Al-Hawasy and Fadhel Iraqi Journal of Science, 2026, Vol. 67, No. 4, pp: 2372- 2386
DOI: 10.24996/ijs.2026.67.4.37

ISSN: 0067-2904

Classical Quaternary Boundary Optimal Control Problem of Quaternary Nonlinear Hyperbolic System

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Received: 12/ 12/2024 Accepted: 10/4/2025 Published: 30/4/2026

Abstract

This work concerns with the study of the continuous classical quaternary boundary optimal control problem or, for brief quaternary boundary optimal control problem (QBOCP) controlling by quaternary nonlinear hyperbolic system (QNLHS). The theorem for existence a unique quaternary state vector solution (QSVS) for the weak formulation (WFO) of the QNLHS is proved in an infinite dimensional space via the method of Galerkin, and the help of the Aubin's Theorem with given a continuous boundary control quaternary vector (CBCQV). The continuity of the operator of Lipschitz between the quaternary state vector and the conforming quaternary boundary control vector is demonstrated. The existence theorem of a continuous boundary optimal control quaternary vector (CBOCQV) which is minimized the objective function (OF), and is controlled by the QNLHS is demonstrated in an infinite dimensional space.

Keywords: Aubin's Theorem, Quaternary Boundary Optimal Control, Quaternary Nonlinear Hyperbolic System, Objective Function.

مسألة السيطرة الامثلية الحدودية التقليدية لنظام زائدي رباعي غير خطي

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الخلاصة

يهتم هذا العمل بدراسة مسألة السيطرة الامثلية الحدود الرباعية التقليدية المستمرة أو باختصار مسألة السيطرة الامثلية للحدود الرباعية المختصرة التي يسيطرعلى نظام زائدي رباعي غير خطي . تم اثبات نظرية وحدانية الوجود لحل متجه الحالة الرباعية للصيغة الضعيفة للنظام الزائدي الرباعي الغير خطي من خلال طريقة كاليركين وبمساعدة نظرية أوبين عندما يكون متجه السيطرة الحدودي الرباعي المستمر معلوما . تم اثبات استمرارية مؤثرليبشيتز بين متجة الحالة الرباعية ومتجة السيطرة بالحدود الرباعية المطابق له . تم اثبات نظرية وجود متجه سيطرة امثلية حدودية رباعية مستمرة الذي يسطر عليه نظام زائدي رباعي غير خطي ويجعل لدالة الهدف قيمة صغرى .

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1. Introduction

Various real-life applications are classified as ideal optimal control problems (OCPs). It has been used in many fields, like robotics [1, 2], thermostats [3, 4], aircrafts [5, 6], automobile cruise [7, 8], environmental control [9,10], transportation [11,12], electricity distribution system [13,14], medical dieses [15,16], management [17,18] and in chemistry [19, 20]. In the field of applied mathematics OCPs usually are controlled by ODEs and PDEs. They were investigated by numerous researchers, as [21-28]. Other investigators [29-31] interested about OCPs controlled by one of the three famous kinds for PDES; elliptic, parabolic and hyperbolic, respectively. Whilst other authors interested about studding boundary OCPs which are controlled by couple [32-34], and triple [35-37], PDES of the above three mentioned kinds, besides the investigation of the quaternary boundary OCP controlling by quaternary PDES of the kind parabolic by [38]. The previous mentioned investigations heartened us to discover the study of QBOCP controlling by QNLHS. This paper is beginning by giving a description about the problem, the weak formulation (WFO) for the QNLHS is formulated, and then the method of Galerkin (MG) [39] is employed with the help of the Aubin's Theorem to demonstrate the theorem of existence a unique QSVS for the WFO of the QNLHS. The continuity of the operator of Lipschitz between the quaternary state vector and the conforming quaternary boundary control vector is demonstrated. The theorem of existence CBOCQV that minimizing the OF and controlling by the QNLHS is demonstrated too.

2. Problem description

Consider $\Omega \subset R^2, I = [0, T], Q = I \times \Omega$, $\Sigma = \partial Q = \partial\Omega \times I$, the QBOCP consists of the QNLHS which is given by:

$$y_{1tt} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{1ij}\frac{\partial y_1}{\partial x_j}\right) + a_1 y_1 - b_2 y_2 + b_3 y_3 - b_4 y_4 = f_1(y_1). \quad (1)$$

$$y_{2tt} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{2ij}\frac{\partial y_2}{\partial x_j}\right) + a_2 y_2 + b_2 y_1 - b_5 y_3 + b_6 y_4 = f_2(y_2). \quad (2)$$

$$y_{3tt} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{3ij}\frac{\partial y_3}{\partial x_j}\right) + a_3 y_3 - b_3 y_1 + b_5 y_2 + b_7 y_4 = f_3(y_3). \quad (3)$$

$$y_{4tt} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{4ij}\frac{\partial y_4}{\partial x_j}\right) + a_4 y_4 + b_4 y_1 - b_6 y_2 - b_7 y_3 = f_4(y_4). \quad (4)$$

$$y_l(x,0) = y_l^0(x)\,,\ \ y_{lt}(x,0) = y_l^1(x)\,, \text{on } \Omega,\ \ l = 1,2,3,4. \quad (5)$$

$$\partial n_l y_l = \sum_{i,j=1}^{2} a_{lij}\ \partial x_j y_l \cos(n_l, x_j) = u_l(x,t), \text{on } \partial Q,\ \ l = 1,2,3,4. \quad (6)$$

Where $x = (x_1, x_2)$, n_l $(l = 1,2,3,4)$ refers to outer normal vector on ∂Q, (n_l, x_j) refers to the angle between the n_l and $x_j - axis$. Further, $\vec{y} = (y_1, y_2, y_3, y_4) \in \left(H^2(Q)\right)^4 = \boldsymbol{H^2(Q)}$ refers to the QSVS, and $\vec{u} = (u_1, u_2, u_3, u_4) \in \left(L^2(\partial Q)\right)^4 = \boldsymbol{L^2(\partial Q)}$ refers the QCBCV, $(f_1, f_2, f_3, f_4) \in \left(L^2(Q)\right)^4 = \boldsymbol{L^2(Q)}$ is given. As well as, $a_{l.ij} = a_{lij}(x,t) \in L^\infty(Q)$, $a_l = a_l(x,t) \in L^\infty(Q)$, $b_k = b_k(x,t) \in L^\infty(Q)$ with $k = 2, 3, 4, 5, 7$, $i = 1, 2$ and $j = 1, 2$.

The admissible set of the QCBCV is $\vec{W} = \{\vec{u} \in L^2(\partial Q): \vec{u} \in \vec{U}\ \ a.e\ \vec{U} \subset \mathcal{R}^4, \vec{U} \text{ convex}\}$.

The objective functions are

$$G(\vec{u}) = \sum_{l=1}^{4}\left(\int_Q (g_l(x,t,y_l))\,dxdt + \int_{\partial Q}(h_l(x,t,u_l))\,d\sigma\right). \quad (7)$$

Define $\vec{V} \in (H^1(\Omega))^4 = \boldsymbol{H^1(\Omega)} = \{\vec{v}: \vec{v} = (v_1, v_2, v_3, v_4) \in \boldsymbol{H^1(\Omega)}\}$.

2.1 The weak formulation

The WFO of ((1)-(6)) is

$\langle y_{1tt}, v_1\rangle + r_1(t, y_1, v_1) + (a_1 y_1, v_1)_{L^2(\Omega)} - (b_2 y_2, v_1)_{L^2(\Omega)} +$

$$(b_3y_3,v_1)_{L^2(\Omega)} - (b_4y_4,v_1)_{L^2(\Omega)} = (f_1(y_1),v_1)_{L^2(\Omega)} + (u_1,v_1)_{L^2(\partial\Omega)}, \tag{8}$$

$$\langle y_{2tt},v_2\rangle + r_2(t,y_2,v_2) + (a_2y_2,v_2)_{L^2(\Omega)} + (b_2y_1,v_2)_{L^2(\Omega)} - (b_5y_3,v_2)_{L^2(\Omega)} + (b_6y_4,v_2)_{L^2(\Omega)} = (f_2(y_2),v_2)_{L^2(\Omega)} + (u_2,v_2)_{L^2(\partial\Omega)}, \tag{9}$$

$$\langle y_{3tt},v_3\rangle + r_3(t,y_3,v_3) + (a_3y_3,v_3)_{L^2(\Omega)} - (b_3y_1,v_3)_{L^2(\Omega)} + (b_5y_2,v_3)_{L^2(\Omega)} + (b_7y_4,v_3)_{L^2(\Omega)} = (f_3(y_3),v_3)_{L^2(\Omega)} + (u_3,v_3)_{L^2(\partial\Omega)}, \tag{10}$$

$$\langle y_{4tt},v_4\rangle + r_4(t,y_4,v_4) + (a_4y_4,v_4)_{L^2(\Omega)} + (b_4y_1,v_4)_{L^2(\Omega)} - (b_6y_2,v_4)_{L^2(\Omega)} - (b_7y_3,v_4)_{L^2(\Omega)} = (f_4(y_4),v_4)_{L^2(\Omega)} + (u_4,v_4)_{L^2(\partial\Omega)}, \tag{11}$$

$$(y_l^0,v_l)_{L^2(\Omega)} = (y_l(0),v_1)_{L^2(\Omega)}\ ,\ \forall l = 1,2,3,4\,, \tag{12a}$$

$$(y_l^1,v_l)_{L^2(\Omega)} = (y_{lt}(0),v_1)_{L^2(\Omega)}\,,\forall l = 1,2,3,4, \tag{12b}$$

where $r_l(t,y_l,v_l) = \iint_\Omega \sum_{i,j=1}^{2} \alpha_{lij} \frac{\partial y_l \partial v_l}{\partial x_i \partial x_j} dx$, for $l = 1,2,3,4$.

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2.1.1 Hypotheses:

i)$f_l(x,t,(y_l)), \forall l = 1,2,3,4$ is of a Carathéodory type on $Q \times \mathbb{R}$, with

a) $|f_l(y_l)| \le \gamma_l(x,t) + c_l|y_l|$, where $(x,t) \in Q, \gamma_l \in L^2(Q,\mathcal{R})\,, y_l \in \mathbb{R}, c_l > 0$.

b) $|f(x,t,y_l) - f(x,t,\bar{y}_l)| \le L_l|y_l - \bar{y}_l|$, where $y_l, \bar{y}_l \in \mathbb{R}$, and $L_l > 0$.

ii) $s(t,\vec{y},\vec{v}) = r_1(t,y_1,v_1) + (a_1y_1,v_1)_{L^2(\Omega)} + r_2(t,y_2,v_2) + (a_2y_2,v_2)_{L^2(\Omega)} + r_3(t,y_3,v_3) + (a_3y_3,v_3)_{L^2(\Omega)} + r_4(t,y_4,v_4) + (a_4y_4,v_4)_{L^2(\Omega)}$.

$$s_t(t,\vec{y},\vec{v}) = \iint_\Omega \sum_{i,j=1}^{2} \frac{\partial \alpha_{lij}}{\partial t} \frac{\partial y_l \partial v_l}{\partial x_i \partial x_j} dx,$$

$|s(t,\vec{y},\vec{v})| \le a\|\vec{y}\|_{\boldsymbol{H}^1(\Omega)}\|\vec{v}\|_{\boldsymbol{H}^1(\Omega)}\,, s(t,\vec{y},\vec{y}) \ge \bar{a}\|\vec{y}\|^2_{\boldsymbol{H}^1(\Omega)},\ a,\bar{a} \in \mathcal{R}^+$.

$|s_t(t,\vec{y},\vec{v})| \le b\|\vec{y}\|_{\boldsymbol{H}^1(\Omega)}\|\vec{v}\|_{\boldsymbol{H}^1(\Omega)}\,, s_t(t,\vec{y},\vec{y}) \ge \bar{b}\|\vec{y}\|^2_{\boldsymbol{H}^1(\Omega)}$, with $b,\bar{b} \in \mathcal{R}^+$.

In this work, the notations $\xrightarrow[L^2(Q)]{}$, $\xrightarrow[\boldsymbol{L^2(Q)}]{}$ $(\xrightarrow[L^2(I,V)]{}, \xrightarrow[\boldsymbol{L^2(I,V)}]{})$ will be referred to the convergent strongly in $L^2(Q)$, $\boldsymbol{L^2(Q)}$,$\left(L^2(I,V),\boldsymbol{L^2(I,V)}\right)$, respectively. And $\underset{L^2(Q)}{\rightharpoonup}$, $\underset{\boldsymbol{L^2(Q)}}{\rightharpoonup}$ $(\underset{L^2(I,V)}{\rightharpoonup}, \underset{\boldsymbol{L^2(I,V)}}{\rightharpoonup})$ to the convergent weakly in $L^2(Q)$, $\boldsymbol{L^2(Q)}$ $\left(L^2(I,V),\boldsymbol{L^2(I,V)}\right)$, respectively.

3. The QSVS for the WFO

The next theorem deals with the existence of a unique QSVS of ((8) – (12)).

Theorem 3.1: If $\vec{u} \in \boldsymbol{L^2(Q)}$, then the WFO ((8) – (12)) has a unique QSVS $\vec{y} \in \left(L^2(I,V)\right)^4 = \boldsymbol{L^2(I,V)}$ and $\vec{y}_t = (y_{1t},y_{2t},y_{3t},y_{4t}) \in \boldsymbol{L^2(I,V^*)}$.

Proof: For any n, take $\vec{V}_n = V_{1n} \times V_{2n} \times V_{3n} \times V_{4n} \subset \vec{V}$, represents the set of piecewise affine function in Ω, $\{\vec{V}_n\}_{n=1}^{\infty}$ be a sequence of subspaces of $\vec{V}$, then by the MG $\forall \vec{v} \in \vec{V}$, there is a sequence $\{\vec{v}_n \in \vec{V}_n\}$, with $\vec{v}_n \xrightarrow[\boldsymbol{H^1}(\Omega)]{} \vec{v}$ (thus $\vec{v}_n \xrightarrow[\boldsymbol{L^2}(\Omega)]{} \vec{v}$). Assume $\{\vec{v}_j = (v_{1j},v_{2j},v_{3j},v_{4j}) : j = 1,2,\ldots\ldots,M(n)\}$ spans $\vec{V}_n$ and $\vec{y}_n = (y_{1n},y_{2n},y_{3n},y_{4n})$ be QSVS, such that

$$y_{ln} = \sum_{j=1}^{n} c_{lj}(t)v_{lj}(x), \tag{13}$$

$$z_{ln} = \sum_{j=1}^{n} d_{lj}(t)v_{lj}(x), \tag{14}$$

where $c_{lj}(t)$, $d_{lj}(t)$ are unknown functions of t, $\forall l = 1,2,3,4$, $j = 1,\ldots,$n.

Utilizing y_{ln} (with $y_{lnt} = z_{ln}$), $\forall l = (1,2,3,4)$ in ((8) – (12)) with $v_l \in V_n$, for $l = 1,2,3,4$, they yield to

$$\langle z_{1nt},v_1\rangle + r_1(t,y_{1n},v_1) + (a_1y_{1n},v_1)_{L^2(\Omega)} - (b_2y_{2n},v_1)_{L^2(\Omega)} + (b_3y_{3n},v_1)_{L^2(\Omega)}$$

$$-(b_4 y_{4n}, v_1)_{L^2(\Omega)} = (f_1(y_{1n}), v_1)_{L^2(\Omega)} + (u_1, v_1)_{L^2(\partial\Omega)}. \tag{15}$$

$$\langle z_{2nt}, v_2\rangle + r_2(t, y_{2n}, v_2) + (a_2 y_{2n}, v_2)_{L^2(\Omega)} + (b_2 y_{1n}, v_2)_{L^2(\Omega)} - (b_5 y_{3n}, v_2)_{L^2(\Omega)}$$
$$+(b_6 y_{4n}, v_2)_{L^2(\Omega)} = (f_2(y_{2n}), v_2)_{L^2(\Omega)} + (u_2, v_2)_{L^2(\partial\Omega)}. \tag{16}$$

$$\langle z_{3nt}, v_3\rangle + r_3(t, y_{3n}, v_3) + (a_3 y_{3n}, v_3)_{L^2(\Omega)} - (b_3 y_{1n}, v_3)_{L^2(\Omega)} + (b_5 y_{2n}, v_3)_{L^2(\Omega)}$$
$$+(b_7 y_{4n}, v_3)_{L^2(\Omega)} = (f_3(y_{3n}), v_3)_{L^2(\Omega)} + (u_3, v_3)_{L^2(\partial\Omega)}. \tag{17}$$

$$\langle z_{4nt}, v_4\rangle + r_4(t, y_{4n}, v_4) + (a_4 y_{4n}, v_4)_{L^2(\Omega)} + (b_4 y_{1n}, v_4)_{L^2(\Omega)} - (b_6 y_{2n}, v_4)_{L^2(\Omega)}$$
$$-(b_7 y_{3n}, v_4)_{L^2(\Omega)} = (f_4(y_{4n}), v_4)_{L^2(\Omega)} + (u_4, v_4)_{L^2(\partial\Omega)}. \tag{18}$$

$$(y^0_{ln}, v_l)_{L^2(\Omega)} = (y^0_l, v_1)_{L^2(\Omega)},\ \forall v_l \in V_l\ , l = 1,2,3,4, \tag{19a}$$

$$(z^1_{ln}, v_l)_{L^2(\Omega)} = (y^1_l, v_1)_{L^2(\Omega)}, \forall v_l \in V_l\ , l = 1,2,3,4, \tag{19b}$$

where $y^0_{ln}(x) = y_{ln}(x,0) \in V_n$ ($z^0_{ln} = y^1_{ln}(x) = y_{lnt}(x,0) \in L^2(\Omega)$) be the projection of y^0_l onto V (of $y^1_l = y_{lt}$ onto $L^2(\Omega)$), $\forall l = 1,2,3,4$, i.e.,

$$y^0_{ln} \xrightarrow[L^2(V)]{} y^0_l, \text{ with } \|\vec{y}^0_n\|_{L^2(V)} \le b_0, \forall l = 1,2,3,4, \tag{20}$$

$$y^1_{ln} \xrightarrow[L^2(\Omega)]{} y^1_l, \text{ with } \|\vec{y}^1_n\|_{L^2(\Omega)} \le b_1, \forall l = 1,2,3,4. \tag{21}$$

Replacing (13)-(14) in ((15) - (19)), respectively with inserting $v_l = v_{li}$, $\forall i = 1,2,3,4,\dots,n$, then the secured equations will be equivalent to the following 1st order nonlinear ODEs system with ICs

$$A_1\acute{D}_1(t) + M_1C_1(t) - FC_2(t) + GC_3(t) - HC_4(t) = d_1, \tag{22}$$
$$A_2\acute{D}_2(t) + M_2C_2(t) + FC_1(t) - OC_3(t) + RC_4(t) = d_2, \tag{23}$$
$$A_3\acute{D}_3(t) + M_3C_3(t) - GC_1(t) + OC_2(t) + PC_4(t) = d_3, \tag{24}$$
$$A_4\acute{D}_4(t) + M_4C_4(t) + HC_1(t) - RC_2(t) - PC_3(t) = d_4, \tag{25}$$
$$A_lC_l(0) = m^0_l \ \ \& \ A_lD_l(0) = m^1_l, \tag{26}$$

where

$A_l = (a_{lij})_{n\times n} = (v_{lj}, v_{li}), M_l = (m_{lij})_{n\times n} = r_l(t, v_{lj}, v_{li}) + (a_l v_{lj}, v_{li})_{L^2(\Omega)}$,
$F = (f_{ij})_{n\times n} = (b_2 v_{2j}, v_{2i})_{L^2(\Omega)}, G = (g_{ij})_{n\times n} = (b_3 v_{3j}, v_{1i})_{L^2(\Omega)}$,
$H = (h_{ij})_{n\times n} = (b_4 v_{4j}, v_{1i})_{L^2(\Omega)}\ , O = (o_{ij})_{n\times n} = (b_5 v_{2j}, v_{3i})_{L^2(\Omega)}$,
$P = (p_{ij})_{n\times n} = (b_7 v_{4j}, v_{3i})_{L^2(\Omega)}\ , R = (r_{ij})_{n\times n} = (b_6 v_{2j}, v_{4i})_{L^2(\Omega)}$,
$d_{lj} = (f_l(y_l), v_{li})_{L^2(\Omega)} + (u_l, v_{li})_{L^2(\partial\Omega)}, m^0_{li} = (y^0_l, v_{li})_{L^2(\Omega)}$,
$m^1_{li} = (y^1_l, v_{li})_{L^2(\Omega)}\ ,\ C_l(0) = (c_{lj}(0))_{n\times 1}\ ,\ C_l(t) = (c_{lj}(0))_{n\times 1}$,
$D_l(0) = (d_{lj}(0))_{n\times 1}\ , D_l(t) = (d_{lj}(t))_{n\times 1}, \forall l = 1,2,3,4\ , j = 1,2,\dots,n.$

Since A_l^{-1}exists $\forall l = 1,2,3,4$ and $f_l(y_l), \frac{\partial f_l(y_l)}{\partial y_l}\ , \forall l = 1,2,3,4$ are continuous. Then ((22) – (26)) has a unique QSVS $\vec{y}_n$.

In the next steps the following norms will be proved bounded
$\|\vec{y}^0_n\|_{L^2(\Omega)}$, $\|\vec{y}_n(t)\|_{L^2(\Omega)}$ and $\|\vec{y}_n(t)\|_{L^2(I,V)}$.

Since $y^0_l = y^0_l(x) \in L^2(\Omega)$, $\forall l = 1,2,3,4$, then the following result secured from the Projection theorem $y^0_{ln} \xrightarrow[L^2(Q)]{} y^0_l$ with $\|y^0_{ln}\|_{L^2(\Omega)} \le d_l$.

Setting $v_l = y_{lnt}\ , \forall l = 1,2,3,4$ in ((12) – (15)), respectively and then gathering the outcome equations, i.e.,

$$\langle y_{1ntt}, y_{1nt}\rangle + r_1(t, y_{1n}, y_{1nt}) + (a_1 y_{1n}, y_{1nt})_{L^2(\Omega)} - (b_2 y_{2n}, y_{1nt})_{L^2(\Omega)}$$
$$+ (b_3 y_{3n} - b_4 y_{4n}, y_{1nt})_{L^2(\Omega)} = (f_1(y_{1n}), y_{1nt})_{L^2(\Omega)} + (u_1, y_{1nt})_{L^2(\partial\Omega)} \tag{27}$$

$$\langle y_{2ntt}, y_{2nt}\rangle + r_2(t, y_{2n}, y_{2nt}) + (a_2 y_{2n}, y_{2nt})_{L^2(\Omega)} + (b_2 y_{1n}, y_{2nt})_{L^2(\Omega)}.$$
$$+(b_5 y_{3n} + b_6 y_{4n}, y_{2nt})_{L^2(\Omega)} = (f_2(y_{2n}), y_{2nt})_{L^2(\Omega)} + (u_2, y_{2nt})_{L^2(\partial\Omega)} \tag{28}$$

$$\langle y_{3ntt}, y_{3nt}\rangle + r_3(t, y_{3n}, y_{3nt}) + (a_3 y_{3n}, y_{3nt})_{L^2(\Omega)} - (b_3 y_{1n}, y_{3nt})_{L^2(\Omega)}.$$

$$+(b_5y_{2n}+b_7y_{4n},y_{3nt})_{L^2(\Omega)}=(f_3(y_{3n}),y_{3nt})_{L^2(\Omega)}+(u_3,y_{3nt})_{L^2(\partial\Omega)} \tag{29}$$

$$\langle y_{4ntt},y_{4nt}\rangle+r_4(t,y_{4n},y_{4nt})+(a_4y_{4n},y_{4nt})_{L^2(\Omega)}+(b_4y_{1n},y_{4nt})_{L^2(\Omega)}.$$

$$-(b_6y_{2n}+b_7y_{3n},y_{4nt})_{L^2(\Omega)}=(f_4(y_{4n}),y_{4nt})_{L^2(\Omega)}+(u_4,y_{4nt})_{L^2(\partial\Omega)}. \tag{30}$$

$$(y_{ln}^0,y_{lnt})_{L^2(\Omega)}=(y_{ln}(0),y_{1nt})_{L^2(\Omega)},\ \ \forall\, l=1,2,3,4, \tag{31a}$$

$$(y_{ln}^1,y_{lnt})_{L^2(\Omega)}=(y_{lnt}(0),y_{1nt})_{L^2(\Omega)}\ ,\forall\, l=1,2,3,4. \tag{31b}$$

By applying Lemma 2.1 in [40] for the first two term in the LHS for each equality, to bring

$$\frac{d}{dt}\Big[\|\vec{y}_{nt}(t)\|^2_{\boldsymbol{L^2(\Omega)}}+s(t,\vec{y}_n,\vec{y}_n)\Big]-s_t(t,\vec{y}_n,\vec{y}_n)=$$
$$2[(f_1(y_{1n}),y_{1nt})_{L^2(\Omega)}+(u_1,y_{1nt})_{L^2(\partial\Omega)}+(b_2y_{2n}+b_3y_{3n}-b_4y_{4n},y_{1nt})_{L^2(\Omega)}$$
$$+(f_2(y_{2n}),y_{2nt})_{L^2(\Omega)}+(u_2,y_{2nt})_{L^2(\partial\Omega)}+(b_2y_{1n}-b_5y_{3n}+b_6y_{4n},y_{2nt})_{L^2(\Omega)}$$
$$+(f_3(y_{3n}),y_{3nt})_{L^2(\Omega)}+(u_3,y_{3nt})_{L^2(\partial\Omega)}+(b_5y_{2n}-b_3y_{1n}+b_7y_{4n},y_{3nt})_{L^2(\Omega)}$$
$$+(f_4(y_{4n}),y_{4nt})_{L^2(\Omega)}+(u_4,y_{4nt})_{L^2(\partial\Omega)}+(b_4y_{1n}-b_6y_{2n}-b_7y_{3n},y_{4nt})_{L^2(\Omega)}]. \tag{32}$$

Employing Hypotheses 2.1.1 after take the absolute values of (32), it yields to:

$$\frac{d}{dt}\Big[\|\vec{y}_{nt}(t)\|^2_{\boldsymbol{L^2(\Omega)}}+\bar{a}\|\vec{y}_n\|^2_{\boldsymbol{H^1(\Omega)}}\Big]\le b\|\vec{y}_n\|^2_{\boldsymbol{H^1(\Omega)}}+2\big(\big|(b_2y_{2n},y_{1nt})_{L^2(\Omega)}\big|$$
$$+\big|(b_3y_{3n},y_{1nt})_{L^2(\Omega)}\big|+\big|(b_4y_{4n},y_{1nt})_{L^2(\Omega)}\big|+\big|(b_5y_{2n},y_{3nt})_{L^2(\Omega)}\big|+\big|(b_7y_{4n},y_{3nt})_{L^2(\Omega)}\big|$$
$$+\big|(b_3y_{1n},y_{3nt})_{L^2(\Omega)}\big|+\big|(b_2y_{1n},y_{2nt})_{L^2(\Omega)}\big|+\big|(b_6y_{4n},y_{2nt})_{L^2(\Omega)}\big|+\big|(b_5y_{3n},y_{2nt})_{L^2(\Omega)}\big|$$
$$+\big|(b_4y_{1n},y_{4nt})_{L^2(\Omega)}\big|+\big|(b_6y_{2n},y_{4nt})_{L^2(\Omega)}\big|+\big|(b_7y_{3n},y_{4nt})_{L^2(\Omega)}\big|+\big|(f_1(y_{1n}),y_{1nt})_{L^2(\Omega)}\big|$$
$$+\big|(u_1,y_{1nt})_{L^2(\partial\Omega)}\big|+\big|(\mathrm{f}_2(y_{2n}),\mathrm{y}_{2\mathrm{nt}})_{L^2(\Omega)}\big|+\big|(\mathrm{u}_2,\mathrm{y}_{2\mathrm{nt}})_{L^2(\partial\Omega)}\big|+\big|(f_3(y_{3n}),y_{3nt})_{L^2(\Omega)}\big|$$
$$+\big|(u_3,y_{3nt})_{L^2(\partial\Omega)}\big|+\big|(f_4(y_{4n}),y_{4nt})_{L^2(\Omega)}\big|+\big|(u_4,y_{4nt})_{L^2(\partial\Omega)}\big|\big). \tag{33}$$

Applying the inequality of Shwartz for the RHS of (33), then integrating on (0,t), and take in account that $\|y_{lnt}\|_{L^2(\Omega)}\le\|\vec{y}_{nt}\|_{\boldsymbol{L^2(\Omega)}}$, $\|y_{lnt}\|_{L^2(\partial\Omega)}\le\bar{c}_l\|\vec{y}_{nt}\|_{\boldsymbol{H^1(\Omega)}}$, $\|u_l\|_{L^2(\partial\mathrm{Q})}\le\bar{e}_l$, $\|f_l\|_{L^2(\partial\mathrm{Q})}\le e_l$, $\|y_{ln}\|_{L^2(\Omega)}\le\|y_{ln}\|_{H^1(\Omega)}\le\|\vec{y}_{ln}\|_{\boldsymbol{H^1(\Omega)}}$.

Then utilizing the Trace theorem, and Hypotheses 2.1.1, to secure

$$\int_0^t\frac{d}{dt}\Big[\|\vec{y}_{nt}(t)\|^2_{\boldsymbol{L^2(\Omega)}}+\bar{a}\|\vec{y}_n\|^2_{\boldsymbol{H^1(\Omega)}}\Big]dt$$
$$\le\int_0^t\big[\bar{h}_2\|\vec{y}_{nt}\|^2{}_{\boldsymbol{L^2(\Omega)}}+\bar{h}_3\ \|\vec{y}_n\|^2{}_{\boldsymbol{H^1(\Omega)}}\big]dt+\textstyle\sum_{l=1}^4(\|f_l\|_Q^2+\|u_l\|^2_{L^2(\partial\mathrm{Q})})$$
$$\le\bar{h}_5+\bar{h}_4\int_0^t\Big(\|\vec{y}_{nt}\|^2_{\boldsymbol{L^2(\Omega)}}+\|\vec{y}_{nt}\|^2_{\boldsymbol{H^1(\Omega)}}\Big)dt, \tag{34}$$

where $|b_i|\le h_i$, $i=2,3,4,5,6,7$, $h_8=3\max\limits_{2\le i\le 6}h_i$, $h_9=\max\limits_{1\le l\le 4}c_l$, $\bar{h}_1=\max\limits_{1\le l\le 4}\bar{c}_l$, $\bar{h}_2=h_8+h_9+\bar{h}_1+1$, $\bar{h}_3=h_8+h_9+b$, $\bar{h}_5=4\,e_l+4\bar{e}_l$, $\bar{h}_4=\max(\bar{h}_2,\bar{h}_3)$.

Since $\|\vec{y}_n^0\|_{\boldsymbol{H^1(Q)}}\le b_1$ and $\|\vec{y}_n^1\|_{\boldsymbol{L^2(\Omega)}}\le b_0$ with $\bar{h}_6=b_0+b_1+\bar{h}_5$, hence (34) turn into

$$\|\vec{y}_{nt}(t)\|^2_{\boldsymbol{L^2(\Omega)}}+\|\vec{y}_n(t)\|^2_{\boldsymbol{H^1(Q)}}\le\bar{h}_6+\bar{h}_4\int_0^t[\ \|\vec{y}_{nt}\|^2_{\boldsymbol{L^2(\Omega)}}+\|\vec{y}_n\|^2_{\boldsymbol{H^1(Q)}}]dt.$$

Applying the inequality for Belman-Gronwall in [41], it yields that $\forall\ t\in[0,T]$

$$\|\vec{y}_{nt}(t)\|^2_{\boldsymbol{L^2(\Omega)}}+\|\vec{y}_n(t)\|^2_{\boldsymbol{H^1(Q)}}\le\bar{h}_6e^{\bar{h}_4}=b^2(c)\Rightarrow$$
$$\|\vec{y}_{nt}(t)\|^2_{\boldsymbol{L^2(\Omega)}}\le b^2(c)\text{ and }\|\vec{y}_n\|^2_{\boldsymbol{H^1(Q)}}\le b^2(c),\forall t\in[0,T].$$

Therefore, $\|\vec{y}_{nt}(t)\|_{\boldsymbol{L^2(Q)}}\le b_1(c)$ and $\|\vec{y}_n(t)\|_{\boldsymbol{L^2(I,V)}}\le b(c)$.

The strongly convergence for the QSVS:

Assume $\vec{V}$ has a sequence of subspace $\{\vec{V}_n\}_{n=1}^{\infty}$, such that $\forall\vec{v}=(v_1,v_2,v_3,v_4)\in\vec{V}$, there is a subsequence $\{\vec{v}_n\}$ with $\vec{v}_n=(v_{1n},v_{2n},v_{3n},v_{4n})\in\vec{V}_n\subset V,\forall n$, for which $\vec{v}_n\xrightarrow[\boldsymbol{H^1(\Omega)}]{}\vec{v}$ and $\vec{v}_n\xrightarrow[\boldsymbol{L^2(Q)}]{}\vec{v}$, problem ((8)-(12)) has a unique QSVS $\vec{y}_n=(y_{1n},y_{2n},y_{3n},y_{4n})$, hence conforming to the sequence of spaces $\{\vec{V}_n\}_{n=1}^{\infty}$, there is a sequence of problems like ((8) – (12)), so replacing $\vec{v}_l=\vec{v}_{ln}=(v_{1n},v_{2n},v_{3n},v_{4n})$ and $\vec{y}_l=\vec{y}_{ln}=(y_{1n},y_{2n},y_{3n},y_{4n})$ in these equations, to secure

$$\langle y_{1ntt}, v_{1n}\rangle + r_1(t, y_{1n}, v_{1n}) + (a_1 y_{1n}, v_{1n})_{L^2(\Omega)} - (b_2 y_{2n}, v_{1n})_{L^2(\Omega)} + (b_3 y_{3n} - b_4 y_{4n}, v_{1n})_{L^2(\Omega)} = (f_1(y_{1n}), v_{1n})_{L^2(\Omega)} + (u_1, v_{1n})_{L^2(\partial\Omega)}. \tag{35}$$

$$\langle y_{2ntt}, v_{2n}\rangle + r_2(t, y_{2n}, v_{2n}) + (a_2 y_{2n}, v_{2n})_{L^2(\Omega)} + (b_2 y_{1n}, v_{2n})_{L^2(\Omega)} - (b_5 y_{3n} + b_6 y_{4n}, v_{2n})_{L^2(\Omega)} = (f_2(y_{2n}), v_{2n})_{L^2(\Omega)} + (u_2, v_{2n})_{L^2(\partial\Omega)}. \tag{36}$$

$$\langle y_{3ntt}, v_{3n}\rangle + r_3(t, y_{3n}, v_{3n}) + (a_3 y_{3n}, v_{3n})_{L^2(\Omega)} - (b_3 y_{1n}, v_{3n})_{L^2(\Omega)} + (b_5 y_{2n} + b_7 y_{4n}, v_{3n})_{L^2(\Omega)} = (f_3(y_{3n}), v_{3n})_{L^2(\Omega)} + (u_3, v_{3n})_{L^2(\partial\Omega)}. \tag{37}$$

$$\langle y_{4ntt}, v_{4n}\rangle + r_4(t, y_{4n}, v_{4n}) + (a_4 y_{4n}, v_{4n})_{L^2(\Omega)} + (b_4 y_{1n}, v_{4n})_{L^2(\Omega)} - (b_6 y_{2n} + b_7 y_{3n}, v_{4n})_{L^2(\Omega)} = (f_4(y_{4n}), v_{4n})_{L^2(\Omega)} + (u_4, v_{4n})_{L^2(\partial\Omega)}. \tag{38}$$

$$(y_{ln}^0, v_{ln})_{\boldsymbol{L}^2(\Omega)} = (y_{ln}(0), v_{1n})_{\boldsymbol{L}^2(\Omega)} \ , \forall v_{ln} \in V_{ln} \ , l = 1,2,3,4, \tag{39a}$$

$$(y_{ln}^1, v_{ln})_{\boldsymbol{L}^2(\Omega)} = (y_{lnt}(0), v_{1n})_{\boldsymbol{L}^2(\Omega)} \ , \forall v_{ln} \in V_{ln} \ , l = 1,2,3,4. \tag{39b}$$

Problem ((35) – (39)) has a sequence of QSVS $\{\vec{y}_n\}_{n=1}^{\infty}$ with $\|\vec{y}_n(t)\|_{\boldsymbol{L}^2(\boldsymbol{Q})}$ and $\|\vec{y}_n(t)\|_{\boldsymbol{L}^2(\boldsymbol{I},\boldsymbol{V})}$ are bounded, by applying of Alaglou's Theorem in [40] we found there is a subsequence of $\{\vec{y}_n\}_{n\in N}$, let for simplicity be $\{\vec{y}_n\}_{n\in N}$, such that $\vec{y}_n \xrightarrow[\boldsymbol{L}^2(\boldsymbol{Q})]{} \vec{y}$ and $\vec{y}_n \xrightarrow[\boldsymbol{L}^2(\boldsymbol{I},\boldsymbol{V})]{} \vec{y}$.
And since
$\boldsymbol{L}^2(\boldsymbol{I},\boldsymbol{V}) \subset \boldsymbol{L}^2(\boldsymbol{Q}) \cong (\boldsymbol{L}^2(\boldsymbol{Q}))^* \subset \boldsymbol{L}^2(\boldsymbol{I},\boldsymbol{V}^*)$.
Then by applying the Aubin's Theorem in [39], there is a subsequence of $\{\vec{y}_n\}_{n\in N}$ let for simplicity be $\{\vec{y}_n\}_{n\in N}$ such that $\vec{y}_n \xrightarrow[\boldsymbol{L}^2(\boldsymbol{Q})]{} \vec{y}$.
Now, multiplying both sides of ((35) – (39)) by $\varphi_l(t) \in C^2[0,T]$, $\forall\, l = 1,2,3,4$, such that $\varphi_l(T) = \acute{\varphi}_l(T) = 0$,$\varphi_l(0) \neq 0$ $\acute{\varphi}_l(0) \neq 0$. By taking the integral on [0,T], then integrating "twice" by part for the 1st expression in each acquired equalities, to bring

$$-\int_0^T \frac{d}{dt}(y_{1nt}, v_{1n})\acute{\varphi}_1(t)\mathrm{d}t + \int_0^T [r_1(t, y_{1n}, v_{1n}) + (a_1 y_{1n}, v_{1n})_{L^2(\Omega)} + (-b_2 y_{2n} + b_3 y_{3n} - b_4 y_{4n}, v_{1n})_{L^2(\Omega)}]\varphi_1(t)dt = \int_0^T ((f_1(y_{1n}), v_{1n})_{L^2(\Omega)} + (u_1, v_{1n})_{L^2(\partial\Omega)})\varphi_1(t)dt + (y_{1n}^1, v_{1n})_{L^2(\Omega)}\varphi_1(0). \tag{40}$$

$$\int_0^T (y_{1nt}, v_{1n})\varphi''(t)dt + \int_0^T [r_1(t, y_{1n}, v_{1n}) + (a_1 y_{1n}, v_{1n})_{L^2(\Omega)} + (-b_2 y_{2n} + b_3 y_{3n} - b_4 y_{4n}, v_{1n})_{L^2(\Omega)}]\varphi_1(t)dt = \int_0^T ((f_1(y_{1n}), v_{1n})_{L^2(\Omega)} + (u_1, v_{1n})_{L^2(\partial\Omega)})\varphi_1(t)dt + (y_{1n}^1, v_{1n})_{L^2(\Omega)}\varphi_1(0) - (y_{1n}^0, v_{1n})_{L^2(\Omega)}\varphi'_1(0). \tag{41}$$

$$-\int_0^T \frac{d}{dt}(y_{2nt}, v_{2n})\acute{\varphi}_2(t)\mathrm{d}t + \int_0^T [r_2(t, y_{2n}, v_{2n}) + (a_2 y_{2n}, v_{2n})_{L^2(\Omega)} + (b_2 y_{1n} - b_5 y_{3n} + b_6 y_{4n}, v_{2n})_{L^2(\Omega)}]\varphi_2(t)dt = \int_0^T ((f_2(y_{2n}), v_{2n})_{L^2(\Omega)} + (u_2, v_{2n})_{L^2(\partial\Omega)})\varphi_2(t)dt + (y_{2n}^1, v_{2n})_{L^2(\Omega)}\varphi_2(0). \tag{42}$$

$$\int_0^T (y_{2nt}, v_{2n})\varphi''_2(t)\mathrm{d}t + \int_0^T [r_2(t, y_{2n}, v_{2n}) + (a_2 y_{2n}, v_{2n})_{L^2(\Omega)} + (b_2 y_{1n} - b_5 y_{3n} + b_6 y_{4n}, v_{2n})_{L^2(\Omega)}]\varphi_2(t)dt = \int_0^T ((f_2(y_{2n}), v_{2n})_{L^2(\Omega)} + (u_2, v_{2n})_{L^2(\partial\Omega)})\varphi_2(t)dt + (y_{2n}^1, v_{2n})_{L^2(\Omega}\varphi_2(0) - (y_{2n}^0, v_{2n})_{L^2(\Omega)}\varphi'_2(0). \tag{43}$$

$$-\int_0^T \frac{d}{dt}(y_{3nt}, v_{3n})\acute{\varphi}_3(t)\mathrm{d}t + \int_0^T [r_3(t, y_{3n}, v_{3n}) + (a_3 y_{3n}, v_{3n})_{L^2(\Omega)} + (-b_3 y_{1n} + b_5 y_{2n} + b_7 y_{4n}, v_{3n})_{L^2(\Omega)}]\varphi_3(t)dt = \int_0^T ((f_3(y_{3n}), v_{3n})_{L^2(\Omega)} + (u_3, v_{3n})_{L^2(\partial\Omega)})\varphi_3(t)dt + (y_{3n}^1, v_{3n})_{L^2(\Omega)}\varphi_3(0). \tag{44}$$

$$\int_0^T (y_{3nt}, v_{3n})\varphi''_3(t)\mathrm{d}t + \int_0^T [r_3(t, y_{3n}, v_{3n}) + (a_3 y_{3n}, v_{3n})_{L^2(\Omega)} + (-b_3 y_{1n} + b_5 y_{2n} + b_7 y_{4n}, v_{3n})_{L^2(\Omega)}]\varphi_3(t)dt = \int_0^T ((f_3(y_{3n}), v_{3n})_{L^2(\Omega)} + (u_3, v_{3n})_{L^2(\partial\Omega)})\varphi_3(t)dt + (y_{3n}^1, v_{3n})_{L^2(\Omega)}\varphi_3(0) - (y_{3n}^0, v_{3n})_{L^2(\Omega)}\varphi'_3(0). \tag{45}$$

$$-\int_0^T \frac{d}{dt}(y_{4nt}, v_{4n})\acute{\varphi}_4(t)\mathrm{d}t + \int_0^T [r_4(t, y_{4n}, v_{4n}) + (a_4 y_{4n}, v_{4n})_{(L^2(\Omega)} + (b_4 y_{1n} - b_6 y_{2n}$$

$-b_7y_{3n},v_{4n})_{(L^2(\Omega)}]\varphi_4(t)dt = \int_0^T((f_4(y_{4n}),v_{4n})_{L^2(\Omega)} + (u_4,v_{4n})_{L^2(\partial\Omega)})\varphi_4(t)dt +$
$(y_{4n}^1,v_{4n})_{(L^2(\Omega)}\varphi_4(0).$ (46)

$\int_0^T (y_{4nt},v_{4n})\varphi''_4(t)\mathrm{dt}\int_0^T[r_4(t,y_{4n},v_{4n}) + (a_4y_{4n},v_{4n})_{L^2(\Omega)} + b_4y_{1n} - b_6y_{2n}$
$-b_7y_{3n},v_{4n})_{L^2(\Omega)}]\varphi_4(t)dt = \int_0^T((f_4(y_{4n}),v_{4n})_{L^2(\Omega)} + (u_4,v_{4n})_{L^2(\partial\Omega)})\varphi_4(t)dt +$
$(y_{4n}^1,v_{4n})_{L^2(\Omega)}\varphi_4(0) - (y_{4n}^0,v_{4n})_{L^2(\Omega)}\varphi'_4(0).$ (47)

First, since $v_{ln} \xrightarrow[V]{} v_l$ and $v_{ln} \xrightarrow[L^2(\Omega)]{} v_l$ $\forall l = 1,2,3,4$, then following are respectively held

$$\begin{cases} v_{\mathrm{l}n}\varphi_l(\mathrm{t}) \xrightarrow[L^2(I,\vec{V})]{} v_l\varphi_l(\mathrm{t}) \\ v_{\mathrm{l}n}\varphi'_l(\mathrm{t}) \xrightarrow[L^2(I,\vec{V})]{} v_l\varphi'_l(\mathrm{t}) \end{cases}, \begin{cases} v_{ln}\varphi'_l(t) \xrightarrow[\mathrm{L}^2(\mathrm{Q})]{} v_l\varphi'_l(t) \\ v_{ln}\varphi''_l(t) \xrightarrow[\mathrm{L}^2(\mathrm{Q})]{} v_l\varphi''_l(t) \end{cases} \text{ and } \begin{cases} v_{\mathrm{l}n}\varphi_l(0) \xrightarrow[\mathrm{L}^2(\Omega)]{} v_l\varphi_l(0) \\ v_{ln}\varphi'_l(0) \xrightarrow[L^2(\Omega)]{} v_l\varphi'_l(0) \end{cases}.$$

Second, $y_{lnt} \xrightarrow[L^2(Q)]{} y_{lt}$ and $y_{ln} \xrightarrow[L^2(I,V)]{} y_i$ and $y_{ln} \xrightarrow[L^2(Q)]{} y_l$.

Third, since $v_{ln}\varphi_l \xrightarrow[L^2(Q)]{} v_l\varphi_l$, then from the assumptions on f_l, $\forall\, l = 1,2,3,4$, and through Proposition 2.1 in [37], it acquires the continuity of $\int_Q(f_l(x,t,y_l)\,dxdt$ on $L^2(\mathrm{Q})$ but $y_{ln} \xrightarrow[L^2(Q)]{} y_l$, therefore

$\int_0^T(f_l(y_{ln}),v_{ln})_{L^2(\Omega)}\varphi_l(t)dt \longrightarrow \int_0^T(f_l(y_l),v_l)_{L^2(\Omega)}\varphi_l(t)dt, \forall\, l = 1,2,3,4$

In addition, since $v_{ln} \to v_l$ in $L^2(\partial\mathrm{Q})$, and hence

$\int_0^T(u_l,v_{ln})_{(\partial\Omega)}\varphi_l(t)dt \longrightarrow \int_0^T(u_l,v_l)_{(\partial\Omega)}\varphi_l(t)dt, \forall l = 1,2,3,4$

The above three points helps us to passage the limits in ((40) – (47)), to get

$-\int_0^T \frac{d}{dt}(y_{1t},v_1)\acute{\varphi}_1(t)dt + \int_0^T[r_1(t,y_1,v_1) + (a_1y_1,v_1)_{L^2(\Omega)} + (-b_2y_2 + b_3y_3$
$-b_4y_4,v_1)_{L^2(\Omega)}]\varphi_1(t)dt = \int_0^T((f_1(y_1),v_1)_{L^2(\Omega)} + (u_1,v_1)_{L^2(\partial\Omega)})\varphi_1(t)dt$
$+(y_1^1,v_1)_{L^2(\Omega)}\varphi_1(0).$ (48)

$\int_0^T (y_1,v_1)\varphi''(t)dt + \int_0^T[r_1(t,y_1,v_1) + (a_1y_1,v_1)_{L^2(\Omega)} + (-b_2y_2 + b_3y_3$
$-b_4y_4,v_1)_{L^2(\Omega)}]\varphi_1(t)dt = \int_0^T((f_1(y_1),v_1)_{L^2(\Omega)} + (u_1,v_1)_{L^2(\partial\Omega)})\varphi_1(t)dt +$
$(y_1^1,v_1)_{L^2(\Omega)}\varphi_1(0) - (y_1^0,v_1)_{L^2(\Omega)}\acute{\varphi}_1(0).$ (49)

$-\int_0^T \frac{d}{dt}(y_{2t},v_2)\acute{\varphi}_2(t)\mathrm{dt} + \int_0^T[r_2(t,y_2,v_2) + (a_2y_2,v_2)_{L^2(\Omega)} + (b_2y_1 - b_5y_3$
$+b_6y_4,v_2)_{L^2(\Omega)}]\varphi_2(t)dt = \int_0^T((f_2(y_2),v_2)_{L^2(\Omega)} + (u_2,v_2)_{L^2(\partial\Omega)})\varphi_2(t)dt +$
$(y_2^1,v_2)_{L^2(\Omega)}\varphi_2(0).$ (50)

$\int_0^T (y_2,v_2)\varphi''_2(t)\mathrm{dt} + \int_0^T[r_2(t,y_2,v_2) + (a_2y_2,v_2)_{L^2(\Omega)} + (b_2y_1 - b_5y_3$
$+b_6y_4,v_2)_{L^2(\Omega)}]\varphi_2(t)dt = \int_0^T((f_2(y_2),v_2)_{L^2(\Omega)} + (u_2,v_2)_{L^2(\partial\Omega)})\varphi_2(t)dt +$
$(y_2^1,v_2)_{L^2(\Omega}\varphi_2(0) - (y_2^0,v_2)_{L^2(\Omega)}\acute{\varphi}_2(0).$ (51)

$-\int_0^T \frac{d}{dt}(y_{3t},v_3)\acute{\varphi}_3(t)dt + \int_0^T[r_3(t,y_3,v_3) + (a_3y_3,v_3)_{L^2(\Omega)} + (-b_3y_1 + b_5y_2$
$+b_7y_4,v_3)_{L^2(\Omega)}]\varphi_3(t)dt = \int_0^T((f_3(y_3),v_3)_{L^2(\Omega)} + (u_3,v_3)_{L^2(\partial\Omega)})\varphi_3(t)dt +$
$(y_3^1,v_3)_{L^2(\Omega)}\varphi_3(0).$ (52)

$\int_0^T(y_3,v_3)\varphi''_3(t)\mathrm{dt} + \int_0^T[r_3(t,y_{3n},v_{3n}) + (a_3y_{3n},v_{3n})_{L^2(\Omega)} + (-b_3y_{1n} + b_5y_2$
$+b_7y_4,v_{3n})_{L^2(\Omega)}]\varphi_3(t)dt = \int_0^T((f_3(y_3),v_3)_{L^2(\Omega)} + (u_3,v_3)_{L^2(\partial\Omega)})\varphi_3(t)dt +$
$(y_3^1,v_3)_{L^2(\Omega)}\varphi_3(0) - (y_3^0,v_3)_{L^2(\Omega)}\acute{\varphi}_3(0).$ (53)

$-\int_0^T \frac{d}{dt}(y_{4t},v_4)\acute{\varphi}_4(t)\mathrm{dt} + \int_0^T[r_4(t,y_4,v_4) + (a_4y_4,v_4)_{L^2(\Omega)} + (b_4y_1 - b_6y_2$
$-b_7y_3,v_4)_{L^2(\Omega)}]\varphi_4(t)dt = \int_0^T((f_4(y_4),v_4)_{L^2(\Omega)} + (u_4,v_4)_{L^2(\partial\Omega)})\varphi_4(t)dt +$
$(y_4^1,v_4)_{(L^2(\Omega)}\varphi_4(0).$ (54)

$\int_0^T (y_4, v_4)\varphi''_4(t)\mathrm{dt} + \int_0^T [r_4(t, y_4, v_4) + (a_4 y_4, v_4)_{L^2(\Omega)} + (b_4 y_1 - b_6 y_2$
$-b_7 y_3, v_4)_{L^2(\Omega)}]\varphi_4(t)dt = \int_0^T ((f_4(y_4), v_4)_{L^2(\Omega)} + (u_4, v_4)_{L^2(\partial\Omega)})\varphi_4(t)dt +$
$(y_4^1, v_4)_{L^2(\Omega)}\varphi_4(0) - (y_4^0, v_4)_{L^2(\Omega)}\acute{\varphi}_4(0).$ (55)

At this point, the following cases are presented
Case1:Pick out $\varphi_l \in C^2[0,T]$ $(l = 1,2,3,4)$, in (49), (51), (53) and (55) with $\varphi_l(0) = \acute{\varphi}_l(0) = \acute{\varphi}_l(T) = \varphi_l(T) = 0$ then integrating "twice" by part the 1st expression in their LHS, they yield to
$\int_0^T (y_{1tt}, v_1)\varphi(t)\mathrm{dt} + \int_0^T [r_1(t, y_1, v_1) + (a_1 y_1, v_1)_{L^2(\Omega)} + (-b_2 y_2 + b_3 y_3$
$-b_4 y_4, v_1)_{L^2(\Omega)}]\varphi_1(t)dt = \int_0^T ((f_1(y_1), v_1)_{L^2(\Omega)} + (u_1, v_1)_{L^2(\partial\Omega)})\varphi_1(t)dt.$ (56)
$\int_0^T (y_{2tt}, v_2)\varphi_2(t)\mathrm{dt} + \int_0^T [r_2(t, y_2, v_2) + (a_2 y_2, v_2)_{L^2(\Omega)} + (b_2 y_1 - b_5 y_3$
$+b_6 y_4, v_2)_{L^2(\Omega)}]\varphi_2(t)dt = \int_0^T ((f_2(y_2), v_2)_{L^2(\Omega)} + (u_2, v_2)_{L^2(\partial\Omega)})\varphi_2(t)dt.$ (57)
$\int_0^T (y_{3tt}, v_3)\varphi_3(t)\mathrm{dt} + \int_0^T [r_3(t, y_{3n}, v_{3n}) + (a_3 y_{3n}, v_{3n})_{L^2(\Omega)} + (-b_3 y_{1n} + b_5 y_2$
$+b_7 y_4, v_{3n})_{L^2(\Omega)}]\varphi_3(t)dt = \int_0^T ((f_3(y_3), v_3)_{L^2(\Omega)} + (u_3, v_3)_{L^2(\partial\Omega)})\varphi_3(t)dt.$ (58)
$\int_0^T (y_{4tt}, v_4)\varphi_4(t)\mathrm{dt} + \int_0^T [r_4(t, y_4, v_4) + (a_4 y_4, v_4)_{L^2(\Omega)} + (b_4 y_1 - b_6 y_2$
$-b_7 y_3, v_4)_{L^2(\Omega)}]\varphi_4(t)dt = \int_0^T ((f_4(y_4), v_4)_{L^2(\Omega)} + (u_4, v_4)_{L^2(\partial\Omega)})\varphi_4(t)dt.$ (59)
Hence, $\vec{y}$ is a QSVS of ((8)-(11)) almost everywhere on *I*.
Case2: Pick out $\varphi_l \in C^2[0,T], \forall l = 1,2,3,4,$ such that $\varphi_l(T) = \acute{\varphi}_l(0) = \acute{\varphi}_l(T) = 0$ & $\varphi_l(0) \neq 0$, multiplying both sides of ((8)-(11)) by $\varphi_1(t), \varphi_2(t), \varphi_3(t)$ and $\varphi_4(t)$, respectively then take the integral on [0, *T*], utilizing integrating by parts for the 1st expression in LHS in each acquired equalities, then subtracting each equality from the each corresponding one of (48),(50), (52) and (54), respectively to acquire
$(y_{lt}(0), v_l)_{L^2(\Omega)}\varphi_l(0) = (y_l^1, v_l)_{L^2(\Omega)}\varphi_l(0), \ \forall l = 1,2,3,4 .$
Case3: Pick out $\varphi_l \in C^2[0,T], \forall l = 1,2,3,4,$ with $\varphi_l(0) = \varphi_l(T) = \varphi'_l(T) = 0$,
$\varphi'_l(0) \neq 0$, multiplying ((8)-(11)) by $\varphi_l, \forall l = 1,2,3,4$, then take the integral on [0, *T*], and then integrating "twice" by part the 1st expression in LHS in each acquired equality, then subtracting each equality from the each conforming one of (49), (51), (53) and (55) respectively, to acquire
$(y_l^0, v_l)_{L^2(\Omega)}\acute{\varphi}_l(0) = (y_l(0), v_l)_{L^2(\Omega)}\varphi'_l(0)$, $\forall l = 1,2,3,4$.
From the above two previous cases, one acquires the ICs (12).
To prove $\vec{y}_n \xrightarrow[L^2(I,V)]{} \vec{y}$, take the integral for (32) on [0, *T*], to acquire that
$\|\vec{y}_{nt}(T)\|^2_{L^2(Q)} - \|\vec{y}_{nt}(0)\|^2_{L^2(Q)} + s(t, \vec{y}_n, \vec{y}_n)(T) -$
$s(t, \vec{y}_n, \vec{y}_n)(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y}_{nt})dt = \int_0^T ((60\text{a}) + (60\text{b}))dt,$ (60)
where
$(60\text{a}) = 2((b_2 y_{2n}, y_{1nt})_{L^2(\Omega)} - (b_3 y_{3n}, y_{1nt})_{L^2(\Omega)} + (b_4 y_{4n}, y_{1nt})_{L^2(\Omega)} + (b_5 y_{2n}, y_{3nt})_{L^2(\Omega)}$
$+(b_2 y_{1n}, y_{2nt})_{L^2(\Omega)} - (b_5 y_{3n}, y_{2nt})_{L^2(\Omega)} + (b_6 y_{4n}, y_{2nt})_{L^2(\Omega)} - (b_3 y_{1n}, y_{3nt})_{L^2(\Omega)}$
$+(b_7 y_{4n}, y_{3nt})_{L^2(\Omega)} - (b_4 y_{1n}, y_{4nt})_{L^2(\Omega)} + (b_6 y_{2n}, y_{4nt})_{L^2(\Omega)} - (b_7 y_{3n}, y_{4nt})_{L^2(\Omega)})$
$(60\text{b}) = 2((f_1(y_{1n}), y_{1nt})_{L^2(\Omega)} + (u_1, y_{1nt})_{L^2(\partial\Omega)} + (f_2(y_{2n}), y_{2nt})_{L^2(\Omega)} + (u_2, y_{2nt})_{L^2(\partial\Omega)}$
$+(f_3(y_{3n}), y_{3nt})_{L^2(\Omega)} + (u_3, y_{3nt})_{L^2(\partial\Omega)} + (f_4(y_{4n}), y_{4nt})_{L^2(\Omega)} + (u_4, y_{4nt})_{L^2(\partial\Omega)}).$
Now, replace $y_{ln} = y_l, \forall l = 1,2,3,4$, in (32), then take the integral on [0, *T*] to acquire
$\|\vec{y}_t(T)\|^2_{L^2(Q)} - \|\vec{y}_t(0)\|^2_{L^2(Q)} + s(t, \vec{y}, \vec{y})(T) -$
$s(t, \vec{y}, \vec{y})(0) - \int_0^T s_t(t, \vec{y}, \vec{y}_t)dt = \int_0^T ((61a) + (61b))dt.$ (61)

$$(61a) = 2((b_2y_2, y_{1t})_{L^2(\Omega)} - (b_3y_3, y_{1t})_{L^2(\Omega)} + (b_4y_4, y_{1t})_{L^2(\Omega)} + (b_5y_2, y_{3t})_{L^2(\Omega)}$$
$$+(b_2y_1, y_{2t})_{L^2(\Omega)} - (b_5y_3, y_{2t})_{L^2(\Omega)} + (b_6y_4, y_{2t})_{L^2(\Omega)} - (b_3y_1, y_{3t})_{L^2(\Omega)} +$$
$$(b_7y_4, y_{3t})_{L^2(\Omega)} - (b_4y_1, y_{4t})_{L^2(\Omega)} - (b_7y_3, y_{4t})_{L^2(\Omega)})$$
$$(61b) = 2((f_1(y_1), y_{1t})_{L^2(\Omega)} + (u_1, y_{1t})_{L^2(\partial\Omega)} + (f_2(y_2), y_{2t})_{L^2(\Omega)} + (u_2, y_{2t})_{L^2(\partial\Omega)}$$
$$+(f_3(y_3), y_{3t})_{L^2(\Omega)} + (u_3, y_{3t})_{L^2(\partial\Omega)} + (f_4(y_4), y_{4t})_{L^2(\Omega)} + (u_4, y_{4t})_{L^2(\partial\Omega)}).$$

Since

$$\|\vec{y}_{nt}(T) - \vec{y}_t(T)\|^2_{\boldsymbol{L^2}(\Omega)} - \|\vec{y}_{nt}(0) - \vec{y}_t(0)\|^2_{\boldsymbol{L^2}(\Omega)} + s(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})(T)$$
$$-s(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})(0) - \int_0^T s_t(t, \vec{y}_n - \vec{y}, \vec{y}_{nt} - \vec{y}_t)dt = (62a) - (62b) - (62c). \qquad (62)$$
$$(62a) = \|\vec{y}_{nt}(T)\|^2_{\boldsymbol{L^2(Q)}} - \|\vec{y}_n(0)\|^2_{\boldsymbol{L^2}(\Omega)} + s(t, \vec{y}_n, \vec{y}_n)(T) -$$
$$s(t, \vec{y}_n, \vec{y}_n)(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y}_{nt})dt.$$
$$(62b) = \big(\vec{y}_{nt}(T), \vec{y}_t(T)\big) - \big(\vec{y}_{nt}(0), \vec{y}_t(0)\big) + s(t, \vec{y}_n, \vec{y})(T) -$$
$$s(t, \vec{y}_n, \vec{y})(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y}_t)dt.$$
$$(62c) = (\vec{y}_t(T), \vec{y}_{nt} - \vec{y}_t(T)) - \big(\vec{y}_t(0), \vec{y}_{nt}(0) - \vec{y}_t(0)\big)$$
$$+s(t, \vec{y}_n, \vec{y}_n - \vec{y})(T) - s(t, \vec{y}_n, \vec{y}_n - \vec{y})(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y}_{nt} - \vec{y}_t)dt.$$

Since $\vec{y}_n \xrightarrow[\boldsymbol{L^2(Q)}]{} \vec{y}$ & $\vec{y}_{nt} \xrightarrow[\boldsymbol{L^2(Q)}]{} \vec{y}_t$, thus by (60) and Hypotheses 2.1.1 one acquires that

$$(62a) = \int_0^T [(60a) + (60b)]dt \to \int_0^T [(61a) + (61b)]dt.$$

A similar manner which utilized to get (21), utilizes also here to obtain

$$\vec{y}_{nt}(T) \xrightarrow[\mathbf{L^2}(\Omega)]{} \vec{y}_t(T). \qquad (63)$$

On the other hand, since $\vec{y}_n \xrightarrow[\boldsymbol{L^2(I,V)}]{} \vec{y}$, then utilizing (21) and (63) in (62b), it yields that

$$(62b) \to \int_0^T (61a) + (61b)dt.$$

From $\vec{y}_n \xrightarrow[\boldsymbol{L^2(Q)}]{} \vec{y}$ and the continuity of $\int_0^T (f_l(y_{ln}), y_{ln})_{L^2(\Omega)}\, dt$ with respect to y_l , one has:

$(60b) \to (61b)$.

Beside these convergences, all the expressions in (62c) imply to zero, in addition the first two expressions in LHS of (62); hence (62) gives

$\int_0^T \|\vec{y}_n(t) - \vec{y}(t)\|^2_{\boldsymbol{H^1(Q)}} dt \to 0$ as $n \to \infty$ so we get that $\vec{y}_n \xrightarrow[\mathbf{L^2(I,V)}]{} \vec{y}$.

The Uniqueness of the QSVS:

Assume that $\vec{y} = (y_1, y_2, y_3, y_4)$ & $\vec{\bar{y}} = (\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4)$ are two QSVS of the WFO ((8) - (12)), then subtracting each equality form its conforming one and letting $v_l = y_l - \bar{y}_l$ for $l = 1,2,3,4$, utilizing Lemma 2.1 in [40] for the 1st expression in each equality and then gathering all the above equalities, and Hypothesis 2.1.1, (ib) to acquire

$$\frac{d}{dt}\left[\left\|(\vec{y} - \vec{\bar{y}})_t\right\|^2_{\boldsymbol{L^2(\Omega)}} + s(t, (\vec{y} - \vec{\bar{y}}), (\vec{y} - \vec{\bar{y}})\right] \leq s_t\left(t, (\vec{y} - \vec{\bar{y}}), (\vec{y} - \vec{\bar{y}})\right)$$
$$+2\int_\Omega [L_1|y_1 - \bar{y}_1|^2 + L_2|y_2 - \bar{y}_2|^2 + L_3|(y_3 - \bar{y}_3)|^2 + L_4|y_4 - \bar{y}_4|^2].$$

Using Hypothesis 2.1.1 (ii) and take the integral from 0 to t,

$$\int_0^t \frac{d}{dt}\left[\left\|(\vec{y} - \vec{\bar{y}})_t\right\|^2_{\boldsymbol{L^2(Q)}} + \bar{a}\left\|(\vec{y} - \vec{\bar{y}})\right\|^2_{\boldsymbol{H^1}(\Omega)}\right] dt \leq$$
$$\int_0^t b\|(\vec{y} - \vec{\bar{y}})\|^2_{\boldsymbol{H^1}(\Omega)}\, dt + \int_0^t L\left\|(\vec{y} - \vec{\bar{y}})\right\|^2_{\boldsymbol{L^2(Q)}} dt.$$

Utilizing the ICs,

$$\left\|(\vec{y} - \vec{\bar{y}})_t(t)\right\|^2_{\boldsymbol{L^2(Q)}} + \left\|(\vec{y} - \vec{\bar{y}})(t)\right\|^2_{\boldsymbol{H^1}(\Omega)} \leq 0 + \tilde{L}\int_0^t [\|(\vec{y} - \vec{\bar{y}})\|^2_{\boldsymbol{H^1}(\Omega)} + \left\|(\vec{y} - \vec{\bar{y}})\right\|^2_{\boldsymbol{L^2(Q)}}]\, dt.$$

Where $L = 2\max\{L_1, L_2, L_3, L_4\}$, $\bar{\bar{L}} = \max\{L, b\}$, $\bar{L} = \min\{1, \bar{a}\}$, and $\tilde{L} = \frac{\bar{\bar{L}}}{\bar{L}}$.
Employing the inequality of Belman-Gronwal in [20], to acquire

$$\left\|\left(\vec{y} - \vec{\bar{y}}\right)(t)\right\|^2_{\boldsymbol{H^1}(\Omega)} = 0\,, \forall t \in I \quad \Rightarrow \left\|\left(\vec{y} - \vec{\bar{y}}\right)(t)\right\|^2_{\boldsymbol{L^2(I,V)}} = 0 \quad \Rightarrow \quad \vec{y} = \vec{\bar{y}}.$$

Thus, the QSVS is unique.

4. Existence of a CBOCQV:

Lemma 4.1: Besides to Hypothesis 2.1.1, consider $\vec{y}$ and $\vec{y} + \delta\vec{y}$ are the QSVS conforming to the bounded CBCQV $\vec{u}$ and $\vec{u} + \overrightarrow{\delta u}$ respectively, then for $K \in \mathcal{R}^+$
$\|\delta\vec{y}\|_{\boldsymbol{L^\infty(I,L^2(\Omega))}} \le K\|\delta\vec{u}\|_{\boldsymbol{L^2(\partial Q)}}$, $\|\delta\vec{y}\|_{\boldsymbol{L^2(Q)}} \le K\,\|\delta\vec{u}\|_{\boldsymbol{L^2(\partial Q)}}$, $\|\delta\vec{y}\|_{\boldsymbol{L^2(I,V)}} \le K\,\|\delta\vec{u}\|_{\boldsymbol{L^2(\partial Q)}}$.

Proof: Assume $\vec{u} = (u_1, u_2, u_3, u_4)$ and $\vec{\bar{u}} = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4) \in \boldsymbol{L^2(\partial Q)}$, then $\vec{u}_\sigma = \vec{u} + \sigma\delta\vec{u} \in \boldsymbol{L^2(\partial Q)}$, with $\overrightarrow{\delta u} = \vec{\bar{u}} - \vec{u}$ and > 0 , then by Theorem 3.1, $\vec{y} = \vec{y}_{\vec{u}} = (y_1, y_2, y_3, y_4)$, and $\vec{y}_\sigma = \vec{y}_{\vec{u}_\sigma} = (y_{1\sigma}, y_{2\sigma}, y_{3\sigma}, y_{4\sigma})$ are their conforming QSVS, and ((8) – (12)) for $\vec{y}_\sigma$, they turns to with $\vec{u}_\sigma = \vec{u} + \sigma\delta\vec{u}$ which are held ((8) – (12)), i.e.,

$$\langle y_{1\sigma tt}, v_1\rangle + r_1(t, y_{1\sigma}, v_1) + (a_1 y_{1\sigma}, v_1)_{L^2(\Omega)} - (b_2 y_{2\sigma}, v_1)_{L^2(\Omega)} + (b_3 y_{3\sigma}, v_1)_{L^2(\Omega)} - (b_4 y_{4\sigma}, v_1)_{L^2(\Omega)} = (f_1(y_{1\sigma}), v_1)_{L^2(\Omega)} + (u_{1\sigma}, v_1)_{L^2(\partial\Omega)}. \tag{64}$$

$$\langle y_{2\sigma tt}, v_2\rangle + r_2(t, y_{2\sigma}, v_2) + (a_2 y_{2\sigma}, v_2)_{L^2(\Omega)} + (b_2 y_{1\sigma}, v_2)_{L^2(\Omega)} - (b_5 y_{3\sigma}, v_2)_{L^2(\Omega)} + (b_6 y_{4\sigma}, v_2)_{L^2(\Omega)} = (f_2(y_{2\sigma}), v_2)_{L^2(\Omega)} + (u_{2\sigma}, v_2)_{L^2(\partial\Omega)}. \tag{65}$$

$$\langle y_{3\sigma tt}, v_3\rangle + r_3(t, y_{3\sigma}, v_3) + (a_3 y_{3\sigma}, v_3)_{L^2(\Omega)} - (b_3 y_{1\sigma}, v_3)_{L^2(\Omega)} + (b_5 y_{2\sigma}, v_3)_{L^2(\Omega)} + (b_7 y_{4\sigma}, v_3)_{L^2(\Omega)} = (f_3(y_{3\sigma}), v_3)_{L^2(\Omega)} + (u_{3\sigma}, v_3)_{L^2(\partial\Omega)}. \tag{66}$$

$$\langle y_{4\sigma tt}, v_4\rangle + r_4(t, y_{4\sigma}, v_4) + (a_4 y_{4\sigma}, v_4)_{L^2(\Omega)} + (b_4 y_{1\sigma}, v_4)_{L^2(\Omega)} - (b_6 y_{2\sigma}, v_4)_{L^2(\Omega)} - (b_7 y_{3\sigma}, v_4)_{L^2(\Omega)} = (f_4(y_{4\sigma}), v_4)_{L^2(\Omega)} + (u_{4\sigma}, v_4)_{L^2(\partial\Omega)}. \tag{67}$$

$$(y^0_{l\sigma}, v_l)_{L^2(\Omega)} = (y_l(0), v_1)_{L^2(\Omega)} \text{ and } (y^1_{l\sigma}, v_l)_{L^2(\Omega)} = (y_{lt}(0), v_1)_{L^2(\Omega)}\ ,\ \forall v_l \in V_l\ , l = 1,2,3,4. \tag{68}$$

By subtracting each one of ((8) – (12)) from its conforming one of ((64) – (68)), putting $\vec{y}_\sigma - \vec{y} = \delta\vec{y}_\sigma = (y_{1\sigma}, y_{2\sigma}, y_{3\sigma}, y_{4\sigma})$, they acquire

$$\langle \delta y_{1\sigma tt}, v_1\rangle + r_1(t, \delta y_{1\sigma}, v_1) + (a_1 \delta y_{1\sigma}, v_1)_{L^2(\Omega)} - (b_2 \delta y_{2\sigma}, v_1)_{L^2(\Omega)} + (b_3 \delta y_{3\sigma}, v_1)_{L^2(\Omega)} - (b_4 \delta y_{4\sigma}, v_1)_{L^2(\Omega)} = (f_1(y_1 + \delta y_{1\sigma}) - f_1(y_1), v_1)_{L^2(\Omega)} + (\sigma u_1, v_1)_{L^2(\partial\Omega)}. \tag{69}$$

$$\langle \delta y_{2\sigma tt}, v_2\rangle + r_2(t, \delta y_{2\sigma}, v_2) + (a_2 \delta y_{2\sigma}, v_2)_{L^2(\Omega)} + (b_2 \delta y_{1\sigma}, v_2)_{L^2(\Omega)} + (b_5 \delta y_{3\sigma}, v_2)_{L^2(\Omega)} + (b_6 \delta y_{4\sigma}, v_2)_{L^2(\Omega)} = (f_2(y_2 + \delta y_{2\sigma}) + f_2(y_2), v_2)_{L^2(\Omega)} + (\sigma u_2, v_2)_{L^2(\partial\Omega)}. \tag{70}$$

$$\langle \delta y_{3\sigma tt}, v_3\rangle + r_3(t, \delta y_{3\sigma}, v_3) + (a_3 \delta y_{3\sigma}, v_3)_{L^2(\Omega)} - (b_3 \delta y_{1\sigma}, v_3)_{L^2(\Omega)} + (b_5 \delta y_{2\sigma}, v_3)_{L^2(\Omega)} + (b_7 \delta y_{4\sigma}, v_3)_{L^2(\Omega)} = (f_3(y_3 + \delta y_{3\sigma}) - f_3(y_3), v_3)_{L^2(\Omega)} + (\sigma u_3, v_3)_{L^2(\partial\Omega)}. \tag{71}$$

$$\langle \delta y_{4\sigma tt}, v_4\rangle + r_4(t, \delta y_{4\sigma}, v_4) + (a_4 \delta y_{4\sigma}, v_4)_{L^2(\Omega)} + (b_4 \delta y_{1\sigma}, v_4)_{L^2(\Omega)} - (b_6 \delta y_{2\sigma}, v_4)_{L^2(\Omega)} - (b_7 \delta y_{3\sigma}, v_4)_{L^2(\Omega)} = (f_4(y_4 + \delta y_{4\sigma}) - f_4(y_4), v_4)_{L^2(\Omega)} + (\sigma u_4, v_4)_{L^2(\partial\Omega)}. \tag{72}$$

$$(\delta y^0_{l\sigma}, v_l)_{L^2(\Omega)} = 0 \text{ and } (\delta y^1_{l\sigma}, v_l)_{L^2(\Omega)} = 0\,, \forall v_l \in V_l\ , l = 1,2,3,4. \tag{73}$$

Using $v_l = \delta y_{l\sigma t}$, for $l = 1,2,3,4$ in ((69) – (73)), employ Lemma 2.1 in [40] for the 1st expressions in the LHS of each resulting equality, then employing the same steps which utilized to acquire (32), ones get analogs equality with $\delta\vec{y}_\sigma$ in position of $\vec{y}_n$, and by employing Hypothesis 2.1.1, (ii) for the 2nd expression in the LHS, after take the absolute values of (32), it yields to:

$$\frac{d}{dt}\left[\left\|\overrightarrow{\delta y_{\sigma t}}\right\|^2_{\boldsymbol{L^2(\Omega)}} + \bar{a}\left\|\overrightarrow{\delta y_\sigma}\right\|^2_{\boldsymbol{H^1(\Omega)}}\right] \le$$

$$b\left\|\overrightarrow{\delta y_\sigma}\right\|^2_{\boldsymbol{H^1(\Omega)}} + 2\Big(\left|(b_2\delta y_{2\sigma} + b_3\delta y_{3\sigma} + b_4\delta y_{4\sigma}, \delta y_{1\sigma t})_{L^2(\Omega)}\right| +$$
$$\left|(b_2\delta y_{1\sigma} + b_5\delta y_{3\sigma} + b_6\delta y_{4\sigma}, \delta y_{2\sigma t})_{L^2(\Omega)}\right| + \left|(b_3\delta y_{1\sigma}, \delta y_{3\sigma t})_{L^2(\Omega)}\right| +$$
$$\left|(b_5\delta y_{2\sigma} + b_7\delta y_{4\sigma}, \delta y_{3\sigma t})_{L^2(\Omega)}\right| + \left|(b_4\delta y_{1\sigma} + b_6\delta y_{2\sigma}, \delta y_{4\sigma t})_{L^2(\Omega)}\right| +$$
$$\left|(b_7\delta y_{3\sigma}, \delta y_{4\sigma t})_{L^2(\Omega)}\right| + \left|(f_1(y_1 + \delta y_{1\sigma}) - f_1(y_1), \delta y_{1\sigma t})_{L^2(\Omega)}\right| +$$
$$\left|(\sigma\delta u_1, \delta y_{1\sigma t})_{L^2(\partial\Omega)}\right| + \left|(f_2(y_2 + \delta y_{2\sigma}) - f_2(y_2), \delta y_{2\sigma t})_{L^2(\Omega)}\right| +$$
$$\left|(\sigma\delta u_2, \delta y_{2\sigma t})_{L^2(\partial\Omega)}\right| + \left|(f_3(y_3 + \delta y_{3\sigma}) - f_3(y_3), \delta y_{3\sigma t})_{L^2(\Omega)}\right| +$$
$$\left|(\sigma\delta u_3, \delta y_{3\sigma t})_{L^2(\partial\Omega)}\right| + \left|(f_4(y_4 + \delta y_{4\sigma}) - f_4(y_4), \delta y_{4\sigma t})_{L^2(\Omega)}\right| +$$
$$\left|(\sigma\delta u_4, \delta y_{4\sigma t})_{L^2(\partial\Omega)}\right|\Big).$$

Taking the integral on [0, t], the inequality of Shwartz and Hypothesis 2.1.1, (ib) on the f_l in RHS, setting $|b_i| \le c_i > 0,\ i = 2,3,4,5,6,7$, to acquire

$$\left\|\overrightarrow{\delta y}_{\sigma t}(t)\right\|^2_{\boldsymbol{L^2(\Omega)}} + \bar{a}\left\|\overrightarrow{\delta y_\sigma}(t)\right\|^2_{\boldsymbol{H^1(\Omega)}} \le$$
$$b\int_0^t\left\|\overrightarrow{\delta y_\sigma}\right\|^2_{\boldsymbol{H^1(\Omega)}}dt +$$
$$2\int_0^t[\, c_2\|\delta y_{2\sigma}\|_{L^2(\Omega)}\|\delta y_{1\sigma t}\|_{L^2(\Omega)} + c_3\|\delta y_{3\sigma}\|_{L^2(\Omega)}\|\delta y_{1\sigma t}\|_{L^2(\Omega)} + c_4\|\delta y_{4\sigma}\|_{L^2(\Omega)}\|\delta y_{1\sigma t}\|_{L^2(\Omega)}$$
$$+ L_1\|\delta y_{1\sigma}\|_{L^2(\Omega)}\|\delta y_{1\sigma t}\|_{L^2(\Omega)} + \|\sigma\delta u_1\|_{L^2(\partial\Omega)}\|\delta y_{1\sigma t}\|_{L^2(\partial\Omega)}]dt +$$
$$2\int_0^t[\, c_2\|\delta y_{1\sigma}\|_{L^2(\Omega)}\|\delta y_{2\sigma t}\|_{L^2(\Omega)} + c_5\|\delta y_{3\sigma}\|_{L^2(\Omega)}\|\delta y_{2\sigma t}\|_{L^2(\Omega)} +$$
$$c_6\|\delta y_{4\sigma}\|_{L^2(\Omega)}\|\delta y_{2\sigma t}\|_{L^2(\Omega)} + L_2\|\delta y_{2\sigma}\|_{L^2(\Omega)}\|\delta y_{2\sigma t}\|_{L^2(\Omega)} + \|\sigma\delta u_2\|_{L^2(\partial\Omega)}\|\delta y_{2\sigma t}\|_{L^2(\partial\Omega)}]dt$$
$$+2\int_0^t[\, c_3\|\delta y_{1\sigma}\|_{L^2(\Omega)}\|\delta y_{3\sigma t}\|_{L^2(\Omega)} + c_5\|\delta y_{2\sigma}\|_{L^2(\Omega)}\|\delta y_{3\sigma t}\|_{L^2(\Omega)} +$$
$$c_7\|\delta y_{4\sigma}\|_{L^2(\Omega)}\|\delta y_{3\sigma t}\|_{L^2(\Omega)} + L_3\|\delta y_{3\sigma}\|_{L^2(\Omega)}\|\delta y_{3\sigma t}\|_{L^2(\Omega)} + \|\sigma\delta u_3\|_{L^2(\partial\Omega)}\|\delta y_{3\sigma t}\|_{L^2(\partial\Omega)}]dt$$
$$+\ 2\int_0^t[\, c_4\|\delta y_{1\sigma}\|_{L^2(\Omega)}\|\delta y_{4\sigma t}\|_{L^2(\Omega)} + c_6\|\delta y_{2\sigma}\|_{L^2(\Omega)}\|\delta y_{4\sigma t}\|_{L^2(\Omega)} +$$
$$c_7\|\delta y_{3\sigma}\|_{L^2(\Omega)}\|\delta y_{4\sigma t}\|_{L^2(\Omega)} + c_6\|\delta y_{2\sigma}\|_{L^2(\Omega)}\|\delta y_{4\sigma t}\|_{L^2(\Omega)} + c_7\|\delta y_{3\sigma}\|_{L^2(\Omega)}\|\delta y_{4\sigma t}\|_{L^2(\Omega)} +$$
$$L_4\|\delta y_{4\sigma}\|_{L^2(\Omega)}\|\delta y_{4\sigma t}\|_{L^2(\Omega)} + \|\sigma\delta u_4\|_{L^2(\partial\Omega)}\|\delta y_{4\sigma t}\|_{L^2(\partial\Omega))}]dt.$$

Utilizing the trace operator, Young's inequality, the relations between the norms and then gathering the same terms and the definition of the vector norm, it yields to

$$\left\|\overrightarrow{\delta y}_{\sigma t}(t)\right\|^2_{\boldsymbol{L^2(\Omega)}} + \bar{a}\left\|\overrightarrow{\delta y_\sigma}(t)\right\|^2_{\boldsymbol{H^1(\Omega)}} \le b\int_0^t\left\|\overrightarrow{\delta y_\sigma}\right\|^2_{\boldsymbol{H^1(\Omega)}}dt + \int_0^t[\bar{c}_1\|\delta y_{1\sigma}\|^2_{L^2(\Omega)} + \bar{c}_2\|\delta y_{2\sigma}\|^2_{L^2(\Omega)}$$
$$+\bar{c}_3\|\delta y_{3\sigma}\|^2_{L^2(\Omega)} + \bar{c}_4\|\delta y_{4\sigma}\|^2_{L^2(\Omega)} + \sigma\|\delta u_1\|^2_{L^2(\partial\Omega)} + c_8\|\delta y_{1\sigma t}\|^2_{L^2(\Omega)} + c_9\|\delta y_{2\sigma t}\|^2_{L^2(\Omega)} +$$
$$c_{10}\|\delta y_{3\sigma t}\|^2_{L^2(\Omega)} + c_{11}\|\delta y_{4\sigma t}\|^2_{L^2(\Omega)}]dt + \sigma\left\|\overrightarrow{\delta u}\,(t)\right\|^2_{L^2(\partial\Omega)}.$$

Where $\bar{c}_1 = c_2 + L_1 + c_3 + c_4$, $\bar{c}_2 = c_2 + L_2 + c_5 + c_6$, $\bar{c}_3 = c_3 + c_5 + c_7 + L_3$, $\bar{c}_4 = c_4 + c_6 + c_7 + L_4$, $c_8 = \bar{c}_1 + \sigma$, $c_9 = \bar{c}_2 + \sigma$, $c_{10} = \bar{c}_3 + \sigma$, $c_{11} = \bar{c}_4 + \sigma$.

Hence,

$$\bar{c}_7\left[\left\|\overrightarrow{\delta y}_{\sigma t}(t)\right\|^2_{\boldsymbol{L^2(\Omega)}} + \left\|\overrightarrow{\delta y_\sigma}(t)\right\|^2_{\boldsymbol{H^1(\Omega)}}\right] \le \sigma\left\|\overrightarrow{\delta \mathrm{u}}\right\|^2_{\boldsymbol{L^2(\partial\Omega)}} + \bar{c}_8\int_0^t\left[\left\|\overrightarrow{\delta y_\sigma}\right\|^2_{\boldsymbol{H^1(\Omega)}} + \left\|\overrightarrow{\delta y}_{\sigma t}\right\|^2_{\boldsymbol{L^2(\Omega)}}\right]dt,$$

where $\bar{c}_5 = \max(c_8, c_9, c_{10}, c_{11})$, $\bar{c}_6 = \max(\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4, b)$, $\bar{c}_7 = \max\,(1, \bar{a})$, $\bar{c}_8 = \max\,(\bar{c}_5, \bar{c}_6)$.

Therefore, with $L = \frac{\bar{c}_8}{\bar{c}_7}$, it becomes

$$\left\|\overrightarrow{\delta y}_{\sigma t}(t)\right\|^2_{\boldsymbol{L^2(\Omega)}} + \left\|\overrightarrow{\delta y_\sigma}(t)\right\|^2_{\boldsymbol{H^1(\Omega)}} \le \frac{\sigma}{\bar{c}_7}\left\|\overrightarrow{\delta u}(t)\right\|^2_{\boldsymbol{L^2(\partial Q)}} + L\int_0^t\left(\left\|\overrightarrow{\delta y}_{\sigma t}\right\|^2_{\boldsymbol{L^2(\Omega)}} + \left\|\overrightarrow{\delta y_\sigma}\right\|^2_{\boldsymbol{H^1(\Omega)}}\right)dt.$$

By employing the BGI, with $K^2 = \frac{\sigma}{\bar{c}_7}e^{TL} > 0$, it yields

$$\left\|\overrightarrow{\delta y}_{\sigma t}(t)\right\|^2_{\boldsymbol{L^2(\Omega)}} + \left\|\overrightarrow{\delta y_\sigma}(t)\right\|^2_{\boldsymbol{H^1(\Omega)}} \le K^2\left\|\overrightarrow{\delta u}(t)\right\|^2_{\boldsymbol{L^2(\partial Q)}},\ \forall t \in I$$

$$\Rightarrow \left\|\overrightarrow{\delta y_\sigma}(t)\right\|^2_{\boldsymbol{H^1(\Omega)}} \le K^2\left\|\overrightarrow{\delta u}(t)\right\|^2_{\boldsymbol{L^2(\partial Q)}} \text{ and } \left\|\overrightarrow{\delta y}_{\sigma t}(t)\right\|^2_{\boldsymbol{L^2(\Omega)}} \le K^2\left\|\overrightarrow{\delta u}(t)\right\|^2_{\boldsymbol{L^2(\partial Q)}},\quad \forall t \in I$$

The other results acquire immediately.

Hypotheses 4.2:
Consider g_l , h_l , $l = 1,2,3,4$ is of Carathéodory Type on $(Q \times \mathbb{R})$ and on $(\partial\Omega \times \mathbb{R})$, respectively and satisfy the following condition with respect to y_l, u_l
$|g_l(x,t,y_l)| \leq \gamma_l(x,t) + c_l(y_l)^2$ and $|h_l(x,t,u_l)| \leq \delta_l(x,t) + d_l(u_l)^2$
Where $y_l , u_l \in \mathbb{R}$, $\gamma_l \in L^1(\text{Q})$, $\delta_l \in L^1(\partial\Omega)$

Lemma 4.3: The OF $G(\vec{u})$ is continuous on $L^2(\partial\Omega)$.
Proof: From Hypotheses 4.2,
$\|g_l(x,t,y_l)\| \leq \gamma_l(x,t) + c_l\|y_l\|^2$ and $\|h_l(x,t,u_l)\| \leq \delta_l(x,t) + d_l\|u_l\|^2$.
Hence, the continuity of $\int_Q g_l(x,t,y_l)dxdt$ and $\int_{\partial Q} h_l(x,t,u_l)d\sigma$ on $L^2(\text{Q})$ & $L^2(\partial\text{Q})$ respectively, $\forall l = 1,2,3,4$ are secured after applying Proposition 2.1 in [14], and thus $G(\vec{u})$ is continuous on $\mathbf{L^2(\partial Q)}$.

Theorem 4.4: If $\vec{U}$ compact, $\vec{W} \neq \emptyset$, $G(\vec{u})$ is convex with resp. to $\vec{u}$ for fixed $(x,t,\vec{y})$. Then there exists a CBOCQV.
Proof: The assumptions on $\vec{U}$ give that $\vec{W}$ is compact weakly, but $\vec{W} \neq \emptyset$, then a minimizing sequence $\{\vec{u}_k\}_{k\in N} \in \vec{W}$, $\forall k$ exists, such that $\lim_{n\to\infty} G(\vec{u}_k) = \inf_{\vec{\vec{u}}_k \in \vec{W}} G(\vec{\bar{u}})$, hence $\|\vec{u}_k\|_{\boldsymbol{L^2(\partial\Omega)}} \leq c$, $\forall k$, and by Alaglou's Theorem $\{\vec{u}_k\}_{k\in N}$ has a subsequence say again $\{\vec{u}_k\}_{k\in N}$ such that $\vec{u}_k \xrightarrow[\boldsymbol{L^2(\partial\Omega)}]{} \vec{u}$.
Then by Theorem 3.1, there exists a sequence $\{\vec{y}_k\}_{k\in N}$ of QSVS corresponding to the sequence of the QBOCV $\{\vec{u}_k\}_{k\in N}$ and that $\|\vec{y}_{kt}\|_{\boldsymbol{L^2(Q)}}$ $\|\vec{y}_k\|_{\boldsymbol{L^\infty(I,L^2(\Omega))}}$ and $\|\vec{y}_k\|_{\boldsymbol{L^2(I,V)}}$ are bounded, hence by Alaglou's Theorem there exists a subsequence of $\{\vec{y}_k\}_{k\in N}$, let for simplicity be $\{\vec{y}_k\}_{k\in N}$ and $\{\vec{y}_{kt}\}_{k\in N}$ such that $\vec{y}_k = \vec{y}_{\vec{u}_k} \xrightarrow[\boldsymbol{L^2(I,V)}]{} \vec{y} = \vec{y}_{\vec{u}}$ and $\vec{y}_{kt} = \vec{y}_{\vec{u}_{kt}} \xrightarrow[\boldsymbol{L^2(Q)}]{} \vec{y}_t = \vec{y}_{\vec{u}_t}$.
But we have $\boldsymbol{L^2(I,V) \subset L^2(Q) \cong (L^2(Q))^* \subset L^2(I,V^*)}$, hence by Aubin's Theorem in [40] there exists a subsequence of $\{\vec{y}_k\}_{k\in N}$, let for simplicity be $\{\vec{y}_k\}_{k\in N}$ such that $\vec{y}_k \xrightarrow[\boldsymbol{L^2(Q)}]{} \vec{y}$.
Now, since $\vec{y}_k$ is the QSVS corresponding to the $\vec{u}_k$, then the WFO in ((8) – (11)) become as

$$\langle y_{1ktt}, v_1\rangle + r_1(t, y_{1k}, v_1) + (a_1 y_{1k}, v_1)_{L^2(\Omega)} - (b_2 y_{2k}, v_1)_{L^2(\Omega)} + (b_3 y_{3k}, v_1)_{L^2(\Omega)} \\ -(b_4 y_{4k}, v_1)_{L^2(\Omega)} = (f_1(y_{1k}), v_1)_{L^2(\Omega)} + (u_{1k}, v_1)_{L^2(\partial\Omega)}. \tag{74}$$

$$\langle y_{2ktt}, v_2\rangle + r_2(t, y_{2k}, v_2) + (a_2 y_{2k}, v_2)_{L^2(\Omega)} + (b_2 y_{1k}, v_2)_{L^2(\Omega)} - (b_5 y_{3k}, v_2)_{L^2(\Omega)} \\ +(b_6 y_{4k}, v_2)_{L^2(\Omega)} = (f_2(y_{2k}), v_2)_{L^2(\Omega)} + (u_{2k}, v_2)_{L^2(\partial\Omega)}. \tag{75}$$

$$\langle y_{3ktt}, v_3\rangle + r_3(t, y_{3k}, v_3) + (a_3 y_{3k}, v_3)_{L^2(\Omega)} - (b_3 y_{1k}, v_3)_{L^2(\Omega)} + (b_5 y_{2k}, v_3)_{L^2(\Omega)} \\ +(b_7 y_{4k}, v_3)_{L^2(\Omega)} = (f_3(y_{3k}), v_3)_{L^2(\Omega)} + (u_{3k}, v_3)_{L^2(\partial\Omega)}. \tag{76}$$

$$\langle y_{4ktt}, v_4\rangle + r_4(t, y_{4k}, v_4) + (a_4 y_{4k}, v_4)_{L^2(\Omega)} + (b_4 y_{1k}, v_4)_{L^2(\Omega)} - (b_6 y_{2k}, v_4)_{L^2(\Omega)} \\ -(b_7 y_{3k}, v_4)_{L^2(\Omega)} = (f_4(y_{4k}), v_4)_{L^2(\Omega)} + (u_{4k}, v_4)_{L^2(\partial\Omega)}. \tag{77}$$

Let $\varphi_l \in C^2[0,T]$, such that $\varphi_l(T) = \dot{\varphi}_l(T) = 0$, $\varphi_l(0) \neq 0$ and $\dot{\varphi}_l(0) \neq 0, \forall l = 1,2,3,4$.
Multiply of ((74) – (77)) by $\varphi_1, \varphi_2, \varphi_3$ & φ_4, respectively such that take the integral on [0, *T*], and applying integrating by parts for the 1st term in each acquired equalities, to bring

$$\int_0^T \frac{d}{dt}(y_{1kt}, v_1)\varphi_1 dt + \int_0^T [r_1(t, y_{1k}, v_1) + (a_1 y_{1k} - b_2 y_{2k} + b_3 y_{3k} \\ -b_4 y_{4k}, v_1)_{L^2(\Omega)}]\varphi_1(t)dt = \int_0^T ((f_1(y_{1k}), v_1)_{L^2(\Omega)} + (u_{1k}, v_1)_{L^2(\partial\Omega)})\varphi_1(t)dt + \\ (y_{1k}(0), v_1)_{L^2(\Omega)}\varphi_1(0). \tag{78}$$

$$\int_0^T \tfrac{d}{dt}\,(y_{2kt}, v_2)\varphi_2(t)\mathrm{d}t + \int_0^T [r_2(t, y_{2k}, v_2) + (a_2 y_{2k} + b_2 y_{1k} - b_5 y_{3k}$$
$$+ b_6 y_{4k}, v_2)_{L^2(\Omega)}]\varphi_2(t)dt = \int_0^T ((f_2(y_{2k}), v_2)_{L^2(\Omega)} + (u_{2k}, v_2)_{L^2(\partial\Omega)})\varphi_2(t)dt +$$
$$(y_{2k}, v_2)_{L^2(\Omega)}\varphi_2(0). \tag{79}$$

$$\int_0^T \tfrac{d}{dt}\,(y_{3kt}, v_3)\varphi_3(t)\mathrm{d}t + \int_0^T [r_3(t, y_{3k}, v_3) + (a_3 y_{3k} - b_3 y_{1k} + b_5 y_{2k}$$
$$+ b_7 y_{4k}, v_3)_{L^2(\Omega)}]\varphi_3(t)dt = \int_0^T ((f_3(y_{3k}), v_3)_{L^2(\Omega)} + (u_{3k}, v_3)_{L^2(\partial\Omega)})\varphi_3(t)dt +$$
$$(y_{3k}, v_3)_{L^2(\Omega)}\varphi_3(0). \tag{80}$$

$$\int_0^T \tfrac{d}{dt}\,(y_{4kt}, v_4)\varphi_4(t)\mathrm{d}t + \int_0^T [r_4(t, y_{4k}, v_4) + (a_4 y_{4k} + b_4 y_{1k} - b_6 y_{2k}$$
$$- b_7 y_{3k}, v_4)_{L^2(\Omega)}]\varphi_4(t)dt = \int_0^T ((f_4(y_{4k}), v_4)_{L^2(\Omega)} + (u_{4k}, v_4)_{L^2(\partial\Omega)})\varphi_4(t)dt +$$
$$(y_{4k}, v_4)_{(L^2(\Omega)}\varphi_4(0). \tag{81}$$

In this point, one can utilize the same manner that was utilized in the proof of Theorem 3.1 to passage the limits. So, in the LHS of ((83)-(86)) and in the first expression of the RHS of them, to passage the limit in the remaining expressions in the RHS of them, since $w_{lk} \xrightarrow[L^2(\partial Q)]{} w_l$ then

$$\int_{\partial\Omega} w_{lk} u_l dxdt \longrightarrow \int_{\partial\Omega} w_l u_l dxdt\ , \forall u_l \in C[\bar{\Omega}], \text{for } l = 1,2,3,4. \tag{82}$$

This convergence is held also for every $v_l \in V$, since $C[\bar{\Omega}]$ is dense in V.
Hence, $\vec{y}$ is the QSVS of the WFO ((8) – (11)), ∀ $v_l \in V$, almost everywhere on *I*.
Finally, to show the passage to the limit in the IC, the same steps that employed in the proof of Theorem 3.1 also they can be utilized here and hence $\vec{y}$ is the QSVS of the WFO of the NLHPBVP.
Beside these, from the proof of Lemma 4.3, $\int_Q g_l(y_{lk})dxdt$is continuous with respect to y_{lk}, $\forall l = 1,2,3,4$, and but $\vec{y}_{lk} \xrightarrow[\mathbf{L^2(Q)}]{} \vec{y}_l$, hence through Proposition 1 in [33], it acquires that

$$\int_Q g_l(y_{lk})dxdt \ \longrightarrow\ \int_Q g_l(y_l)dxdt. \tag{83}$$

From the continuity of $h_l(u_l)$ by Lemma 4.3 and $h_l(u_l)$ has the convex property, its result is weakly lower semi continuous (WLSC) with respect to u_l , $\forall l = 1,2,3,4$, then

$$\int_Q g_l(y_l)dxdt + \int_{\partial Q} h_l(u_l)d\sigma \le \lim_{k\to\infty}\inf \int_{\partial Q} h_l(u_{lk})d\sigma + \int_Q g_l(y_{lk})dxdt$$
$$= \lim_{k\to\infty}\inf \int_{\partial Q} h_l(u_{lk})d\sigma + \lim_{k\to\infty} \int_Q (g_l(y_l) - g_l(y_{lk}))dxdt + \int_Q g_l(y_{lk})dxdt$$
$$= \lim_{k\to\infty}\inf \int_{\partial Q} h_l(u_{lk})d\sigma + \lim_{k\to\infty}\inf \int_Q g_l(y_{lk})dxdt, \text{ (by (83))}$$

i.e., $G(\vec{u}) \le \lim_{k\to\infty}\inf G(\vec{u}_k) = \lim_{k\to\infty} G(\vec{u}_k) = \inf_{\vec{u}\in\vec{U}} G(\vec{\vec{u}}_k) \Rightarrow G(\vec{u}) = \min_{\vec{u}_k\in\vec{W}} G(\vec{\vec{u}})$.
Therefore, $\vec{u}$ is a CBCOQV.

5. Conclusions

From the study of the QBOCP controlling by QNLHS one can conclude that; the theorem of existence a unique QSVS for the WFO of the QNLHS is proved in infinite dimensional space via the MG and the help of the theorem of Aubin with given CBCQV. The continuity of the operator of Lipchitz between the QSVS and the conforming CBOCQV is demonstrated. The theorem of existence a CBOCQV which is minimized the OF and is controlled by the QNLHS is demonstrated in infinite dimensional space.

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