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On Analytical Solutions of Operator Equation Constructed by The Adjointable Operator

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Abstract

Recently, Operator Equation Theory (OET) has a leading demonstrated potentiality applicable in numerous scientific ranges of engineering, physical and mathematical. In a Hilbert C^* -module, OET has enhanced by expanding upon extensive research. In this study, for the general situation of adjointable operators, the solvability of the operator equation $\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} = \Omega$, where X and Y are unknown operators, are investigated based on Moore-Penrose inverse. Necessary and sufficient conditions for founding a solution to this equation are proposed. Moreover, by utilizing matrix approaches, four general expressions for the solutions are derived depending on the states of the operators \mathcal{P} and Φ involved in the equation.

Keywords: Operator, Hilbert C^* -module, Operator equation, Invertibility, Adjoint operator, Self-adjoint, Moore-Penrose invers.

حول الحلول التحليلية لمعادلة المؤثر التي تم انشاؤها بواسطة المؤثر القابل للترافق

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الخلاصة:

حديثاً اثبتت نظرية معادلة المؤثر (OET) إمكانية تطبيقها في العديد من المجالات العلمية منها الهندسية والفيزيائية والرياضية. في وحدة هيلبرت- C^* (Hilbert C^* -module)، تم تعزيز نظرية معادلة المؤثر من خلال التوسع في البحث المكثف. في هذه الدراسة، بالنسبة للوضع العام للمؤثرات المترافقة، يتم التحقيق في قابلية حل معادلة المؤثر $\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} = \Omega$ بناءً على معكوس مور-بينروز (Moore-Penrose invers)، حيث X, Y مؤثرين مجهولين. وسوف يتم اقتراح الشروط الضرورية والكافية لوجود حل لهذه المعادلة، علاوة على ذلك وباستخدام تقنية المصفوفة يتم اشتقاق أربع تعبيرات عامة للحلول اعتماداً على حالات المؤثرين \mathcal{P}, Φ المتضمنين في المعادلة.

1. Introduction

The realm of Matrix Equations has intrinsic role in various domains of engineering and mathematics. In system theory, the matrix equation formulated by $\mathcal{P}XD + FX^T\Phi = C$ has been vastly utilized, for instance, to eigen structure assignment [1, 2]. Following, the Φ revival

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of Operator Equations on Hilbert space or Hilbert C^* -module, which is the extended formula of M-Eqs. Indeed, the Hilbert C^* -module is a nature generalized formula of C^* -algebras and Hilbert space, was explored by Kaplansky [3] in 1953. The Operator Equations has contributed to algebraic theory, particularly in non-abelian geometry and quantum groups and KK-theory, see [4-9]. There are a variety of previous studies that paved the way or contributed to presenting this study. Interestingly, the mathematicians Baksalary and Kala [10] in 1979 provided the requisite and sufficient stipulations for the existence of a solution to equation $PX - Y = \Omega$ and constructed the general formula of the solution. In 1998, the researcher Xu, Wei, and Zheng [11] presented the requisite and sufficient stipulations for the existence and uniqueness of the solution to equation $PX\Phi + CYD = \Omega$. They investigated the general formula of the solution. Then, Wang, Zhang and Yu [12] in 2008 expanded attempts made investigating and deduced the requisite and sufficient stipulations for the existence of real and imaginary solutions to aforementioned equation. In 2009, Dehghan and Hajarian [13] deduced the reflexive solutions to the Matrix Equation $PX\Phi + CYD = \Omega$ by providing the requisite and sufficient stipulations for the existence of these solutions. Subsequently, Karizaki and Djordjevic [14] in 2016 provided solutions to the Operator Equation $PX\Phi - SYQ = \Omega$, on a Hilbert C^* -module, $\text{ran}(\mathcal{P}) = \text{ran}(S)$ and $\text{ran}(\Phi^*) = \text{ran}(Q^*)$ are closed. For more information, see [15-26]. In this sequel, F and D are Hilbert modules over the same C^* -algebra. Denote by $B(F, D)$ the set including the adjointable operators defined on F to D . For case $F = D$, $B(F, D)$ coincides with $B(F)$. $\mathcal{P}, \Phi, \Omega$ represents known operators. For $\mathcal{P} \in B(F, D)$, let $\text{Ker}(\mathcal{P})$ and $R(\mathcal{P})$ represent the zero-space and range, respectively.

The following several principles are required in this study.

Definition 1.1: [27, 28] Let F and D be vector spaces over the same field then the operator $\mathcal{P}: F \rightarrow D$ called invertibility if there exists an operator $\Phi: D \rightarrow F$ where $\mathcal{P}\Phi = \Phi\mathcal{P} = I$, where I is an identity operator.

Definition 1.2: [29, 31] Let F and D be Hilbert spaces and $\mathcal{P}: F \rightarrow D$ be a linear operator then the operator $\mathcal{P}^*: D \rightarrow F$ is called adjoint of the operator \mathcal{P} if $\langle \mathcal{P}x, y \rangle = \langle x, \mathcal{P}^*y \rangle$ for each $x \in F, y \in D$.

Definition 1.3: [32] An operator $\mathcal{P}: F \rightarrow D$ where F and D are Hilbert spaces called self-adjoint if $\mathcal{P} = \mathcal{P}^*$.

Definition 1.4: [33, 34] Let $\mathcal{P} \in B(F, D)$, the range of \mathcal{P} , is denoted by $R(\mathcal{P})$ such that $R(\mathcal{P}) = \{\mathcal{P}x: x \in F\}$.

Definition 1.5: [35] Let $\mathcal{P} \in B(F, D)$, the kernel of \mathcal{P} , is denoted by $\text{ker}(\mathcal{P})$ such that $\text{ker}(\mathcal{P}) = \{x \in F: \mathcal{P}x = 0\}$.

Definition 1.6: [36] Let $\mathcal{P} \in B(F, D)$, the Moore-Penrose invers of \mathcal{P} indicated by \mathcal{P}^+ such that \mathcal{P}^+ is unique in $B(F, D)$ and fulfills:

$$\mathcal{P}\mathcal{P}^+\mathcal{P} = \mathcal{P}, \quad \mathcal{P}^+\mathcal{P}\mathcal{P}^+ = \mathcal{P}^+, \quad (\mathcal{P}\mathcal{P}^+)^* = \mathcal{P}\mathcal{P}^+, \quad (\mathcal{P}^+\mathcal{P})^* = \mathcal{P}^+\mathcal{P}.$$

The existence of the Moore-Penrose bounded inverse of a continuous operator between two Hilbert C^* -modules is guaranteed if and only if the operator has a closed range. For more about the properties and applications of Moore-Penrose invers, see [37, 38]. The operator $\mathcal{P} \in B(F, D)$ is called regular if there exists $\Phi \in B(D, F)$ such that $\mathcal{P}\Phi\mathcal{P} = \mathcal{P}$. It is clear that regular operators are almost regular and that regular operator have close range [39].

Theorem 1.7: [40] Let $\Phi \in B(F, D)$ and $\mathcal{P} \in B(Z, D)$ be invertibility operators and $\Omega \in B(D)$, F, D, Z be Hilbert C^* -modules. Then the next assumptions are comparable: (i) There is a solution $X \in B(F, Z)$ to $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$. (ii) Ω is self-adjoint. If (i) or (ii) are available, therefore any solution to $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$. When $X \in B(F, Z)$ represents in the form $X = \frac{1}{2}\mathcal{P}^{-1}\Omega(\Phi^*)^{-1} - \mathcal{P}^{-1}Z(\Phi^*)^{-1}$, where $Z \in B(D)$ satisfies $Z^* = -Z$.

For the case Φ is invertible and \mathcal{P} is regular, the following theorem gives an explicit solution to the equation $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$.

Theorem 1.8: [40] Suppose $\Phi \in B(F, D)$ is an invertibility and $\mathcal{P} \in B(Z, D)$ is regular and $\Omega \in B(D)$, such that F, D, Z are Hilbert C^* -modules. After that, next claims are comparable: (i) There exists $X \in B(F, Z)$ is a solution to the operator equation $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$. (ii) Ω is self-adjoint and $(I - \mathcal{P}\mathcal{P}^+)\Omega(I - \mathcal{P}\mathcal{P}^+) = 0$. If (i) or (ii) are available Thus, the solution to $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$. It is written in order: $X = \frac{1}{2}\mathcal{P}^+\Omega\mathcal{P}\mathcal{P}^+(\Phi^*)^{-1} + \mathcal{P}^+Z\mathcal{P}\mathcal{P}^+(\Phi^*)^{-1} + \mathcal{P}\Omega(1 - \mathcal{P}\mathcal{P}^*)(\Phi^*)^{-1} + (I - \mathcal{P}\mathcal{P}^+)Y(\Phi^*)^{-1}$, where $Z \in B(D)$ achieves $\mathcal{P}^*(Z + Z^*)\mathcal{P} = 0$ and $Y \in B(D, Z)$ is random.

For \mathcal{P} and Φ both close their regions, the next theorem studies the equation $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$.

Theorem 1.9: [41] Let $\Phi, \Omega \in B(F)$, $\mathcal{P} \in B(D, F)$. So that Φ is self-adjoint and both \mathcal{P} and Φ possess regulars. Let F, D, Z be Hilbert C^* -modules, $\Omega\Phi^+\Phi = \Omega$ and $\mathcal{P}^+\Phi^+\Phi = \mathcal{P}^+$. The next claims are therefore comparable: (i) There is $X \in B(F, D)$ a solution to the operator equation $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$. (ii) $\Omega = \Omega^*$ and $(I - \mathcal{P}\mathcal{P}^+)\Omega(I - \mathcal{P}\mathcal{P}^+) = 0$. If (i) or (ii) are available, thus the solution to $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$. It is written in order: $X = \mathcal{P}^+\Omega\Phi^+ - \frac{1}{2}\mathcal{P}^+\Omega\mathcal{P}\mathcal{P}^+\Phi^+ + \mathcal{P}^+Z\mathcal{P}\mathcal{P}^+\Phi^+ + V - \mathcal{P}^+V\mathcal{P}\Phi\Phi^+$, where $Z \in B(F)$ achieves $\mathcal{P}^*(Z + Z^*)\mathcal{P} = 0$, $V \in B(F, D)$ is random.

2. Main results

This section investigates and provides four formulations for general solutions along with their prerequisites and conditions. Regarding the operator equation

$$\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} = \Omega. \quad (1)$$

And it is considered a generalization of the equation that appeared in [41].

For \mathcal{P} and Φ are invertible, the following theorem discusses the state solution of Equation (1).

Theorem 2.1: Let $\mathcal{P} \in B(B(F, D))$ and $\Phi \in B(D, F)$ be invertible and $\Omega \in B(F)$, where F, D be Hilbert C^* -modules, then the operator Equation (1) has a solution $(X, Y) \in B(D) \times B(D)$, in this case any solution to Equation (1) in this situation is shown as follows:

$$X = \frac{1}{2}(\mathcal{P}^*)^{-1}\Omega(\Phi^*)^{-1} - (\mathcal{P}^*)^{-1}K(\Phi^*)^{-1}$$

$$Y = \frac{1}{2}(\Phi)^{-1}\Omega(\mathcal{P})^{-1} + (\Phi)^{-1}K(\mathcal{P})^{-1},$$

where $K \in B(F)$ is arbitrary.

Proof: Suppose $M = \begin{bmatrix} \mathcal{P}^* & 0 \\ 0 & \mathcal{P}^* \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \rightarrow \mathcal{F} \oplus \mathcal{F}$, $N = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \rightarrow \mathcal{F} \oplus \mathcal{F}$, $\hat{X} = \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \rightarrow \mathcal{D} \oplus \mathcal{D}$, $L = \begin{bmatrix} 0 & \Omega \\ \Omega^* & 0 \end{bmatrix} : \mathcal{F} \oplus \mathcal{F} \rightarrow \mathcal{F} \oplus \mathcal{F}$.

It is clear that L is self-adjoint. Therefore, using Theorem 1.7, there is a solution like \hat{X} for equation $M\hat{X}N^* + N\hat{X}^*M^* = L$, from which we will obtain

$$\begin{bmatrix} \mathcal{P}^* & 0 \\ 0 & \mathcal{P}^* \end{bmatrix} \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} \begin{bmatrix} \Phi^* & 0 \\ 0 & \Phi^* \end{bmatrix} + \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} 0 & Y \\ X^* & 0 \end{bmatrix} \begin{bmatrix} \mathcal{P} & 0 \\ 0 & \mathcal{P} \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ \Omega^* & 0 \end{bmatrix} = L,$$

from this, we conclude $\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} = \Omega$. That is (X, Y) represents a solution to this equation. It is clear that M and N are reversible. Referring to Theorem 1.7 and utilizing the data that has emerged, we find that the solution to equation $M\hat{X}N^* + N\hat{X}^*M^* = L$ is as follows

$$\hat{X} = \frac{1}{2}M^{-1}L(N^*)^{-1} - M^{-1}Z(N^*)^{-1}, \quad (2)$$

where $Z \in B(F \oplus F)$ satisfied $Z^* = -Z$. If we take $Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$, and after substituting each value with its equivalent in Equation (2), produces

$$\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & (\mathcal{P}^*)^{-1} \Omega (\Phi^*)^{-1} \\ (\mathcal{P}^*)^{-1} \Omega^* (\Phi^*)^{-1} & 0 \end{bmatrix} - \begin{bmatrix} (\mathcal{P}^*)^{-1} Z_1 (\Phi^*)^{-1} & (\mathcal{P}^*)^{-1} Z_2 (\Phi^*)^{-1} \\ (\mathcal{P}^*)^{-1} Z_3 (\Phi^*)^{-1} & (\mathcal{P}^*)^{-1} Z_4 (\Phi^*)^{-1} \end{bmatrix}.$$

from this data and by utilizing the properties of matrices, leads to

$$X = \frac{1}{2} (\mathcal{P}^*)^{-1} \Omega (\Phi^*)^{-1} - (\mathcal{P}^*)^{-1} Z_2 (\Phi^*)^{-1} \quad (3)$$

$$Y^* = \frac{1}{2} (\mathcal{P}^*)^{-1} \Omega^* (\Phi^*)^{-1} - (\mathcal{P}^*)^{-1} Z_3 (\Phi^*)^{-1} \quad (4)$$

$$(\mathcal{P}^*)^{-1} Z_1 (\Phi^*)^{-1} = 0, (\mathcal{P}^*)^{-1} Z_4 (\Phi^*)^{-1}. \quad (5)$$

We have that both \mathcal{P}^* and Φ^* are invertible, so by utilizing Equation (5), it becomes clear that $Z_1 = 0, Z_4 = 0$. Additionally, we have $Z^* = -Z$, and this relationship provides us with the relationship $Z_3^* = -Z_2$. Therefore, when we assume that $K = Z_2$, both (3) and (4) become as the following form, which represents a solution to Equation (1).

$$X = \frac{1}{2} (\mathcal{P}^*)^{-1} \Omega (\Phi^*)^{-1} - (\mathcal{P}^*)^{-1} K (\Phi^*)^{-1}$$

$$Y = \frac{1}{2} (\Phi)^{-1} \Omega (\mathcal{P})^{-1} + (\Phi)^{-1} K (\mathcal{P})^{-1}.$$

If Φ is a regular and \mathcal{P} is invertible, the following theorem gives another solution of Equation (1).

Theorem 2.2: Let $\mathcal{P} \in B(F, D)$, $\Phi \in B(D, F)$ and $\Omega \in B(F)$, where F, D be Hilbert C^* -modules, \mathcal{P} be invertible and Φ is regular, such that $\Phi \Phi^+ \Omega = \Omega$, then the statements that follow are interchangeable:

- (i) There is a solution $(X, Y) \in B(D) \times B(D)$ to Equation (1),
- (ii) $(I - \Phi \Phi^+) \Omega (I - \Phi \Phi^+) = 0$.

Any solution to Equation (1) takes the following form if (i) or (ii) are fulfilled.

$$X = \frac{1}{2} \mathcal{P}^{*-1} \Phi \Phi^+ \Omega \Phi^{*+} - \mathcal{P}^{*-1} \Phi \Phi^+ K \Phi^{*+} + \mathcal{P}^{*-1} (I - \Phi \Phi^+) \Omega \Phi^{*+} + \mathcal{P}^{*-1} U^* (I - \Phi^+ \Phi).$$

$$Y = \frac{1}{2} \Phi^+ \Omega \Phi \Phi^+ \mathcal{P}^{-1} + \Phi^+ K \Phi \Phi^+ \mathcal{P}^{-1} + \Phi^+ \Omega (I - \Phi \Phi^+) \mathcal{P}^{-1} + (I - \Phi^+ \Phi) W \mathcal{P}^{-1},$$

where $K \in B(F), W, U \in B(F, D)$.

Proof: (i) \Rightarrow (ii) Suppose the Equation (1) has solution $(X, Y) \in B(D) \times B(D)$. Then we have

$$\begin{aligned} (I - \Phi \Phi^+) \Omega (I - \Phi \Phi^+) &= (I - \Phi \Phi^+) (\mathcal{P}^* X \Phi^* + \Phi Y \mathcal{P}) (I - \Phi \Phi^+) \\ &= (I - \Phi \Phi^+) \mathcal{P}^* X \Phi^* (I - \Phi \Phi^+) + (I - \Phi \Phi^+) \Phi Y \mathcal{P} (I - \Phi \Phi^+) = 0. \end{aligned}$$

(ii) \Rightarrow (i), Suppose $(I - \Phi \Phi^+) \Omega (I - \Phi \Phi^+) = 0$ and let $N = \begin{bmatrix} \mathcal{P}^* & 0 \\ 0 & \mathcal{P}^* \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \rightarrow \mathcal{F} \oplus \mathcal{F}, M = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \rightarrow \mathcal{F} \oplus \mathcal{F}, \hat{X} = \begin{bmatrix} 0 & X^* \\ Y & 0 \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \rightarrow \mathcal{D} \oplus \mathcal{D}, L = \begin{bmatrix} 0 & \Omega^* \\ \Omega & 0 \end{bmatrix} : \mathcal{F} \oplus \mathcal{F} \rightarrow \mathcal{F} \oplus \mathcal{F}$. Clear L is self-adjoint also,

$$(I - MM^+) L (I - MM^+) = \begin{bmatrix} 0 & (I - \Phi \Phi^+) \Omega^* (I - \Phi \Phi^+) \\ (I - \Phi \Phi^+) \Omega (I - \Phi \Phi^+) & 0 \end{bmatrix} = 0,$$

from this data and referring to Theorem 1.8, \hat{X} represents a solution to equation $M \hat{X} N^* + N \hat{X}^* M^* = L$, and from this equation, consists

$$M \hat{X} N^* + N \hat{X}^* M^* = \begin{bmatrix} 0 & \Phi X^* \mathcal{P} + \mathcal{P}^* Y^* \Phi^* \\ \Phi Y \mathcal{P} + \mathcal{P}^* X \Phi^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Omega^* \\ \Omega & 0 \end{bmatrix} = L,$$

from this, we conclude $\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} = \Omega$, That is, (X, Y) represents a solution to this equation. Referring to Theorem 1.8 and utilizing the data that has emerged, we find that the solution to equation $M\hat{X}N^* + N\hat{X}^*M^* = L$ is as follows

$$\hat{X} = \frac{1}{2}M^+LMM^+(N^*)^{-1} + M^+ZMM^+(N^*)^{-1} + M^+L(I - MM^+)(N^*)^{-1} + (I - M^+M)S(N^*)^{-1}, \quad (6)$$

where $Z \in B(F \oplus F)$ satisfied $M^*(Z + Z^*)M = 0$ and $S \in B(F \oplus F, D \oplus D)$ is arbitrary. If take $Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$, $S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}$ and substituting each value with its equivalent in Equation (6), leads to

$$\begin{bmatrix} 0 & X^* \\ Y & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \Phi^+\Omega^*\Phi\Phi^+\mathcal{P}^{-1} \\ \Phi^+\Omega\Phi\Phi^+\mathcal{P}^{-1} & 0 \end{bmatrix} + \begin{bmatrix} \Phi^+Z_1\Phi\Phi^+\mathcal{P}^{-1} & \Phi^+Z_2\Phi\Phi^+\mathcal{P}^{-1} \\ \Phi^+Z_3\Phi\Phi^+\mathcal{P}^{-1} & \Phi^+Z_4\Phi\Phi^+\mathcal{P}^{-1} \end{bmatrix} + \begin{bmatrix} 0 & \Phi^+\Omega^*(I - \Phi\Phi^+)\mathcal{P}^{-1} \\ \Phi^+\Omega(I - \Phi\Phi^+)\mathcal{P}^{-1} & 0 \end{bmatrix} + \begin{bmatrix} (I - \Phi^+\Phi)S_1\mathcal{P}^{-1} & (I - \Phi^+\Phi)S_2\mathcal{P}^{-1} \\ (I - \Phi^+\Phi)S_3\mathcal{P}^{-1} & (I - \Phi^+\Phi)S_4\mathcal{P}^{-1} \end{bmatrix},$$

this, relying on the properties of matrices, it bears fruit

$$X^* = \frac{1}{2}\Phi^+\Omega^*\Phi\Phi^+\mathcal{P}^{-1} + \Phi^+Z_2\Phi\Phi^+\mathcal{P}^{-1} + \Phi^+\Omega^*(I - \Phi\Phi^+)\mathcal{P}^{-1} + (I - \Phi^+\Phi)S_2\mathcal{P}^{-1} \quad (7)$$

$$Y = \frac{1}{2}\Phi^+\Omega\Phi\Phi^+\mathcal{P}^{-1} + \Phi^+Z_3\Phi\Phi^+\mathcal{P}^{-1} + \Phi^+\Omega(I - \Phi\Phi^+)\mathcal{P}^{-1} + (I - \Phi^+\Phi)S_3\mathcal{P}^{-1} \quad (8)$$

$$\Phi^+Z_1\Phi\Phi^+\mathcal{P}^{-1} + (I - \Phi^+\Phi)S_1\mathcal{P}^{-1} = 0 \quad (9)$$

$$\Phi^+Z_4\Phi\Phi^+\mathcal{P}^{-1} + (I - \Phi^+\Phi)S_4\mathcal{P}^{-1} = 0, \quad (10)$$

and after multiplying Equations (9) and (10) by \mathcal{P} from the right and Φ from the left, provides

$$\Phi\Phi^+Z_1\Phi\Phi^+ = 0 \text{ and } \Phi\Phi^+Z_4\Phi\Phi^+ = 0,$$

and since we have $M^*(Z + Z^*)M = 0$, this gives us the following data.

$$\Phi^*Z_2^*\Phi = -\Phi^*Z_3\Phi \text{ and } \Phi^*Z_3^*\Phi = -\Phi^*Z_2\Phi, \quad (11)$$

and since $\ker(M^+) = \ker(M^*)$, this means that $M^+(Z + Z^*)M = 0$, and this leads us to the following

$$\Phi^+Z_3\Phi = -\Phi^+Z_2^*\Phi \text{ and } \Phi^+Z_2\Phi = -\Phi^+Z_3^*\Phi. \quad (12)$$

Therefore, taking $Z_3 = K, S_2 = U, S_3 = W$ and using Equations (11) and (12). Equations (7) and (8) will become as follows and represent a solution to the Equation (1)

$$X = \frac{1}{2}\mathcal{P}^{*-1}\Phi\Phi^+\Omega\Phi^{*+} - \mathcal{P}^{*-1}\Phi\Phi^+K\Phi^{*+} + \mathcal{P}^{*-1}(I - \Phi\Phi^+)\Omega\Phi^{*+} + \mathcal{P}^{*-1}U^*(I - \Phi^+\Phi)$$

$$Y = \frac{1}{2}\Phi^+\Omega\Phi\Phi^+\mathcal{P}^{-1} + \Phi^+K\Phi\Phi^+\mathcal{P}^{-1} + \Phi^+\Omega(I - \Phi\Phi^+)\mathcal{P}^{-1} + (I - \Phi^+\Phi)W\mathcal{P}^{-1}.$$

If Φ is invertible and \mathcal{P} is a regular, the following theorem provides solution to Equation (1).

Theorem 2.3: Let $\mathcal{P} \in B(F, D)$, $\Phi \in B(D, F)$ and $\Omega \in B(F)$, where F, D be Hilbert C^* -modules, Φ be invertible and \mathcal{P} be regular, such that $\Omega\mathcal{P}^+\mathcal{P} = \Omega$, then the statements that follow are interchangeable:

- (i) There is a solution $(X, Y) \in B(D) \times B(D)$ to Equation (1),
- (ii) $(I - \mathcal{P}^+\mathcal{P})\Omega(I - \mathcal{P}^+\mathcal{P}) = 0$.

Any solution to Equation (1) takes the following form if (i) or (ii) are fulfilled

$$X = \frac{1}{2}\mathcal{P}^{*+}\Omega\mathcal{P}^+\mathcal{P}\Phi^{*-1} + \mathcal{P}^{*+}K\mathcal{P}^+\mathcal{P}\Phi^{*-1} + \mathcal{P}^{*+}\Omega(I - \mathcal{P}^+\mathcal{P})\Phi^{*-1} + (I - \mathcal{P}\mathcal{P}^+)U\Phi^{*-1}$$

$$Y = \frac{1}{2}\Phi^{-1}\mathcal{P}^+\mathcal{P}\Omega\mathcal{P}^+ - \Phi^{-1}\mathcal{P}^+\mathcal{P}K\mathcal{P}^+ + \Phi^{-1}(I - \mathcal{P}^+\mathcal{P})\Omega\mathcal{P}^+ + \Phi^{-1}W^*(I - \mathcal{P}\mathcal{P}^+),$$

where $K \in B(F), W, U \in B(F, D)$

Proof: (i) \Rightarrow (ii) Suppose the Equation (1) has solution $(X, Y) \in B(D) \times B(D)$. This achieves

$$(I - \mathcal{P}^+\mathcal{P})\Omega(I - \mathcal{P}^+\mathcal{P}) = (I - \mathcal{P}^+\mathcal{P})(\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P})(I - \mathcal{P}^+\mathcal{P})$$

$$= (I - \mathcal{P}^+\mathcal{P})\mathcal{P}^*X\Phi^*(I - \mathcal{P}^+\mathcal{P}) + (I - \mathcal{P}^+\mathcal{P})\Phi Y\mathcal{P}(I - \mathcal{P}^+\mathcal{P}) = 0.$$

(ii) \Rightarrow (i) Suppose $(I - \mathcal{P}^+\mathcal{P})\Omega(I - \mathcal{P}^+\mathcal{P}) = 0$ and let $M = \begin{bmatrix} \mathcal{P}^* & 0 \\ 0 & \mathcal{P}^* \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \rightarrow \mathcal{F} \oplus \mathcal{F}$, $N = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \rightarrow \mathcal{F} \oplus \mathcal{F}$, $\hat{X} = \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \rightarrow \mathcal{D} \oplus \mathcal{D}$, $L = \begin{bmatrix} 0 & \Omega \\ \Omega^* & 0 \end{bmatrix} : \mathcal{F} \oplus \mathcal{F} \rightarrow \mathcal{F} \oplus \mathcal{F}$. Clear L is self-adjoint also,

$$(I - MM^+)L(I - MM^+) = \begin{bmatrix} 0 & (I - \mathcal{P}^+\mathcal{P})\Omega(I - \mathcal{P}^+\mathcal{P}) \\ (I - \mathcal{P}^*\mathcal{P}^{*+})\Omega^*(I - \mathcal{P}^*\mathcal{P}^{*+}) & 0 \end{bmatrix} = 0,$$

from this data and referring to Theorem 1.8, \hat{X} will represent a solution to equation $M\hat{X}N^* + N\hat{X}^*M^* = L$, and from this equation, consists

$$M\hat{X}N^* + N\hat{X}^*M^* = \begin{bmatrix} 0 & \mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} \\ \mathcal{P}^*Y^*\Phi^* + \Phi X^*\mathcal{P} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ \Omega^* & 0 \end{bmatrix} = L,$$

from this, produces $\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} = \Omega$, that is (X, Y) represents a solution to this equation. Referring to Theorem 1.8 and utilizing the data that has emerged, we find that the solution to equation $M\hat{X}N^* + N\hat{X}^*M^* = L$ is as follows

$$\hat{X} = \frac{1}{2}M^+LMM^+(N^*)^{-1} + M^+ZMM^+(N^*)^{-1} + M^+L(I - MM^+)(N^*)^{-1} + (I - MM^+)S(N^*)^{-1}, \quad (13)$$

where $Z \in B(F \oplus F)$ satisfied $M^*(Z + Z^*)M = 0$ and $S \in B(F \oplus F, D \oplus D)$ is arbitrary. If take $Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$ and $S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}$ and substituting each value with its equivalent in Equation (13), this achieves

$$\begin{aligned} \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 0 & \mathcal{P}^{*+}\Omega\mathcal{P}^*\mathcal{P}^{*+}\Phi^{*-1} \\ \mathcal{P}^{*+}\Omega^*\mathcal{P}^*\mathcal{P}^{*+}\Phi^{*-1} & 0 \end{bmatrix} \\ &+ \begin{bmatrix} \mathcal{P}^{*+}Z_1\mathcal{P}^*\mathcal{P}^{*+}\Phi^{*-1} & \mathcal{P}^{*+}Z_2\mathcal{P}^*\mathcal{P}^{*+}\Phi^{*-1} \\ \mathcal{P}^{*+}Z_3\mathcal{P}^*\mathcal{P}^{*+}\Phi^{*-1} & \mathcal{P}^{*+}Z_4\mathcal{P}^*\mathcal{P}^{*+}\Phi^{*-1} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & \mathcal{P}^{*+}\Omega(I - \mathcal{P}^*\mathcal{P}^{*+})\Phi^{*-1} \\ \mathcal{P}^{*+}\Omega^*(I - \mathcal{P}^*\mathcal{P}^{*+})\Phi^{*-1} & 0 \end{bmatrix} \\ &+ \begin{bmatrix} (I - \mathcal{P}^{*+}\mathcal{P}^*)S_1\Phi^{*-1} & (I - \mathcal{P}^{*+}\mathcal{P}^*)S_2\Phi^{*-1} \\ (I - \mathcal{P}^{*+}\mathcal{P}^*)S_3\Phi^{*-1} & (I - \mathcal{P}^{*+}\mathcal{P}^*)S_4\Phi^{*-1} \end{bmatrix}, \end{aligned}$$

this, relying on the properties of matrices, yield

$$X = \frac{1}{2}\mathcal{P}^{*+}\Omega\mathcal{P}^*\mathcal{P}^{*+}\Phi^{*-1} + \mathcal{P}^{*+}Z_2\mathcal{P}^*\mathcal{P}^{*+}\Phi^{*-1} + \mathcal{P}^{*+}\Omega(I - \mathcal{P}^*\mathcal{P}^{*+})\Phi^{*-1} + (I - \mathcal{P}^{*+}\mathcal{P}^*)S_2\Phi^{*-1}. \quad (14)$$

$$Y^* = \frac{1}{2}\mathcal{P}^{*+}\Omega^*\mathcal{P}^*\mathcal{P}^{*+}\Phi^{*-1} + \mathcal{P}^{*+}Z_3\mathcal{P}^*\mathcal{P}^{*+}\Phi^{*-1} + \mathcal{P}^{*+}\Omega^*(I - \mathcal{P}^*\mathcal{P}^{*+})\Phi^{*-1} + (I - \mathcal{P}^{*+}\mathcal{P}^*)S_3\Phi^{*-1}. \quad (15)$$

$$\mathcal{P}^{*+}Z_1\mathcal{P}^*\mathcal{P}^{*+}\Phi^{*-1} + (I - \mathcal{P}^{*+}\mathcal{P}^*)S_1\Phi^{*-1} = 0. \quad (16)$$

$$\mathcal{P}^{*+}Z_4\mathcal{P}^*\mathcal{P}^{*+}\Phi^{*-1} + (I - \mathcal{P}^{*+}\mathcal{P}^*)S_4\Phi^{*-1} = 0, \quad (17)$$

and after multiplying Equations (16) and (17) by Φ^* from the right and \mathcal{P}^* from the left, gives

$$\mathcal{P}^+\mathcal{P}Z_1\mathcal{P}^+\mathcal{P} = 0 \text{ and } \mathcal{P}^+\mathcal{P}Z_4\mathcal{P}^+\mathcal{P} = 0,$$

and since we have $M^*(Z + Z^*)M = 0$, this provides us with the following data

$$\mathcal{P}Z_2\mathcal{P}^* = -\mathcal{P}Z_3^*\mathcal{P}^* \text{ and } \mathcal{P}Z_3\mathcal{P}^* = -\mathcal{P}Z_2^*\mathcal{P}^*, \quad (18)$$

and since $\ker(M^+) = \ker(M^*)$, this means that $M^+(Z + Z^*)M = 0$, and this leads us to the following

$$\mathcal{P}^{*+}Z_3\mathcal{P}^* = -\mathcal{P}^{*+}Z_2^*\mathcal{P}^* \text{ and } \mathcal{P}^{*+}Z_2\mathcal{P}^* = -\mathcal{P}^{*+}Z_3^*\mathcal{P}^*. \quad (19)$$

Therefore, taking $Z_2 = K, S_2 = U, S_3 = W$, using Equations (18) and (19). Equations (14) and (15) will become as follows and represent a solution to the Equation (1)

$$X = \frac{1}{2}\mathcal{P}^{*+}\Omega\mathcal{P}^*\mathcal{P}^{*+}\Phi^{*-1} + \mathcal{P}^{*+}K\mathcal{P}^+\mathcal{P}\Phi^{*-1} + \mathcal{P}^{*+}\Omega(I - \mathcal{P}^+\mathcal{P})\Phi^{*-1} + (I - \mathcal{P}\mathcal{P}^+)U\Phi^{*-1}.$$

$$Y = \frac{1}{2}\Phi^{-1}\mathcal{P}^+\mathcal{P}\Omega\mathcal{P}^+ - \Phi^{-1}\mathcal{P}^+\mathcal{P}K\mathcal{P}^+ + \Phi^{-1}(I - \mathcal{P}^+\mathcal{P})\Omega\mathcal{P}^+ + \Phi^{-1}W^*(I - \mathcal{P}\mathcal{P}^+).$$

The next theorem addresses the fourth solution of Equation (1), where both \mathcal{P} and Φ are regulars.

Theorem 2.4: Let $\mathcal{P} \in B(F, D)$, $\Phi \in B(D, F)$ be self-adjoint and $\Omega \in B(F)$, where F, D be Hilbert C^* -modules, Φ and \mathcal{P} are regular, such that $\Omega\mathcal{P}^+\mathcal{P} = \Omega$, $\mathcal{P} = \mathcal{P}\Phi\Phi^+$, $\Phi\Phi^+\Omega = \Omega = \Omega\Phi^+\Phi$ then the statements that follow are interchangeable:

- (i) There is a solution $(X, Y) \in B(F, D) \times B(D, F)$ to Equation (1),
- (ii) $(I - \mathcal{P}^+\mathcal{P})\Omega(I - \mathcal{P}^+\mathcal{P}) = 0$ and $(I - \Phi\Phi^+)\Omega(I - \Phi\Phi^+) = 0$.

Any solution to Equation (1) takes the following form if (i) or (ii) are fulfilled

$$X = \mathcal{P}^{*+}\Omega\Phi^+ - \frac{1}{2}\mathcal{P}^{*+}\Omega\mathcal{P}^+\mathcal{P}\Phi^+ + \mathcal{P}^{*+}K\mathcal{P}^+\mathcal{P}\Phi^+ - U + \mathcal{P}\mathcal{P}^+U\Phi\Phi^+,$$

$$Y = \Phi^+\Omega\mathcal{P}^+ - \frac{1}{2}\Phi^+\mathcal{P}^+\mathcal{P}\Omega\mathcal{P}^+ - \Phi^+\mathcal{P}^+\mathcal{P}K\mathcal{P}^+ - W^* + \Phi^+\Phi W^*\mathcal{P}\mathcal{P}^+,$$

where $K \in B(F)$, $W, U \in B(F, D)$.

Proof: (i) \Rightarrow (ii) Suppose the Equation (1) has solution $(X, Y) \in B(F, D) \times B(F, D)$. Then we have

$$(I - \mathcal{P}^+\mathcal{P})\Omega(I - \mathcal{P}^+\mathcal{P}) = (I - \mathcal{P}^+\mathcal{P})(\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P})(I - \mathcal{P}^+\mathcal{P})$$

$$= (I - \mathcal{P}^+\mathcal{P})\mathcal{P}^*X\Phi^*(I - \mathcal{P}^+\mathcal{P}) + (I - \mathcal{P}^+\mathcal{P})\Phi Y\mathcal{P}(I - \mathcal{P}^+\mathcal{P}) = 0,$$

and

$$(I - \Phi\Phi^+)\Omega(I - \Phi\Phi^+) = (I - \Phi\Phi^+)(\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P})(I - \Phi\Phi^+)$$

$$= (I - \Phi\Phi^+)\mathcal{P}^*X\Phi^*(I - \Phi\Phi^+) + (I - \Phi\Phi^+)\Phi Y\mathcal{P}(I - \Phi\Phi^+) = 0.$$

(ii) \Rightarrow (i) Suppose $(I - \mathcal{P}^+\mathcal{P})\Omega(I - \mathcal{P}^+\mathcal{P}) = 0$, and $(I - \Phi\Phi^+)\Omega(I - \Phi\Phi^+) = 0$ and let

$$M = \begin{bmatrix} \mathcal{P}^* & 0 \\ 0 & \mathcal{P}^* \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \rightarrow \mathcal{F} \oplus \mathcal{F}, N = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} : \mathcal{F} \oplus \mathcal{F} \rightarrow \mathcal{F} \oplus \mathcal{F}, \hat{X} = \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} : \mathcal{F} \oplus \mathcal{F} \rightarrow \mathcal{D} \oplus \mathcal{D}, L = \begin{bmatrix} 0 & \Omega \\ \Omega^* & 0 \end{bmatrix} : \mathcal{F} \oplus \mathcal{F} \rightarrow \mathcal{F} \oplus \mathcal{F}. \text{ Clear } L \text{ is self-adjoint also,}$$

$$(I - MM^+)L(I - MM^+) = \begin{bmatrix} 0 & (I - \mathcal{P}^+\mathcal{P})\Omega(I - \mathcal{P}^+\mathcal{P}) \\ (I - \mathcal{P}^*\mathcal{P}^{*+})\Omega^*(I - \mathcal{P}^*\mathcal{P}^{*+}) & 0 \end{bmatrix} = 0,$$

from this data and referring to Theorem 1.9, \hat{X} represents a solution to equation $M\hat{X}N^* + N\hat{X}^*M^* = L$, and from this equation, consists

$$M\hat{X}N^* + N\hat{X}^*M^* = \begin{bmatrix} 0 & \mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} \\ \mathcal{P}^*Y^*\Phi^* + \Phi X^*\mathcal{P} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ \Omega^* & 0 \end{bmatrix} = L,$$

from this, produces $\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} = \Omega$, that is (X, Y) represents a solution to this equation. Referring to Theorem 1.9 and utilizing the data that has emerged, the solution to equation $M\hat{X}N^* + N\hat{X}^*M^* = L$ is as follows

$$\hat{X} = M^+LN^+ - \frac{1}{2}M^+LMM^+N^+ + M^+ZMM^+N^+ + S - M^+MSNN^+, \quad (20)$$

where $Z \in B(F \oplus F)$ satisfied $M^*(Z + Z^*)M = 0$ and $S \in B(F \oplus F, D \oplus D)$ is arbitrary. If take $Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$ and $S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}$ and substituting each value with its equivalent in Equation (1), this generates

$$\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{P}^{*+}\Omega\Phi^+ \\ \mathcal{P}^{*+}\Omega^*\Phi^+ & 0 \end{bmatrix}$$

$$- \frac{1}{2} \begin{bmatrix} 0 & \mathcal{P}^{*+}\Omega\mathcal{P}^*\mathcal{P}^{*+}\Phi^+ \\ \mathcal{P}^{*+}\Omega^*\mathcal{P}^*\mathcal{P}^{*+}\Phi^+ & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{P}^{*+}Z_1\mathcal{P}^*\mathcal{P}^{*+}\Phi^+ & \mathcal{P}^{*+}Z_2\mathcal{P}^*\mathcal{P}^{*+}\Phi^+ \\ \mathcal{P}^{*+}Z_3\mathcal{P}^*\mathcal{P}^{*+}\Phi^+ & \mathcal{P}^{*+}Z_4\mathcal{P}^*\mathcal{P}^{*+}\Phi^+ \end{bmatrix}$$

$$- \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} + \begin{bmatrix} \mathcal{P}^{*+}\mathcal{P}^*S_1\Phi\Phi^+ & \mathcal{P}^{*+}\mathcal{P}^*S_2\Phi\Phi^+ \\ \mathcal{P}^{*+}\mathcal{P}^*S_3\Phi\Phi^+ & \mathcal{P}^{*+}\mathcal{P}^*S_4\Phi\Phi^+ \end{bmatrix},$$

this, relying on the properties of matrices, yield

$$X = \mathcal{P}^{*+}\Omega\Phi^+ - \frac{1}{2}\mathcal{P}^{*+}\Omega\mathcal{P}^*\mathcal{P}^{*+}\Phi^+ + \mathcal{P}^{*+}Z_2\mathcal{P}^*\mathcal{P}^{*+}\Phi^+ - S_2 + \mathcal{P}^{*+}\mathcal{P}^*S_2\Phi\Phi^+. \quad (21)$$

$$Y^* = \mathcal{P}^{*+} \Omega^* \Phi^+ - \frac{1}{2} \mathcal{P}^{*+} \Omega^* \mathcal{P}^* \mathcal{P}^{*+} \Phi^+ + \mathcal{P}^{*+} Z_3 \mathcal{P}^* \mathcal{P}^{*+} \Phi^+ - S_3 + \mathcal{P}^{*+} \mathcal{P}^* S_3 \Phi \Phi^+. \quad (22)$$

$$\mathcal{P}^{*+} Z_1 \mathcal{P}^* \mathcal{P}^{*+} \Phi^+ - S_1 + \mathcal{P}^{*+} \mathcal{P}^* S_1 \Phi \Phi^+ = 0. \quad (25)$$

$$\mathcal{P}^{*+} Z_4 \mathcal{P}^* \mathcal{P}^{*+} \Phi^+ - S_4 + \mathcal{P}^{*+} \mathcal{P}^* S_4 \Phi \Phi^+ = 0, \quad (26)$$

and after multiplying Equations (25) and (26) by Φ from the right and \mathcal{P}^* from the left, provide

$$\mathcal{P}^* \mathcal{P}^{*+} Z_1 \mathcal{P}^* \mathcal{P}^{*+} = 0, \mathcal{P}^* \mathcal{P}^{*+} Z_4 \mathcal{P}^* \mathcal{P}^{*+} = 0,$$

from the relationship $M^*(Z + Z^*)M = 0$ and because $\ker(M^+) = \ker(M^*)$, gives $M^+(Z + Z^*)M = 0$, and this leads us to the following data

$$\mathcal{P}^{*+} Z_3 \mathcal{P}^* = -\mathcal{P}^{*+} Z_2^* \mathcal{P}^* \text{ and } \mathcal{P}^{*+} Z_2 \mathcal{P}^* = -\mathcal{P}^{*+} Z_3^* \mathcal{P}^*. \quad (27)$$

Therefore, when we assume that $Z_2 = K, S_2 = U, S_3 = W$, using Equation (27), then Equations (21) and (22) will become as follows and represent a solution to the Equation (1)

$$X = \mathcal{P}^{*+} \Omega \Phi^+ - \frac{1}{2} \mathcal{P}^{*+} \Omega \mathcal{P}^+ \mathcal{P} \Phi^+ + \mathcal{P}^{*+} K \mathcal{P}^+ \mathcal{P} \Phi^+ - U + \mathcal{P} \mathcal{P}^+ U \Phi \Phi^+$$

$$Y = \Phi^+ \Omega \mathcal{P}^+ - \frac{1}{2} \Phi^+ \mathcal{P}^+ \mathcal{P} \Omega \mathcal{P}^+ - \Phi^+ \mathcal{P}^+ \mathcal{P} K \mathcal{P}^+ - W^* + \Phi^+ \Phi W^* \mathcal{P} \mathcal{P}^+.$$

3. Conclusions

The aim of this work was to provide some necessary and sufficient conditions for the existence of solutions to the operator equation $\mathcal{P}^* X \Phi^* + \Phi Y \mathcal{P} = \Omega$, in addition to constructing the forms of these solutions under additional assumptions and by using matrix techniques. This technique may also be applicable to studying other equations such as $\mathcal{P}^* X + Y \mathcal{P} = \Omega$ and so on.

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