+3\lambda+24\eta-\gamma-2\beta)\Psi_{4,\alpha}\Psi_{2,\alpha}]}{(1+2\lambda+6\eta-\gamma-2\beta)^2(1+\lambda-\gamma-2\beta)(1+3\lambda+24\eta-\gamma-2\beta)\Psi_{3,\alpha}^2\Psi_{4,\alpha}\Psi_{2,\alpha}}\tau_k^4, \quad (61)$$

where $\Psi_j$ is defined in Eqution (10) for $0 \leq \lambda \leq 1$, $\gamma \geq 0$, $0 \leq \beta$, $\eta \geq 0$.

**Proof.** Since $\mathcal{F}(\zeta) \in \mathfrak{V}\sum_{\sigma,\mathcal{v},q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\gamma,\beta,\lambda,\eta,\tilde{p}_k)$, by Definition 1.11, we have:

$$\left\{(1-\gamma-2\beta)\frac{(\mathcal{N}_{\sigma,\mathcal{v},q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))}{\zeta} + \frac{\lambda\zeta\left(\mathcal{N}_{\sigma,\mathcal{v},q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'}{(\mathcal{N}_{\sigma,\mathcal{v},q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta))} + \eta\zeta^2\left(\mathcal{N}_{\sigma,\mathcal{v},q,\mu,\kappa}^{\alpha,\rho,d,b,n}\mathcal{F}(\zeta)\right)'''\right\} = \mathcal{v}(\zeta) = 1 + t_1\zeta + t_2\zeta^2 + \cdots, \quad (62)$$

and by expanding and equating coefficients in Eqution (62), it follows:

$$a_2 = \frac{t_1}{(1+\lambda-\gamma-2\beta)\Psi_{2,\alpha}},$$

$$a_3 = \frac{t_2}{(1+2\lambda+6\eta-\gamma-2\beta)\Psi_{3,\alpha}}$$

and

$$a_4 = \frac{t_3}{(1+3\lambda+24\eta-\gamma-2\beta)\Psi_{4,\alpha}}.$$

Using Lemma 1.14, we obtain:

$$|a_2a_4 - a_3^2| \leq \frac{[4(1+2\lambda+6\eta-\gamma-2\beta)^2\Psi_{3,\alpha}^2 + 9(1+\lambda-\gamma-2\beta)(1+3\lambda+24\eta-\gamma-2\beta)\Psi_{4,\alpha}\Psi_{2,\alpha}]}{(1+2\lambda+6\eta-\gamma-2\beta)^2(1+\lambda-\gamma-2\beta)(1+3\lambda+24\eta-\gamma-2\beta)\Psi_{3,\alpha}^2\Psi_{4,\alpha}\Psi_{2,\alpha}}\tau_k^4.$$

## 5. Corollaries and consequences

**Corollary 5.1:** Let $\mathcal{F}$ be in class $\sum_{\sigma,\mathcal{v},q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\gamma,\beta,\lambda,\Xi)$. Then

$$|a_2| \leq min\left\{\frac{|r_0|k_1}{|(4+2\gamma-2\beta-2\lambda)\Psi_{2,\alpha}|}, \sqrt{\frac{|4r_0|(k_1+|(k_2-k_1)|}{|(12+6\gamma-6\beta+-6\lambda)\Psi_{3,\alpha}-(4+2\gamma-4\beta^2)\Psi_{2,\alpha}^2|}}\right\},$$

$$|a_3| \leq min\left\{\frac{2k_1|r_0|+|r_1|}{|(12+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}|} - \frac{|r_0^2|k_1^2}{|(4+2\gamma-2\beta-2\lambda)^2\Psi_{2,\alpha}^2|}, \frac{2k_1|r_0|+|r_1|}{|(12+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}|} + \frac{|4r_0|(k_1+|(k_2-k_1)|}{|(12+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}-(4+2\gamma-4\beta^2)\Psi_{2,\alpha}^2|}\right\}.$$

**Corollary 5.2:** Let $\mathcal{F}$ be in class $\sum_{\sigma,\mathcal{v},q,\mu,\kappa}^{\alpha,\rho,d,b,n}(1,\gamma,\beta,\lambda,\Xi)$. Then

$$|a_2| \leq min\left\{\frac{|r_0|k_1}{|(5+2\gamma-2\beta-2\lambda)\Psi_{2,\alpha}|}, \sqrt{\frac{|4r_0|(k_1+|(k_2-k_1)|}{|(14+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}-(5+2\gamma-4\beta^2)\Psi_{2,\alpha}^2|}}\right\},$$

$$|a_3| \leq min\left\{\frac{2k_1|r_0|+|r_1|}{|(14+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}|} - \frac{|r_0^2|k_1^2}{|(5+2\gamma-2\beta-2\lambda)^2\Psi_{2,\alpha}^2|}, \frac{2k_1|r_0|+|r_1|}{|(14+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}|} + \frac{|4r_0|(k_1+|(k_2-k_1)|}{|(14+6\gamma-6\beta-6\lambda)\Psi_{3,\alpha}-(5+2\gamma-4\beta^2)\Psi_{2,\alpha}^2|}\right\}.$$

**Corollary 5.3:** Let $\mathcal{F}$ be in class $\sum_{\sigma,\mathcal{v},q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\delta,\beta,\lambda,\Xi)$. Then

$$|a_2| \leq min\left\{\frac{|r_0|k_1}{|(4+\delta-2\beta-2\lambda)\Psi_{2,\alpha}|}, \sqrt{\frac{|4r_0|(k_1+|(k_2-k_1)|}{|(12+2\delta-6\beta-6\lambda)\Psi_{3,\alpha}-(4+\delta-4\beta^2)\Psi_{2,\alpha}^2|}}\right\},$$

$$|a_3| \leq min\left\{\frac{2k_1|r_0|+|r_1|}{|(12+2\delta-6\beta-6\lambda)\Psi_{3,\alpha}|} - \frac{|r_0^2|k_1^2}{|(4+\delta-2\beta-2\lambda)^2\Psi_{2,\alpha}^2|}, \frac{2k_1|r_0|+|r_1|}{|(12+2\delta-6\beta-6\lambda)\Psi_{3,\alpha}|} + \frac{|4r_0|(k_1+|(k_2-k_1)|}{|(12+2\delta-6\beta-6\lambda)\Psi_{3,\alpha}-(4+\delta-4\beta^2)\Psi_{2,\alpha}^2|}\right\}.$$

**Corollary 5.4:** Let $\mathcal{F}$ be in class $\sum_{\sigma,\mathcal{v},q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\delta,\gamma,\lambda,\Xi)$. Then

$|a_2| \leq min\left\{\frac{|r_0|k_1}{|(4+\delta+2\,\gamma-2\lambda)\Psi_{2,\alpha}|}, \sqrt{\frac{|4r_0|(k_1+|(k_2-k_1)|}{|(12+2\delta+6\gamma-6\lambda)\Psi_{3,\alpha}-(4+\delta+2\gamma)\Psi_{2,\alpha}^2|}}\right\},$

$|a_3| \leq min\left\{\frac{2k_1|r_0|+|r_1|}{|(12+2\delta+6\gamma-6\lambda)\Psi_{3,\alpha}|} - \frac{|r_0^2|k_1^2}{|(4+\delta+2\,\gamma-2\lambda)^2\Psi_{2,\alpha}^2|}, \frac{2k_1|r_0|+|r_1|}{|(12+2\delta+6\gamma-6\lambda)\Psi_{3,\alpha}|} + \frac{|4r_0|(k_1+|(k_2-k_1)|}{|(12+2\delta+6\gamma-6\lambda)\Psi_{3,\alpha}-(4+\delta+2\gamma)\Psi_{2,\alpha}^2|}\right\}.$

**Corollary 5.5:** Let $\mathcal{F}$ be in class $\sum_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\delta,\gamma,1,\lambda,\Xi)$ . Then

$|a_2| \leq min\left\{\frac{|r_0|k_1}{|(2+\delta+2\,\gamma-2\lambda)\Psi_{2,\alpha}|}, \sqrt{\frac{|4r_0|(k_1+|(k_2-k_1)|}{|(6+2\delta+6\gamma-6\lambda)\Psi_{3,\alpha}-(\delta+2\gamma)\Psi_{2,\alpha}^2|}}\right\},$

$|a_3| \leq min\left\{\frac{2k_1|r_0|+|r_1|}{|(6+2\delta+6\gamma-6\lambda)\Psi_{3,\alpha}|} - \frac{|r_0^2|k_1^2}{|(2+\delta+2\,\gamma-2\lambda)^2\Psi_{2,\alpha}^2|}, \frac{2k_1|r_0|+|r_1|}{|(5+2\delta+6\gamma-6\lambda)\Psi_{3,\alpha}|} + \frac{|4r_0|(k_1+|(k_2-k_1)|}{|(6+2\delta+6\gamma-6\lambda)\Psi_{3,\alpha}-(\delta+2\gamma)\Psi_{2,\alpha}^2|}\right\}.$

**Corollary 5.6:** Let $\mathcal{F}$ be in class $\sum_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\delta,\gamma,\beta,\Xi)$ . Then

$|a_2| \leq min\left\{\frac{|r_0|k_1}{|(4+\delta+2\,\gamma-2\beta)\Psi_{2,\alpha}|}, \sqrt{\frac{|4r_0|(k_1+|(k_2-k_1)|}{|(12+2\delta+6\gamma-6\beta)\Psi_{3,\alpha}-(4+\delta+2\gamma-4\beta^2)\Psi_{2,\alpha}^2|}}\right\},$

$|a_3| \leq min\left\{\frac{2k_1|r_0|+|r_1|}{|(12+2\delta+6\gamma-6\beta)\Psi_{3,\alpha}|} - \frac{|r_0^2|k_1^2}{|(4+\delta+2\,\gamma-2\beta)^2\Psi_{2,\alpha}^2|}, \frac{2k_1|r_0|+|r_1|}{|(12+2\delta+6\gamma-6\beta)\Psi_{3,\alpha}|} + \frac{|4r_0|(k_1+|(k_2-k_1)|}{|(12+2\delta+6\gamma-6\beta)\Psi_{3,\alpha}-(4+\delta+2\gamma-4\beta^2)\Psi_{2,\alpha}^2|}\right\}.$

**Corollary 5.7:** Let $\mathcal{F}$ be in class $\sum(\delta,\gamma,\beta,\lambda,\Xi)$. Then

$|a_2| \leq min\left\{\frac{|r_0|k_1}{|(4+\delta+2\,\gamma-2\beta-2\lambda)|}, \sqrt{\frac{|4r_0|(k_1+|(k_2-k_1)|}{|(12+2\delta+6\gamma-6\beta-6\lambda)-(4+\delta+2\gamma-4\beta^2)|}}\right\},$

$|a_3| \leq min\left\{\frac{2k_1|r_0|+|r_1|}{|(12+2\delta+6\gamma-6\beta-6\lambda)|} - \frac{|r_0^2|k_1^2}{|(4+\delta+2\,\gamma-2\beta-2\lambda)^2|}, \frac{2k_1|r_0|+|r_1|}{|(12+2\delta+6\gamma-6\beta-6\lambda)|} + \frac{|4r_0|(k_1+|(k_2-k_1)|}{|(12+2\delta+6\gamma-6\beta-6\lambda)-(4+\delta+2\gamma-4\beta^2)|}\right\}.$

**Corollary 5.8:** Let $\mathcal{F}(\zeta)$ defined in (1) be in class $\mathcal{Q}\sum_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(0,0,10,\tilde{p}_k)$. Then

$$|a_2| \leq \frac{2k|k||\mathcal{L}_k|}{\sqrt{|k^2\mathcal{L}_k[6\Psi_{3,\alpha}-\Psi_{2,\alpha}^2]+8N_K\Psi_{2,\alpha}^2|}},$$

$$|a_3| \leq \frac{|k||\mathcal{L}_k|}{6\Psi_{3,\alpha}} + \frac{k^3\mathcal{L}_k^2}{|k^2\mathcal{L}_k[6\Psi_{3,\alpha}-\lambda\Psi_2^2]+6N_K\Psi_{2,\alpha}^2|},$$

and for $\varepsilon \in \mathbb{R}$

$$|a_3-\varepsilon a_2^2| \leq \begin{cases} \dfrac{|k||\mathcal{L}_k|}{3\Psi_{3,\alpha}}, 0 \leq \mathfrak{J}(\varepsilon) \leq \dfrac{|k||\mathcal{L}_k|}{6\Psi_{3,\alpha}} \\ 2|\mathfrak{J}(\varepsilon)||k||\mathcal{L}_k|, \mathfrak{J}(\varepsilon) \geq \dfrac{|k||\mathcal{L}_k|}{6\Psi_{3,\alpha}}. \end{cases}$$

Where $N_K = k-(k^2+2)\mathcal{L}_k$,

$$\mathfrak{J}(\xi) = \frac{(1-\varepsilon)k^3\mathcal{L}_k^2}{2k^2\mathcal{L}_k[6\Psi_{3,\alpha}-\Psi_{2,\alpha}^2]+8N_K\Psi_{2,\alpha}^2}.$$

If $\gamma=\beta=n=\eta=0$ and $\lambda=1$, Theorem 4 gives the next example.

**Example 5.9:** If $\mathcal{F}(\zeta) \in \mathcal{Q}\sum_{\sigma,\upsilon,q,\mu,\kappa}^{\alpha,\rho,d,b,n}(\gamma,\beta,\lambda,\eta,\tilde{p}_k)$ , then

$$|a_2a_4-a_3^2| \leq \frac{108}{24}\tau_k^4.$$

## 6. Conclusions

This examine introduces and employs a singular operator wonderful from preceding investigations, which allowed the extraction of quasi-subordination consequences and the derivation of new effects within the unit disk. Specifically, the studies explored new subclasses: $\sum_{\sigma,\mathscr{v},\mathscr{q},\mu,\kappa}^{\alpha,\rho,d,\mathscr{b},n}(\delta,\gamma,\beta,\lambda,\Xi)$, $\mathfrak{V}\sum_{\sigma,\mathscr{v},\mathscr{q},\mu,\kappa}^{\alpha,\rho,d,\mathscr{b},n}(\gamma,\beta,\lambda,\eta,\tilde{p}_k)$ and $\mathcal{Q}\sum_{\sigma,\mathscr{v},\mathscr{q},\mu,\kappa}^{\alpha,\rho,d,\mathscr{b},n}(\gamma,\beta,\lambda,\eta,\tilde{p}_k)$. These subclasses awareness on univalent and bi-univalent mathematical features, particularly emphasizing geometric residences and holomorphic conduct. Notable results consist of computations of $|a_2|$, $|a_3|$, the Fekete-Szegö inequality, and the second one Hankel determinant for univalent and bi-univalent capabilities in the framework of the newly described operator.

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