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Timewise Coefficient Identification Inverse Problem for Parabolic Equation with Dirichlet Boundary Condition and Over Specified Condition of Integral Kind

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Abstract

Through the application of a Dirichlet boundary condition and under an additional integral-type condition, the recovery of the time-dependent coefficient in a one -dimensional parabolic equation is investigated in this paper. When data is entered, the solution is affected to a precarious status during exposure to random errors and noise. The Crank-Nicolson finite difference approach is implemented for the direct solution of the problem, while nonlinear numerical optimization is employed for the inverse problem. *lsqnonlin*, the MATLAB routine optimization tool, is applied to compute the last problem. The Tikhonov regularization approach must be used to produce smooth, stable answers. The evaluation and comparison with their identical answers were performed by running the root mean square error formula. It conclude that, the numerical results are consistent and accurate.

Keywords: Nonlinear minimization, Crank-Nicolson approach, Non-local integration condition, Dirichlet boundary conditions, Heat equation, Inverse problem.

المسألة العكسية لتحديد المعامل الزمني لمعادلة القطع المكافئ مع شرط حدود ديريشليت وشرط اضافي من النوع التكاملي

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الخلاصة :

من خلال تطبيق شرط ديريشليت الحدودي وتحت شرط إضافي من نوع التكاملي، تمت في هذه الورقة البحثية استرداد المعامل المعتمد على الزمن في معادلة مكافئة أحادية البعد. عند إدخال البيانات، يتأثر الحل بحالة غير مستقرة أثناء التعرض للأخطاء العشوائية والضوضاء. يتم تنفيذ نهج كرانك-نيكولسون للفروقات المنتهية للحل المباشر للمسألة، بينما تم استخدام التحسين العددي غير الخطي للمسألة العكسية. تم تطبيق أداة التحسين الروتيني في ماتلاب lsqnonlin لحساب المسألة الأخيرة. يجب استخدام

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نهج تسوية تيخونوف للحصول على نتائج سلسة ومستقرة. تم إجراء التقييم والمقارنة مع الحلول المظبوطة من خلال صيغة خطأ جذر معدل التربيعات . والنتائج العددية كانت ثابتة ودقيقة.

1. Introduction

Partial differential equations have played a leading role in various fields of science for many decades due to their importance in applications, including physics, engineering, and quantum chemistry. On the other hand, the role and importance of partial differential equations are evident in forming models of natural phenomena and establishing numerical methods for approximate solutions and their application to concrete fields of life [1-4]. In the last six decades of the previous century, inverse problems and their study have earned a growing prominence due to their importance in the crystallization of their mathematical models and their spacious framework of applications. These models have become essential in formulating physical problems in engineering and industry [5-7]. At the beginning of the millennium, the concept of inverse problems was expanded due to its broad role in important application areas and the reconstruction of unknown parameters [8-11]. Under Dirichlet boundary conditions, in addition to the integral overdetermination condition, the inverse problem of assigning the spreading coefficient in a 1-D parabolic heat equation was investigated in [12]. A new method has been developed in [13] for the first time, which involves the implementation of two algorithms in stages to recreate the time-dependent diffusion coefficient in a 1-D parabolic equation. For these unknown coefficients, the procedure for finding them is necessary to re-create the numerically balanced inverse problem in the one-dimensional time-dependent thermal equation. $u_t = k(x,t)u_{xx} + f(x,t)$, [14]. In [15], a new formulation for the numerical solution of the inverse problem of the simultaneous determination of the time-dependent right-hand side and the fundamental coefficient in the parabolic equation. In a similar method, the researchers tackled the problem of finding the unknown time- and space-dependent parameters in the right part of a 2-D heat equation [16]. In [17], the authors address the inverse source problem for parabolic equations; they introduced and developed a new approach and implementation to solve the inverse problem for the nonlinear parameter. As well as finding the time-dependent diffusion coefficient of a one-dimensional inverse problem in the parabolic equation $\frac{\partial w}{\partial \tau}(x,\tau) = k(\tau)\frac{\partial^2 w}{\partial^2 x}(x,\tau)$, was also investigated by employing the Crank-Nicholson technique [18]. In the same context, a numerical investigation was carried out using the approach previously mentioned in [18] in the heat equation to identify the time-dependent reaction coefficients through Stefan boundary conditions and thermal moments available from the supplementary data [19]. In these works [20, 21], a proposed method for calculating the parabolic thermal coefficients is followed in two inverse problems with a non-local boundary condition. In addition, the authors in [22, 23] considered concurrently determining two time-dependent parameters for a one-dimensional nonlinear IP. Furthermore, the numerical determination of the time-dependent heat conduction coefficients was studied by applying the Dirichlet boundary condition and the non-local heat flux as over-specification and initial boundary conditions in the two-dimensional heat equation. $u_t = a_1(t)u_{xx} + a_2(t)u_{yy} + f(x,y,t)$, [24]. Also, by relying on Dirichlet boundary conditions and an over-specification condition of the integral kind for the 1-D parabolic equation, the time-dependent functions $b_1 = b_1(t), b_2 = b_2(t)$ were specified in [25], as well as finding the functions $u(x,\tau), s(\tau), a(\tau), b(x,\tau)$ in [26]. In [27], with the second-order 1-D parabolic equation, the non-local integration condition, and additional conditions, the reconstruction of the unknown time-dependent coefficients associated with the Neumann boundary condition is examined. Finding the time-dependent coefficient of a two-dimensional second-order parabolic equation was considered under the mixed homogeneous boundary conditions in [28, 29]. Meanwhile, an inverse problem for a 2-D parabolic equation

for reconstructing a time-dependent coefficient subject to non-local conditions has been dealt with in [30]. A non-local linear combination of heat flux overdetermination is employed to reconstruct the conductivity and temperature in the two-dimensional heat equation [31]. Huntul discussed in [32] the restoration of time-wise heat conduction coefficient, the free boundary of heat flux, and non-local integrated control as over-specification conditions in a two-dimensional parabolic equation. For the first time numerically and under overdetermination conditions, the inverse problem of recovering both the temperature $u(x,y,t)$ as well as the time-dependent heat source $f_{i,j}(t), i,j=\overline{0,1}$ a two-dimensional parabolic equation subject to a Neumann boundary condition was examined in [33]. This paper aims to find the numerical reconstruction for the time-wise coefficient in the total variable coefficient parabolic heat equation with integral mass/ heat specification condition as the overdetermination condition. The structure of this work is described below:
The mathematical setting of a one-dimensional parabolic heat equation in Section 2 was introduced. In Section 3, the numerical solution of the direct problem (DP) $(1)-(3)$ is introduced. In Section 4, the regularized minimization problem is presented, which is treated with the MATLAB implementation of *lsqnonlin* with the inverse problem (IP) solving technique $(1)-(4)$ .In addition, the numerical results are discussed in Section 5. In Section 6, the conclusions were provided.

2. Mathematical Statement of (IP)

Under the region $\Omega_T=\{(x,t): 0<x<h, 0<t<T\}$, were concerned with the IP of identifying the anonymous timewise coefficient $c(t)$ That satisfies a one-dimensional parabolic equation alongside the unknown temperature $u(x,t)$.

$$u_t=a(x,t)u_{xx}+b(x,t)u_x+c(t)u+f(x,t)\,,\quad (x,t)\in\Omega_T \tag{1}$$

according to the initial condition

$$u(x,0)=\omega(x),\qquad x\in[0,h]\,, \tag{2}$$

associated to the non-homogenous Dirichlet boundary conditions;

$$u(0,t)=\mathcal{M}_3(t)\,,\ u(h,t)=\mathcal{M}_4(t),\ t\in[0,T], \tag{3}$$

Afterward, with an over-specified condition of the integral type.

$$\int_0^h u(x,t)dx=\mathcal{M}_5(t)\,,\quad t\in[0,T]. \tag{4}$$

The compatibility conditions are fulfilled by the functions $\omega(x), \mathcal{M}_3(t), \mathcal{M}_4(t)$ and $\mathcal{M}_5(t)$ that are provided. The unique solvability of the investigated problem has been proved in [34]. However, no computational solution was performed on this, so this is the major purpose of the present work.

Definition 2.1: [34] Assume the pair $(c(t),u(x,t))\in C[0,T]\times C^{2,1}(\Omega_T)$ be a solution to the IP $(1)-(4)$, when equation (1) and conditions $(2)-(4)$ are met.
The IP $(1)-(4)$ have an existence and unique solvability illustrated in [34] and are as below:

Theorem 2.2: [34] Assume the following conditions are satisfied:

$B_1)$ $\mathcal{M}_j\in C^1[0,T], j=\overline{3,5}, \omega\in C^2[0,h], a\in C^{1,0}(\overline{\Omega}_T), b,f\in H^{\alpha,0}(\overline{\Omega}_T)$;

$B_2)$ $a(x,t)>0, (x,t)\in\overline{\Omega}_T, \mathcal{M}_5(t)\neq 0, t\in[0,T]$;

$B_3)$ $\omega(0)=\mathcal{M}_3(0),\quad \omega(h)=\mathcal{M}_4(0), \int_0^h \omega(x)=\mathcal{M}_5\,(0),$

$\mathcal{M}_3'(0)=a(0,0)\omega''(0)+b(0,0)\omega'(0)+c(0)\omega(0)+f(0,0),$

$\mathcal{M}_4'(0)=a(h,0)\omega''(h)+b(h,0)\omega'(h)+c(0)\omega(h)+f(h,0),$

where, $c(0) = \frac{1}{\mathcal{M}_5(0)}[\mathcal{M}_5'(0) - \int_0^h (a(x,0)\omega''(x) + b(x,0)\omega'(x) + f(x,0))dx]$ (5)

Afterwards, it can be identified by the number $0 < t_0 \le T$, which is determined by the initial data, that the solution to the IP $(1)-(4)$ exists for $(x,t) \in \overline{\Omega}_{t_0}$.

Theorem 2.3: [34] Suppose the below conditions are fulfilled.
$a(x,t) > 0, (x,0) \in \overline{\Omega}_T, \mathcal{M}_5(t) \neq 0, t \in [0.T]$, and $a, b \in H^{\alpha,0}(\overline{\Omega_T})$. The solution of the (IP) $(1)-(4)$ is unique.

3. Numerical technique for the direct problem $(1)-(3)$ based on FDM.

Dealing with the DP boundary value problem represented by Equations $(1)-(3)$ in this context. The functions $a(x,t), b(x,t), c(t), \omega(x), \mathcal{M}_3(t), \mathcal{M}_4(t)$ and $f(x,t)$ are given, additionally, required to find the solution $u(x,t)$. Also, via employing the Crank-Nicholson FDM, which is second-order accurate in space and time, the problem is solved as follows: a description of the discrete form of the DP $(1)-(3)$. Slice the domain Ω_T into two subintervals with positive numbers, M, N with equal step lengths Δx and Δt , at $\Delta x = \frac{h}{M}$ and $\Delta t = \frac{T}{N}$, respectively. Indicating $u_{i,j} := u(x_i, t_j)$,$a(x_i, t_j) := a_{i,j}$, $b(x_i, t_j) := b_{i,j}$, $c(t_j) := c_j$, and $f(x_i, t_j) := f_{i,j}$ where $x_i = i\Delta x$,$t_j = j\Delta t$, at each specific node (i,j) for $i = \overline{0, M}$,$j = \overline{0, N}$.

To implement the Crank-Nicholson procedure, assume the right-hand side of the heat Equation (1) as, $G(x,t,u,u_x,u_{xx}) = a(x,t)u_{xx} + b(x,t)u_x + c(t)u + f(x,t)$

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{1}{2}\left(G_{i,j+1} + G_{i,j}\right), \tag{6}$$

where,

$$\left(\frac{(-\Delta t)a_{i,j+1}}{2(\Delta x)^2} - \frac{(\Delta t)b_{i,j+1}}{4\Delta x}\right)u_{i+1,j+1} + \left(1 + \frac{(\Delta t)a_{i,j+1}}{(\Delta x)^2} - \frac{(\Delta t)c_{j+1}}{2}\right)u_{i,j+1} + \left(-\frac{(\Delta t)a_{i,j}}{2(\Delta x)^2} + \frac{(\Delta t)b_{i,j}}{4\Delta x}\right)u_{i-1,j+1}$$
$$= \left(\frac{(\Delta t)a_{i,j}}{2(\Delta x)^2} + \frac{(\Delta t)b_{i,j}}{4\Delta x}\right)u_{i+1,j} + \left(1 - \frac{(\Delta t)a_{i,j}}{(\Delta x)^2} + \frac{\Delta t c_j}{2}\right)u_{i,j} + \left(\frac{(\Delta t)a_{i,j}}{2(\Delta x)^2} - \frac{(\Delta t)b_{i,j}}{4\Delta x}\right)u_{i-1,j} + \frac{\Delta t}{2}\left(f_{i,j} + f_{i,j+1}\right). \tag{7}$$

Concerning the value of $i = \overline{1,(M-1)}$ and $j = \overline{0,N}$, let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ as

$$\tilde{A}_{i,j} = \left(\frac{\Delta t a_{i,j}}{2(\Delta x)^2} + \frac{\Delta t b_{i,j}}{4\Delta x}\right), \tilde{B}_{i,j} = \left(\frac{-\Delta t a_{i,j}}{(\Delta x)^2} + \frac{\Delta t c_j}{2}\right), \tilde{C}_{i,j} = \left(\frac{\Delta t a_{i,j}}{2(\Delta x)^2} - \frac{\Delta t b_{i,j}}{4\Delta x}\right). \tag{8}$$

By substituting Equation (8) in Equation (7) , obtaining

$$-\tilde{A}_{i,j+1}u_{i+1,j+1} + [1 - \tilde{B}_{i,j+1}]u_{i,j+1} - \tilde{C}_{i,j+1}u_{i-1,j+1} = \tilde{A}_{i,j}u_{i+1,j} + [1 + \tilde{B}_{i,j}]u_{i,j} + \tilde{C}_{i,j}u_{i-1,j} + \frac{\Delta t}{2}\left(f_{i,j} + f_{i,j+1}\right). \tag{9}$$

In Equations (2) and (3) , expressing the initial and boundary conditions as below.

$$u(x_i, 0) = \omega(x_i)\ , i = \overline{0, M}\ , \quad u(0, t_j) = \mathcal{M}_3(t_j), u(M, t_j) = \mathcal{M}_4(t_j), \quad j = \overline{0, N}. \tag{10}$$

In the Equation (9) typically, the values are known on the right-hand side, while those are unknown on the left. After that, into M quad intervals, the interval X was partitioned. Then, for each, $= 1, \dots \dots \dots, M-1$, $j = 0$ at the initial time, doing the substitution in the equation (9). Now, substituting the first part of the parabolic thermal equation with the temperature function represented by $u_{i,j}$ and then substitute the second part with the temperature function represented by $u_{i,j+1}$ into the right-hand side by relying on the given Dirichlet boundary conditions $u_{0,1} = ʍ_3(t_0), u_{M,1} = ʍ_4(t_0)$, calculating $u_{0,j+1}, u_{0,j}, u_{M,j}$ and $u_{M,j+1}$. The equation (9) yields a linear equation system $(M-1) \times (M-1)$ for the unknown values. As a linear equation system, the above difference equation can be re-written as form:

$$\boldsymbol{W} u_{j+1} = \boldsymbol{S} u_j + F \,, \tag{11}$$

For which $\boldsymbol{W}$ and $\boldsymbol{S}$ are $(M-1) \times (M-1)$ and $u_{j+1} = (u_{1,j+1}\, u_{2,j+1}, \dots, u_{M-1,j+1})^{tr}$ and $u_j = (u_{1,j}, u_{2,j}, \dots, u_{M-1,j})^{tr}$, are stated as below:

$$\boldsymbol{W} = \begin{bmatrix} 1-\tilde{B}_{1,j+1} & -\tilde{A}_{1,j+1} & 0 & 0 & 0 \\ -\tilde{C}_{2,J+1} & 1-\tilde{B}_{2,j+1} & -\tilde{A}_{2,j+1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -\tilde{C}_{M-2,j+1} & 1-\tilde{B}_{M-2,j+1} & -\tilde{A}_{M-2,j} \\ 0 & 0 & 0 & -\tilde{C}_{M-1,j+1} & 1-\tilde{B}_{M-1,j+1} \end{bmatrix}_{(M-1)\times(M-1)}$$

$$\boldsymbol{S} = \begin{bmatrix} 1+\tilde{B}_{1,j} & \tilde{A}_{1,j} & 0 & 0 & 0 \\ \tilde{C}_{2,j} & 1+\tilde{B}_{2,j} & \tilde{A}_{2,j} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \tilde{C}_{M-2,j} & 1+\tilde{B}_{M-2,j} & \tilde{A}_{M-2,j} \\ 0 & 0 & 0 & \tilde{C}_{M-1,j} & 1+\tilde{B}_{M-1,j} \end{bmatrix}_{(M-1)\times(M-1)}$$

$$F = \begin{bmatrix} \tilde{C}_{1,j+1} u_{0,j+1} + \tilde{C}_{1,j} u_{0,j} + \frac{\Delta t}{2}(f_{1,j} + f_{1,j+1}) \\ \frac{\Delta t}{2}(f_{2,j} + f_{2,j+1}) \\ 0 \\ \vdots \\ 0 \\ \frac{\Delta t}{2}(f_{M-2,j} + f_{M-2,j+1}) \\ \tilde{A}_{M-1,j} u_{M,j} + \tilde{A}_{M-1,j+1} u_{M,j+1} + \frac{\Delta t}{2}(f_{M-1,j} + f_{M-1,j+1}) \end{bmatrix}_{(M-1)\times 1}$$

3.1 Example of a direct problem (whenever $c(t)$ is known).

Suppose having a DP with the entered data, such as when the unknown parameters are available, as shown below, and for simplicity, consider $h = T = 1$.
$a(x,t) = 1,\ b(x,t) = 1 + t^2,\ c(t) = 1 + t,\ \omega(x) = (e^{-x} + x^2),\ ʍ_3(t) = e^t,$
$ʍ_4(t) = (e^{-1} + 1)e^t$,

$f(x,t) = (e^{-x} + x^2)e^t - (e^{-x} + 2)e^t - (1 + t^2)(-e^{-x} + 2x)e^t - (1 + t)(e^{-x} + x^2)e^t$
where the desired answer is

$$u(x,t) = (e^{-x} + x^2)e^t \tag{12}$$

Accurate and steady results for the temperature $u(x,t)$ when the mesh size was determined at $M = N = 40$, Figure 1; according to this figure, it can be concluded that a stable and accurate solution was attained at this appropriate grid size, and the graph indicates the value of the absolute error, which has a value of 10^{-5}. Moreover, it illustrates the excellent result obtained. In addition, Figure 2 presents the computational details of the required outcome $ʍ_5(t)$ for the selected grid points within the computed range. This Figure provides more precise results.

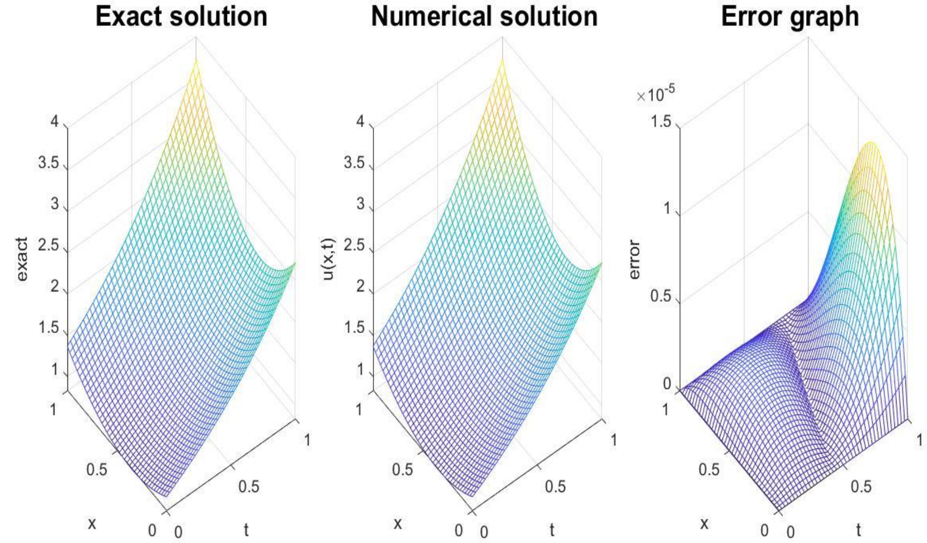

Figure 1: The exact and numerical results temperature $u(x,t)$ for the direct problem was calculated at mesh size $M = N = 40$ and error graph.

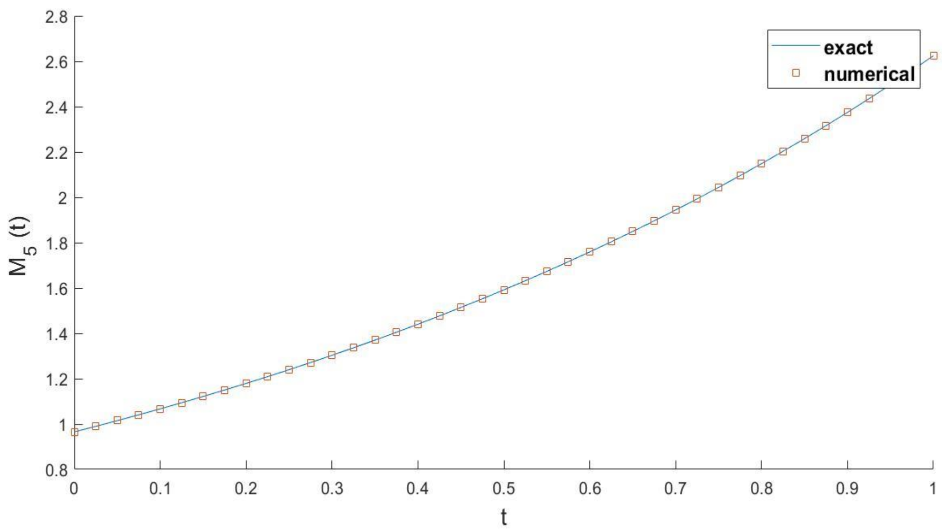

Figure 2: Approximate and analytical values of $M_5(t)$ At the time, the node was calculated as a single size.

Table 1: The approximate values for $M_5(t)$ at different time node evaluated at different mesh sizes.

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
M=N=10	1	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000	1.7000	1.8000	1.9000	000
M=N=20	1	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000	1.7000	1.8000	1.9000	000
M=N=40	1	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000	1.7000	1.8000	1.9000	000
M=N=80	1	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000	1.7000	1.8000	1.9000	000
M=N=100	1	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000	1.7000	1.8000	1.9000	000
Exact	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2

4. A computational technique for IP $(1)-(4)$

Retrieve the unknown time-dependent coefficient $c(t)$ with temperature $u(x,t)$ for a one-dimensional parabolic equation to achieve numerically stable and accurate results, which is our purpose for the inverse problem $(1)-(4)$ presented in Section 2. At the initial time, from the entered data, the unknown coefficient can be retrieved using the Equation (5). To solve the inverse problem iteratively, these values will be implemented as a fixed initial guess. As a nonlinear minimization problem, the inverse problem was reformulated to solve this problem. Seeking to reduce the discrepancy between the measured data and the numerically calculated solution. The usual Tikhonov regularization approach was modified to achieve a stable and smooth solution. From the overdetermination condition (4), the functional error can be enforced as below:

$$\Theta(c) = \left\| \int_0^h u(x,t)dx - \mathcal{M}_5(t) \right\|^2 + \beta \|c(t)\|^2 , \tag{13}$$

It can also be described approximately as follows:

$$\Theta(\underline{c}) = \sum_{j=1}^{N} \left[\int_0^h u(x,t_j)dx - \mathcal{M}_5(t_j) \right]^2 + \beta \sum_{j=1}^{N} c_j^2 , \tag{14}$$

It is possible to determine the regularization parameter for β by relying on the appropriate selection technique, such that $\beta \geq 0$. Therefore, to implement an optimization in the accuracy of the solution, it is necessary to carefully select the value of the regularization parameter for β. In addition, by applying the *lsqnonlin* routine via the MATLAB toolbox with the Trust-Region-Reflective algorithm to find the minimizer of the nonlinear Tikhonov regularization functional (13), check [35] for further details. By embarking on a specific initial guess, this routine seeks to find the minimum sum of squares by making a specific initial guess. Dealing with noisy as well as accurate data (4), the IP $(1)-(4)$ was solved. As follows, random errors are introduced to simulate factual data.

$$\mathcal{M}_5^{\epsilon}(t_j) = \mathcal{M}_5(t_j) + \epsilon_{1,j}, \qquad j = \overline{1,N}, \tag{15}$$

as that ϵ indicates a vector that follows a random Gaussian normal distribution, with a mean for zero and standard deviation of, which is determined through:

$$\sigma = p \times \max_{t\in[0,T]} |\ \mathcal{M}_5(t)| , \tag{16}$$

Since p represents the noise proportion. Our random variables $\underline{\epsilon} = (\epsilon_j)$, $j = \overline{0,N}$ were constructed utilizing the MATLAB bulletin function "*normrnd*" of the form:

$$\underline{\epsilon} = normrnd(0, \sigma, N). \tag{17}$$

5. Computational outcomes and discussion

A numerical experiment example is presented in this section to interpret the accuracy and stability of numerical methods by relying on the Crank-Nicholson procedure. Moreover, to employ Equations (15), (16) and (17) to mimic the actual conditions of measurement errors, introducing noise to the input data (4), reducing the objective function, Θ described in section 4. In addition, root mean square error (RMSE) was applied, which can be characterized as

$$\text{RMSE}(c) = \left[\frac{T}{N} \sum_{j=1}^{N} \left(c^{numerical}(t_j) - c^{exact}(t_j) \right)^2 \right]^{\frac{1}{2}} \tag{18}$$

to validate the accuracy of the computational solution. As follows an analytical example to illustrate the accuracy and stability of numerical solutions in Equations (21) and (22) for the unknown values $u(x,t)$ and $c(t)$ successively was discussed, and for easiness, taking $(h=1)$ and $(T=1)$ in this example and specify the mesh size $(M=N=80)$.

5.1 Test example (for IP)

first, considering the IP represented by Equations $(1)-(4)$ with the unknown timewise coefficient $c(t)$, alongside the entered data expressed as the following:

$a(x,t) = 1 + t\,,\; b(x,t) = -1 - 2t\,,\; \omega(x) = (1-3x)^2,\; \mathcal{M}_3(t) = e^t\,, \mathcal{M}_4(t) = 4e^t\,,$

$$f(x,t) = (1-3x)^2 e^t - 18(1+t)e^t + (1+2t)(-6+18x)e^t, \tag{19}$$

$$\mathcal{M}_5(t) = e^t, \tag{20}$$

The solution has ensured that local existence and uniqueness, subject to Theorems 1 and 2 conditions, must be observed clearly. So, the accurate answer to the problem is

$$u(x,t) = (1-3x)^2 e^t, (x,t) \in \Omega_T \tag{21}$$

$$c(t) = t^2, \quad t \in [0,T] \tag{22}$$

After that, making $M = N = 80$ and verify to reconstruct the coefficient time dependent $c(t)$ where no noise is included in the over-specification (4), i.e., $p = 0$, and without enforcing any regularization in (16). Employing the Isqnonlin subroutine to minimize Θ in (14) with data (19) and (20) investigating the IP $(1) - (4)$. The numerical results for $c(t)$ are presented in Figure $3a$, where obtaining RMSE$(c) = 0.0152$. As a function of the number of iterations, the objective function Θ is interpreted; in Figure $3b$, it is shown that the process requires 9 iterations to obtain a very low variance of about $O(10^{-9})$ while reaching a monotonically decreasing convergence.

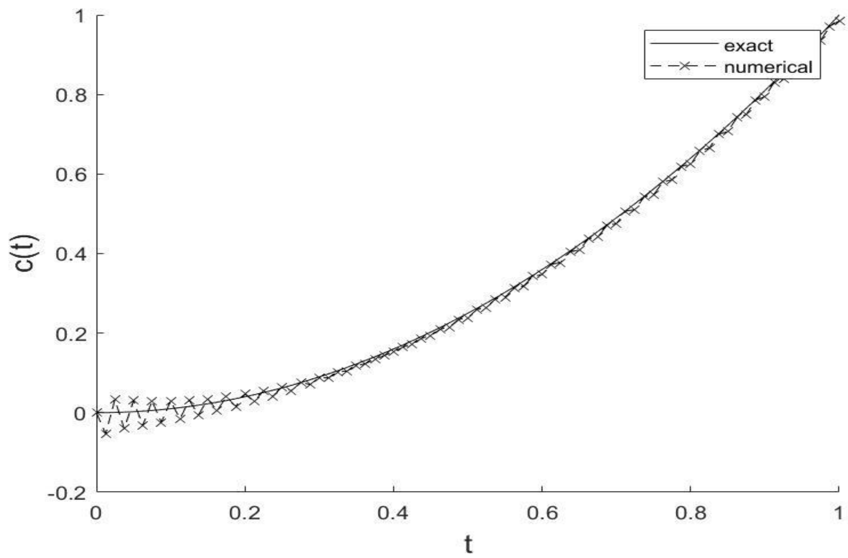

Figure 3: (a). Data for the IP (1)-(4) calculated for a mesh size $M = N = 80$, including both exact and approximate, for c(t).

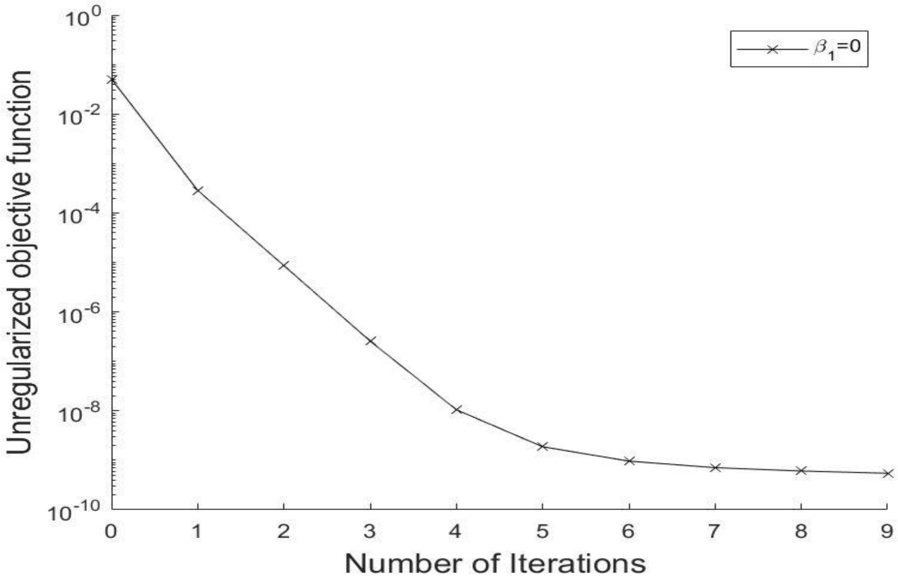

Figure 3: (b). The objective function (14), in the noise-free and regularization-free case with mesh size $M = N = 80$

Also, here, to the over-specification condition $\mu_5(t)$ 1% noise was added, i.e., $(p = 1\%)$as in (15), to check for stability. At (16), also performing the search for larger amounts of noise. The results obtained were less accurate and are therefore not presented. Without including regularization, Figure $4b$ depicts the objective function (14) when it decreases to the value. (10^{-9}) Rapidly at iteration 130. According to Figure $4a$, oscillating and unstable results were produced, indicating that the inverse problem is primarily inaccurate and unstable when it does not contain any regularization, that is, $\beta = 0$.With RMSE $(c) = 41.9859$.

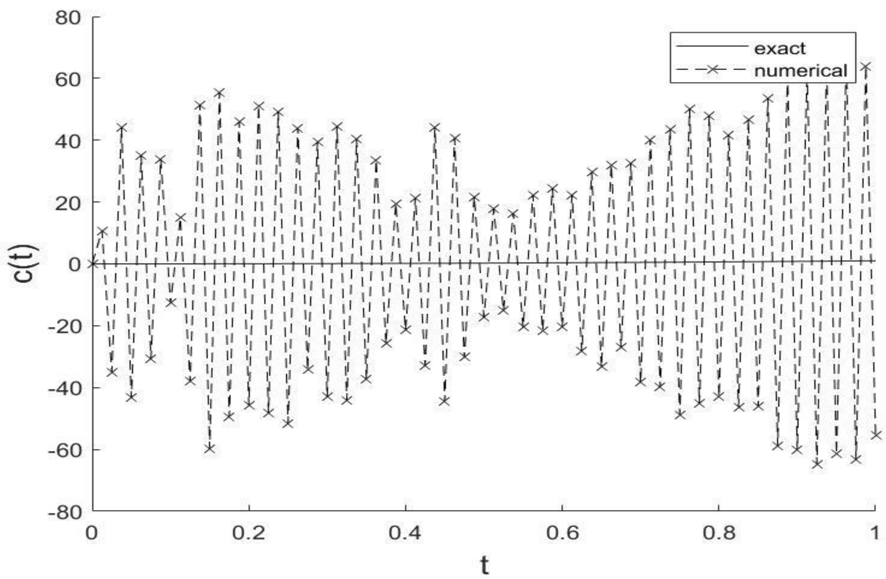

Figure 4: (a) Exact solution and approximate solution for $p = 1\%$ noise and absence of regularization

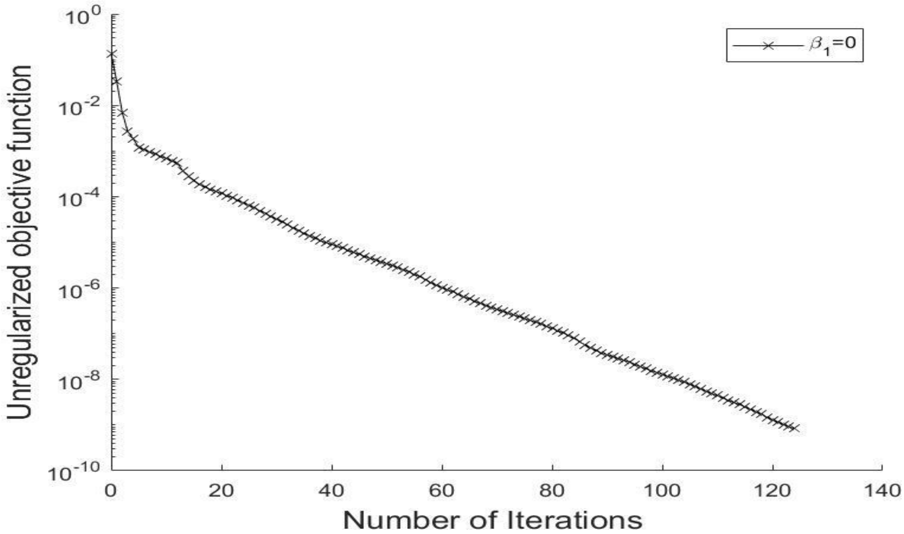

Figure 4: (b) at $p = 1\%$ noise and does not contain regularization, the unregularized objective function 14.

Hence, regularization needs to be implemented to find accurate and stable solutions. Noise with $p = 1\%$ and set β with the parameters = $\{10^{-i}; i = 1, \ldots, 7\}$ was introduced to emulate the noisy data as it is entered. Employing equation (14) for $\kappa_5(t)$ Figure 6,7 depicts the regularized function Θ and additionally displays the coefficient time-dependent $c(t)$. By looking at this figure, when $\beta \in \{10^{-3}, 10^{-2}, 10^{-1}\}$, the results obtained are well-stabilised and acceptable. However, when $\beta \in \{10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}\}$, the values are highly oscillatory and unstable. Table 1 presents the numerical results and details the number of iterations and objective function values, the RMSE values for the unknown time-dependent coefficient $c(t)$, and the various regularisation values. It is evident from the table that the best RMSE result was obtained at 0.3133. Figure 5 indicates that the steady and rapid stabilization at the value of $0(10^{-3})$.

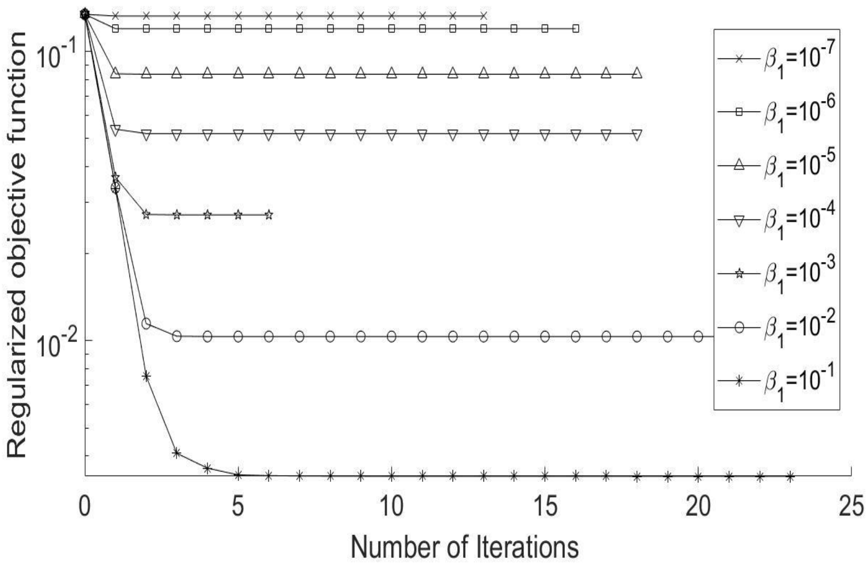

Figure 5: The regularization objective function(14), and a regularization $\beta \in \{10^{-i}; i = 1, \ldots, 7\}$, with input data containing $p = 1\%$ noise

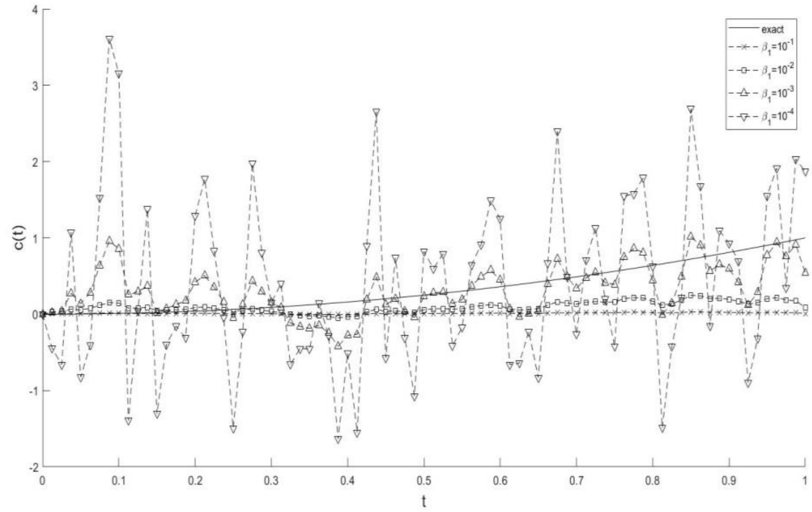

Figure 6: The analytical and the numerical reconstructions for $c(\tau)$, with $p = 1\%$ noise involved in the input data, with a regularization parameter $\beta \in \{10^{-i}; i = 1, \ldots, 4\}$.

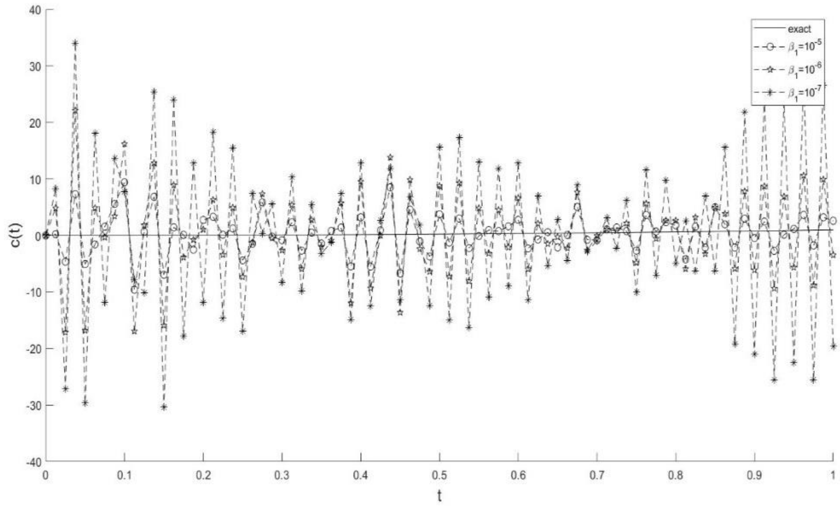

Figure 7: The analytical and the numerical reconstructions for $c(\tau)$, with $p = 1\%$ noise involved in the input data, with a regularization parameter $\beta \in \{10^{-i}; i = 5,6,7\}$.

Table 2: Nnumerically data from some expressing different regularization parameters $\beta \in \{10^{-i}; i = 1, \ldots, 7\}$ with involves noise $= 1\%$.

β	10^{-7}	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
No. of iterations	23	24	6	18	18	16	13
bjective functional 2) at final iteration	0.0034	0.0103	0.0272	0.0520	0.0834	0.1198	0.1325
RMSE(c)	14.9790	7.6310	3.4434	1.1194	0.3133	0.3525	0.4408

6. Conclusions

In a one-dimensional parabolic equation under over-specification conditions, the IP of recovering the time-dependent coefficient with temperature $u(x,t)$ is considered. By adopting the Crank-Nicholson approach, the direct problem was handled. By executing the Isqnonlin routine in MATLAB, the IP was inspected and converted into a least squares optimization problem. Cases of numerical results exist with different noise ratios and with and without regularization. In the first case, consider when no noise is included $(p = 0\%)$ and regularization $(\beta = 0)$. The results are stable and accurate, requiring fewer iterations to reach results quickly. On the contrary, in the second case, when noise is introduced $(p = 1\%)$ with no regularization $(\beta = 0)$, achieving highly fluctuating results for the reconstructed coefficient $c(t)$, which are unstable and require more iterations to reach the results. In the third case, when including noise $(p = 1\%)$ and setting different values of regularization, observing that the results are stable and steady when the values of regularization are $\{10^{-3}, 10^{-2}, 10^{-1}\}$ While obtaining highly oscillatory and unstable results when the values of regularization are $\{10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}\}$ for the reconstructed coefficient on time $c(t)$.

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