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Equilibrium points and their stability of food - chain prey - predator model involving fear and toxin

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Abstract

This paper presents and analyses a food chain eco-toxicant model involving one prey and two predators, incorporating a fear with linear and nonlinear harvesting, utilizing two distinct functional responses (Lotka-Volterra and Sokol-Howell). Prey species are directly exposed to environmental toxins, while first and second predators encounter these toxins through consumption. The model's boundedness and uniqueness have been established. All the equilibrium points of this model have been found. All requisite conditions for achieving the local stability of equilibrium points have been identified, leading to in the conclusion that all points E_i, where $i = 0, 1, \dots 5$, are locally stable. Additionally, a global stability study has been conducted using appropriate Lyapunov functions. Ultimately, numerical simulations for a one set of parameters and varying initial conditions have been conducted to validate the analytical results and to assess the effect of fear, toxins, linear and nonlinear harvesting on the dynamics of the proposed model.

Keywords: Prey-predator, Toxin, Sokol-Howell, Fear, Harvesting, Stability.

نقاط الأتزان واستقراريتها لنموذج سلسلة غذائية لفريسة ومفترس الذي يتضمن الخوف والسمية

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الخلاصة

في هذا البحث تم اقتراح وتحليل نموذج رياضي بيئي–سمي لسلسلة غذائية تتضمن فريسة واثنين من المفترسات, اضافة الى الخوف و الحصاد الخطي وغير الخطي, بأستخدام نوعين مختلفين من دوال الأستجابة الوظيفية (لوتكا فوليتيرا و سوكول هاويل). تتعرض انواع الفرائس بشكل مباشر الى السموم البيئية بينما تواجه المفترسات الاولى و الثانية هذه السموم من خلال الاستهلاك, وجود و وحدانية و قيود الحل للنموذج نوقشت, جميع نقاط الأتزان للنظام وجدت, تم تحديد جميع الشروط اللازمة لتحقيق الاستقرارية المحلية وتم استنتاج ان جميع النقاط مستقرة محليا. اضافة الى ذالك, تم دراسة الاستقرارية الشاملة لجميع نقاط الأتزان بأستخدام دوال

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ليبانوف مناسبة. في النهاية, تم اجراء المحاكاة العددية لمجموعة واحدة من قيم المعلمات وشروط ابتدائية مختلفة لتقييم تأثير الخوف, السموم و الحصاد الخطي وغير الخطي على ديناميكيات النموذج المقترح.

1. Introduction

An ecological community is characterized as a collection of organisms that are either currently or potentially interacting within their common environment. A community is interconnected by the network of influences that species exert on one another, this perspective posits that the impact on one species invariably influences many other species, embodying the concept of the "balance of nature". Knowledge of communities have been developed by analyzing two-way and subsequently multi-way interactions among pairs or groups of species [1]. The relationship that sustains health is termed prey-predator. The interactions among these key components pertain to the impact of predators on prey, as well as the flow of resources and energy, which affect ecosystem structure [2].

Modeling is essential for studying applied mathematics and its influence on biological and environmental processes, prompting biologists to consult mathematicians for solutions to complex equations. As a result, many mathematical models have been proposed and studied [3 - 6].

Lotka and Volterra's works intersected in the examination of predator-prey dynamics. The issue was addressed by Lotka in 1920 [7] and by Volterra in 1926 [8], with both concluding that the interaction between the two species would result in periodic oscillations in their numbers. Volterra recognized Lotka's precedence, yet he noted the distinctions in their respective papers. They corresponded with mutual respect. Regarding predator-prey dynamics, Lotka's priority was unequivocally established, and the equations featuring periodic solutions are referred to as Lotka-Volterra equations [9]. A multitude of researchers have studied and developed mathematical models, including the Lotka-Volterra functional response [10 - 12].

The dynamics of mathematical models representing predator-prey interactions are mostly dependent on the functional response. Holling's type I, type II, and type III functional responses are the most prevalent. Holling type IV has been employed in numerous investigations [13]. This study examines a prey-predator framework utilizing a Sokol-Howell functional response, which is a modification of Holling type III. The Sokol-Howell model posits that predator consumption rates increase with prey density but eventually stabilize due to constraints, such as the time required to locate and consume each prey item. Predators cannot continuously increase their consumption rates, even when prey is abundant, see [14] and [15].

Toxicology is a scientific discipline dedicated to the examination of harmful interactions between chemicals and biological systems. It encompasses various categories of toxic substances, including phytotoxins, zootoxins, bacterial toxins, and synthetic compounds. Given the need for early humans to differentiate between consumable substances and those that posed health risks, toxicology can be regarded as one of the oldest scientific fields [16]. There is a growing interest among researchers in the field of toxicology, leading to the development of numerous mathematical models aimed at improving our understanding and analysis of this critical area. For instance, the research conducted by Hallam et al. , [17] focused on a significant toxicant-population model. This study investigated the ecological uptake rates of toxicants and their impact on population dynamics, as well as the detrimental effects of toxic substances in relation to the body weight of the population. Additionally, other studies have explored various models within toxic ecosystems, further contributing to this important field of inquiry, for example [18] and [19].

Predators always influence prey populations through direct, indirect, or both types of interactions, with the direct impact manifesting in their predatory behavior, which involves capturing and killing prey. At the same time, the fear of the threat of predation can be considered an indirect effect of predators on prey. Certain theoretical ecologists and biologists have acknowledged that a prey-predator model must include not only direct mortality but also the element of fear [20]. Pal et al., [21] have been investigated the impact of fear on the stability of food chain models. Researchers have developed various models that explore the impact of fear across multiple dimensions. Dubey et al., [22] proposed and studied a prey–predator model with fear in prey due to predator and anti-predator behavior by prey against the predator with fear response delay and gestation delay. Also, Pal et al., [23] and Pratama [24] investigated the influence of fear on the growth of both prey and predator.

The effects of harvesting on prey-predator models have been the subject of extensive investigation [25]. Chu et al., [26] examined fisheries model that includes prey harvesting, considers price fluctuations, and accounts for marine protected zones. Assume that price alterations occur at a substantially accelerated rate compared to other variables, such as predation and population increase. Many studies used the same concept to study the harvest in prey – predator models, see [27 - 30]

This paper proposed and studied a food chain model that includes prey, first and second predators with Lotka-Volterra and Sokol-Howell functional responses to describe the feeding processes. Our study includes the effect of fear, toxins, linear and nonlinear harvesting on the dynamics of solutions.

2. Mathematical model

This paper proposed a model of (prey, first predator and second predator) with Sokol-Howell functional response for analysis. The total population densities at time T represented by $P(T), P_1(T), P_2(T)$, respectively.

$$\frac{dP}{dT} = \frac{a_1 P}{1+ m_1 P_1} - b_1 P^2 - c_1 P P_1 - h_1 P - \delta_1 P^3$$
$$\frac{dP_1}{dT} = \frac{a_2 P_1}{1+ m_2 P_2} - b_2 P_1^2 - \frac{c_2 P_1 P_2}{\alpha + \gamma P_1^2} + l_1 P P_1 - \delta_2 P_1^2 - h_2 P_1^2 - d_1 P_1, \qquad (1)$$
$$\frac{dP_2}{dT} = \frac{l_2 P_1 P_2}{\alpha + \gamma P_1^2} - \delta_3 P_2 - h_3 P_2^2 - d_2 P_2 .$$

With the following initial condition $P(0) \geq 0$, $P_1(0) \geq 0$ and $P_2(0) \geq 0$. While the biological meaning of the parameters in system (1) illustrated in Table 1:

Table 1: Biological description of system's (1) parameters

Parameters	**Biological description**
$a_i > 0, i = 1,2$	Intrinsic growth rates of the prey and first predator populations, respectively
$b_i > 0, i = 1,2$	Internal competition rates of the prey and first predator, respectively
$m_i > 0, i = 1,2$	Fear rates of the prey and the first predator, respectively
$c_i > 0, i = 1,2$	Attack rates of the prey and first predator, respectively
$l_i > 0, i = 1,2$	Food transfer rates for first and second predators, respectively
$\alpha > 0.$	Half-saturation constant
$\gamma > 0.$	Inverse measure of inhibitory effect
$\delta_i > 0, i = 1,2,3$	Rates of toxicity for the prey, first predator and second predator, respectively
$h_i > 0, i = 1,2,3$	Harvesting rates of prey, first predator and second predator, respectively
$d_i > 0, i = 1,2$	Natural death rates of first predator and second predator, respectively

It is simple to check that all of system (1) interaction functions with partial derivatives are continuous on R_+^3 for the variables P, P_1 and P_2. Therefore, they are Lipschitzian functions, that is system (1) has a unique solution depending on the initial conditions. The next theorem establishes the boundedness of the system.

Theorem 2.1: All solutions of system (1) that start in R_+^3 are uniformly bounded.

Proof: Let $\big(P(t), P_1(t), P_2(t)\big)$ be any solution of system (1) with non-negative initial conditions $\big(P(0), P_1(0), P_2(0)\big) \in R_+^3$.

Assume that $V(t) = P(t) + P_1(t) + P_2(t)$, then by drive $V(t)$ along the solution of system (1), imply

$$\frac{dV}{dt} = \frac{a_1 P}{1+m_1 P_1} - b_1 P^2 - (c_1 - l_1) P P_1 - h_1 P - \delta_1 P^3 + \frac{a_2 P_1}{1+m_2 P_2} - b_2 P_1^2$$
$$- (c_2 - l_2)\frac{P_1 P_2}{\alpha + \gamma P_1^2} - \delta_2 P_1^2 - d_1 P_1 - h_2 P_1^2 - \delta_3 P_2 - h_3 P_2^2 - d_2 P_2 .$$

Now, by the biological facts, always $l_i < c_i$, $i = 1,2$ hence it is obtained that

$$\frac{dV}{dt} \le a_1 P\left(1 - \frac{b_1}{a_1} P\right) + a_2 P_1\left(1 - \frac{b_2}{a_2} P_1\right) - \big(h_1 P + d_1 P_1 + (\delta_3 + d_2) P_2\big).$$

Thus,

$$\frac{dV}{dt} \le a_1 P\left(1 - \frac{b_1}{a_1} P\right) + a_2 P_1\left(1 - \frac{b_2}{a_2} P_1\right) - D_1 V,$$

where $D_1 = \min\ \{h_1, d_1, \delta_3 + d_2\}$.

Now, since the logistic function $f(P) = a_1 P\left(1 - \frac{b_1}{a_1} P\right)$ is bounded above by the constant $\frac{a_1^2}{4b_1}$.

Also, since the logistic function $g(P_1) = a_2 P_1\left(1 - \frac{b_2}{a_2} P_1\right)$ is bounded above by the constant $\frac{a_2^2}{4b_2}$, so it is observed that

$$\frac{dV}{dt} + D_1 V \le B \ , \text{ where } \mathrm{B} = \frac{1}{4}\left(\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2}\right).$$

Then by solving the above differential inequality for the initial value $V(0) = V_0$, it is obtained that

$$V(t) \le \frac{B}{D_1} + \left(V_0 - \frac{B}{D_1}\right) e^{-D_1 t} .$$

Then,

$$\lim_{t\to\infty} V(t) \le \frac{B}{D_1} .$$

So , $0 \le V(t) \le \frac{B}{D_1}$, that is the solutions are uniformly bounded.

3. Existence of equilibrium points (EPs)

System (1) has six equilibrium points which are summarized below:

- The (EP) $E_0 = (0,0,0)$ always exists.
- The (EP) $E_1 = (\breve{P}, 0, 0)$, where $\breve{P}$ is a unique positive solution of the equation

$$\delta_1 P^2 + b_1 P - (a_1 - h_1) = 0 \tag{2}$$

exists if the following condition holds

$$a_1 > h_1. \tag{3}$$

- The (EP) $E_2 = (0, \bar{P}_1, 0)$ where $\bar{P}_1 = \frac{a_2 - d_1}{(b_2 + \delta_2 + h_2)}$ exists if the following condition holds

$$a_2 > d_1. \tag{4}$$

- The (EP) $E_3 = (\ddot{P}, \ddot{P}_1, 0)$ exists uniquely in R_+^2 if the following system:

$$\frac{a_1}{1+m_1P_1} - b_1P - c_1P_1 - \delta_1P^2 - h_1 = 0, \tag{5.1}$$

$$a_2 - b_2P_1 + l_1P - \delta_2P_1 - d_1 - h_2P_1 = 0, \tag{5.2}$$

has a positive solution
From Equation (5.2) we have

$$P_1 = \frac{l_1P + (a_2 - d_1)}{(b_2 + \delta_2 + h_2)}. \tag{5.3}$$

Now, by substituting Equation (5.3) in Equation (5.1), we obtain the following equation

$$\sigma_1P^3 + \sigma_2P^2 + \sigma_3P + \sigma_4 0, \tag{5.4}$$

where:

$$\sigma_1 = \delta_1m_1l_1 (b_2 + \delta_2 + h_2),$$

$$\sigma_2 = m_1b_1l_1 (b_2 + \delta_2 + h_2) + m_1c_1l_1^2 + \delta_1b_2 + \delta_1\delta_2 + \delta_1h_2 + \delta_1m_1((a_2 - d_1)(b_2 + \delta_2 + h_2)),$$

$$\sigma_3 = (b_2 + \delta_2 + h_2)[b_1 (b_2 + \delta_2 + h_2) + m_1b_1(a_2 - d_1) + m_1l_1h_1 + c_1l_1] + 2c_1m_1l_1(a_2 - d_1),$$

$$\sigma_4 = -a_1(b_2^2 + \delta_2^2 + h_2^2) + (a_2 - d_1)[(b_2 + h_2 + \delta_2)(c_1 + h_1m_1) + c_1m_1(a_2 - d_1)] + h_1(b_2^2 + \delta_2^2 + h_2^2),$$

Clearly, by Descartes's rule, Equation (5.4) has a unique positive root, if in addition to condition (4) the next condition hold

$$a_1(b_2^2 + \delta_2^2 + h_2^2) > N \tag{5.5}$$

Where,

$N = (a_2 - d_1)[(b_2 + h_2 + \delta_2)(c_1 + h_1m_1) + c_1m_1(a_2 - d_1)] + h_1(b_2^2 + \delta_2^2 + h_2^2)$,

So, the (EP) $E_3 = (\ddot{P}, \ddot{P}_1, 0)$ where $\ddot{P}_1 = P_1(\ddot{P})$ exists.

- The (EP) $E_4 = (0, \tilde{P}_1, \tilde{P}_2)$ exists uniquely in R_+^2 if the following system:

$$\frac{a_2}{1+m_2P_2} - b_2P_1 - \frac{c_2P_2}{\alpha+\gamma P_1^2} - \delta_2P_1 - h_2P_1 - d_1 = 0, \tag{6.1}$$

$$\frac{l_2P_1}{\alpha+\gamma P_1^2} - \delta_3 - h_3P_2 - d_2 = 0, \tag{6.2}$$

has a positive solution
from Equation (6.2), we have

$$P_2 = \frac{l_2P_1}{h_3(\alpha+\gamma P_1^2)} - \frac{(\delta_3+d_2)}{h_3}. \tag{6.3}$$

Now, by substituting Equation (6.3) in Equation (6.1) we have

$$¥_1P_1^7 + ¥_2P_1^6 + ¥_3P_1^5 + ¥_4P_1^4 + ¥_5P_1^3 + ¥_6P_1^2 + ¥_7P_1 + ¥_8 = 0, \tag{6.4}$$

where:

$¥_1 = \gamma^3 h_3(b_2+\delta_2+h_2)[m_2(\delta_3+d_2)-h_3]$,
$¥_2 = \gamma^2 h_3[\gamma(a_2-d_1)h_3 - m_2 l_2(b_2+\delta_2+h_2)+\gamma d_1 m_2(\delta_3+d_2)]$,
$¥_3 = \gamma^2 h_3[3\alpha(b_2+\delta_2+h_2)(m_2(\delta_3+d_2)-h_3)-d_1 m_2 l_2]$,

$¥_4 = \alpha\gamma^2 h_3[3(a_2-d_1)h_3 - 2m_2 l_2(b_2+\delta_2+h_2)]+\gamma^2 h_3(c_2+\alpha d_1 m_2)[\delta_3+d_2]+$
$\gamma^2 m_2[c_2(\delta_3^2-d_2^2)+2\alpha d_1\delta_3 h_3]+2\gamma^2 m_2[d_2(\alpha d_1 h_3-c_2\delta_3)]$,

$¥_5 = 3\alpha^2\gamma h_3(b_2+\delta_2+h_2)[m_2(\delta_3+d_2)-h_3]+\gamma l_2[(2c_2(\delta_3+d_2)-\alpha d_1 h_3)-(c_2+\alpha d_1 m_2)h_3]$
,

$¥_6 = \alpha^2 h_3[3\gamma(a_2-d_1)h_3 - m_2 l_2(b_2+\delta_2+h_2)]+2\alpha\gamma(c_2+\alpha d_1 m_2)h_3[\delta_3+d_2]-$
$\alpha\gamma m_2(\delta_3+d_2)[2c_2(\delta_3+d_2)-\alpha d_1 h_3]-c_2 m_2 l_2^2$,

$¥_7 = \alpha\left[\alpha^2 h_3(b_2+\delta_2+h_2)[m_2(\delta_3+d_2)-h_3]+l_2[2c_2 m_2(\delta_3+d_2)-(c_2+\alpha d_1 m_2)h_3]\right]$,

$¥_8 = \alpha^2\left[h_3[\alpha(a_2-d_1)h_3+(c_2+\alpha d_1 m_2)(\delta_3+d_2)]-c_2 m_2[\delta_3(\delta_3+2d_2)-d_2^2]\right]$.

Clearly, by Descartes's rule, Equation (6.4) has a unique positive root, if condition (4) and the next conditions hold

$$\text{Max}\left\{\frac{c_2\delta_3}{\alpha d_1},\frac{m_2 l_2(b_2+\delta_2+h_2)}{3\gamma(a_2-d_1)}\right\} < h_3 < \min\left\{m_2(\delta_3+d_2),\ \frac{2c_2(\delta_3+d_2)}{\alpha d_1}\right\}, \quad (6.5)$$

$$h_3(c_2+\alpha d_1 m_2) < \min\{2c_2(\delta_3+d_2)-\alpha d_1 h_3, 2c_2 m_2(\delta_3+d_2)\}, \quad (6.6)$$

$$m_2 l_2(b_2+\delta_2+h_2) < \min\left\{\gamma(a_2-d_1)h_3+d_1 m_2(\delta_3+d_2),\frac{3(a_2-d_1)h_3}{2}\right\}, \quad (6.7)$$

$$\alpha\gamma m_2(\delta_3+d_2)[2c_2(\delta_3+d_2)-\alpha d_1 h_3]+c_2 m_2 l_2^2 < \alpha^2 h_3[3\gamma(a_2-d_1)h_3 - m_2 l_2(b_2+\delta_2+h_2)]+2\alpha\gamma(c_2+\alpha d_1 m_2)h_3[\delta_3+d_2], \quad (6.8)$$

$$d_2^2 < \min\{\delta_3^2,\ \delta_3(\delta_3+2d_2)\}, \quad (6.9)$$

$$\frac{\alpha(a_2-d_1)h_3^2+h_3(c_2+\alpha d_1 m_2)(\delta_3+d_2)}{c_2(\delta_3(\delta_3+2d_2)-d_2^2)} < m_2 < \frac{3\alpha(b_2+\delta_2+h_2)(m_2(\delta_3+d_2)-h_3)}{d_1 l_2}. \quad (6.10)$$

So, the (EP) $E_4 = (0, \tilde{P}_1, \tilde{P}_2)$, where $\tilde{P}_2 = P_2(\tilde{P}_1)$ if in addition to the above conditions, the next condition hold

$$\frac{l_2\tilde{P}_1}{h_3(\alpha+\gamma\tilde{P}_1^2)} > \frac{(\delta_3+d_2)}{h_3}. \quad (6.11)$$

- The (EP) $E_5 = (P^*, P_1^*, P_2^*)$ exists uniquely in R_+^3 if the following system:

$$\frac{a_1}{1+m_1 P_1} - b_1 P - c_1 P_1 - \delta_1 P^2 - h_1 = 0, \quad (7.1)$$

$$\frac{a_2}{1+m_2 P_2} - b_2 P_1 + l_1 P - \frac{c_2 P_2}{\alpha+\gamma P_1^2} - \delta_2 P_1 - h_2 P_1 - d_1 = 0, \quad (7.2)$$

$$\frac{l_2 P_1}{\alpha+\gamma P_1^2} - h_3 P_2 - \delta_3 - d_2 = 0, \quad (7.3)$$

has a positive solution.

From Equation (7.1), we have

$$K_1(P,P_1)=\frac{a_1}{1+m_1P_1}-b_1P-c_1P_1-h_1-\delta_1P^2=0. \tag{7.4}$$

Next from Equation (7.4), we observe that, when $P_1\to 0$, then $P\to P^{*\prime}$, where $P^{*\prime}$ is a unique positive root of the next equation

$$f(P)=\delta_1P^2+b_1P-(a_1-h_1)=0. \tag{7.5}$$

Moreover, from Equation (7.4) we have $\frac{dP}{dP_1}=-\left(\left(\frac{\partial K_1}{\partial P_1}\right)/\left(\frac{\partial K_1}{\partial P}\right)\right)$. So, $\frac{dP}{dP_1}>0$ if either

$$\frac{\partial K_1}{\partial P_1}>0,\frac{\partial K_1}{\partial P}<0 \text{ or } \frac{\partial K_1}{\partial P_1}<0,\ \frac{\partial K_1}{\partial P}>0. \tag{7.6}$$

Now, from Equation (7.3), we have

$$P_2=\frac{1}{h_3}\left[\frac{l_2P_1-(\delta_3+d_2)(\alpha+\gamma P_1^2)}{(\alpha+\gamma P_1^2)}\right] \tag{7.7}$$

Subtitling (7.7) in Equation (7.2), we get

$$K_2(P,P_1)=\frac{a_2}{1+\frac{m_2}{h_3}\left[\frac{l_2P_1-(\delta_3+d_2)(\alpha+\gamma P_1^2)}{(\alpha+\gamma P_1^2)}\right]}-b_2P_1+l_1P-\frac{c_2\left(l_2P_1-(\delta_3+d_2)(\alpha+\gamma P_1^2)\right)}{h_3\left(\alpha+\gamma P_1^2\right)^2}-\delta_2P_1-d_1-h_2P_1=0, \tag{7.8}$$

Further, from Equation (7.8) we observe that, when $P_1\to 0$, then $P\to P^{*\prime\prime}$, where

$$P^{*\prime\prime}=\frac{1}{l_1}\left[\frac{-a_2}{1-\frac{m_2}{h_3}(\delta_3+d_2)}+\frac{c_2(\delta_3+d_2)}{h_3}\right]+\frac{d_1}{l_1}. \tag{7.9}$$

Moreover, from Equation (7.8) we have $\frac{dP}{dP_1}=-\left(\left(\frac{\partial K_2}{\partial P_1}\right)/\left(\frac{\partial K_2}{\partial P}\right)\right)$. So, $\frac{dP}{dP_1}<0$ if either

$$\frac{\partial K_2}{\partial P_1}<0,\frac{\partial K_2}{\partial P}<0 \text{ or, } \frac{\partial K_2}{\partial P_1}>0,\frac{\partial K_2}{\partial P}>0, \tag{7.10}$$

Then the isocline (7.4) intersect the isocline (7.8) at a unique positive point (P^*,P_1^*) in the int. R_+^3 of PP_1 −space.

So, the (EP) $E_5=(P^*,P_1^*,P_2^*)$ exists where $P_2^*=P_2(\ P_1^*)$ if condition (3) and the next conditions hold:

$$l_2P_1^*>(\delta_3+d_2)(\alpha+\gamma P_1^2), \tag{7.11}$$

$$\frac{m_2}{h_3}(\delta_3+d_2)>1, \tag{7.12}$$

$$P^{*\prime}<P^{*\prime\prime}. \tag{7.13}$$

4. Local stability (LS)

In this section, the local stability (LS) have been discussed at each equilibrium point by obtaining the Jacobian matrix (JM), and determining the eigenvalues of system (1) at all points.

Note that, we denote the eigenvalues of (JM) J_i ; $i = 0, 1, \dots 5$ that characterize the dynamics in the P, P_1, P_2 directions as λ_{iP}, λ_{iP_1}, and λ_{iP_2}, respectively. The Jacobian matrix $J(P, P_1, P_2)$ of the system (1) at each point can be expressed as $J = [\mho_{ij}]_{3\times 3}$, .

$$\mho_{11} = \frac{a_1}{1+m_1 P_1} - 2b_1 P - c_1 P_1 - 3\delta_1 P^2 - h_1,$$
$$\mho_{12} = \frac{-a_1 m_1 P}{(1+m_1 P_1)^2} - c_1 P \ , \quad \mho_{13} = 0\,, \quad \mho_{21} = l_1 P_1,$$
$$\mho_{22} = \frac{a_2}{1+m_2 P_2} - 2b_2 P_1 + l_1 P - \frac{c_2 P_2(\alpha - \gamma P_1^2)}{(\alpha+\gamma P_1^2)^2} - 2\delta_2 P_1 - d_1 - 2h_2 P_1,$$
$$\mho_{23} = \frac{-a_2 m_2 P_1}{(1+m_2 P_2)^2} - \frac{c_2 P_1}{\alpha+\gamma P_1^2}, \quad \mho_{31} = 0, \qquad \mho_{32} = \frac{l_2 P_2(\alpha - \gamma P_1^2)}{(\alpha+\gamma P_1^2)^2},$$
$$\mho_{33} = \frac{l_2 P_1}{\alpha+\gamma P_1^2} - \delta_3 - 2h_3 P_2 - d_2.$$

- The (JM) of system (1) at $E_0 = (0,0,0)$ can be written as

$$J_0 = J(E_0) = \begin{bmatrix} a_1 - h_1 & 0 & 0 \\ 0 & a_2 - d_1 & 0 \\ 0 & 0 & -(\delta_3 + d_2) \end{bmatrix}. \tag{8}$$

So, the characteristic equation of J_0 is

$$(a_1 - h_1 - \lambda)(a_2 - d_1 - \lambda)(-(\delta_3 + d_2) - \lambda) = 0, \tag{9}$$

and hence the eigenvalues of J_0 are

$$\lambda_{0P} = a_1 - h_1\,, \quad \lambda_{0P_1} = a_2 - d_1 \quad \text{and} \quad \lambda_{0P_2} = -(\delta_3 + d_2) < 0\,.$$

Hence E_0 is a locally asymptotically stable, by reversing conditions (3) and (4) and it is unstable otherwise.

- The (JM) of system (1) at $E_1 = (\breve{P}, 0, 0)$ can be written as

$$J_1 = J(E_1) = \begin{bmatrix} a_1 - [\breve{P}(2b_1 + 3\delta_1 \breve{P}) + h_1] & -\breve{P}(a_1 m_1 + c_1) & 0 \\ 0 & a_2 + l_1 \breve{P} - d_1 & 0 \\ 0 & 0 & -(\delta_3 + d_2) \end{bmatrix}. \tag{10}$$

So, the characteristic equation of J_1 is

$$(a_1 - [\breve{P}(2b_1 + 3\delta_1 \breve{P}) + h_1] - \lambda)(a_2 + l_1 \breve{P} - d_1 - \lambda)(-(\delta_3 + d_2) - \lambda) = 0, \tag{11}$$

and hence the eigenvalues of J_1 are:

$$\lambda_{1P} = a_1 - [\breve{P}(2b_1 + 3\delta_1 \breve{P}) + h_1],$$
$$\lambda_{1P_1} = a_2 + l_1 \breve{P} - d_1 \quad \text{and} \quad \lambda_{1P_2} = -(\delta_3 + d_2) < 0.$$

Hence E_1 is a locally asymptotically stable provided that

$$a_1 < \breve{P}(2b_1 + 3\delta_1 \breve{P}) + h_1\,, \tag{12}$$

$$a_2 + l_1 \breve{P} < d_1. \tag{13}$$

It is unstable otherwise.

- The (JM) of system (1) at $E_2 = (0\,, \bar{P}_1, 0\,)$ where $\bar{P}_1 = \frac{a_2 - d_1}{(b_2+\delta_2+h_2)}$ can be written as

$$J_2 = J(E_2) = [u_{ij}]_{3\times 3}$$
$$= \begin{bmatrix} \frac{a_1}{1+m_1\bar{P}_1} - (c_1\bar{P}_1 + h_1) & 0 & 0 \\ l_1\bar{P}_1 & a_2 - [2\bar{P}_1(b_2+\delta_2+h_2)+d_1] & -a_2m_2\bar{P}_1 - \frac{c_2\bar{P}_1}{\alpha+\gamma\bar{P}_1^2} \\ 0 & 0 & \frac{l_2\bar{P}_1}{\alpha+\gamma\bar{P}_1^2} - (\delta_3 + d_2) \end{bmatrix} \qquad (14)$$

where:

$$u_{11} = \frac{a_1}{1+m_1\bar{P}_1} - (c_1\bar{P}_1 + h_1), \quad u_{12} = 0\,, \quad u_{13} = 0$$
$$u_{21} = l_1\bar{P}_1\,, \quad u_{22} = a_2 - [2\bar{P}_1(b_2+\delta_2+h_2)+d_1]\,,$$
$$u_{23} = -a_2m_2\bar{P}_1 - \frac{c_2\bar{P}_1}{\alpha+\gamma\bar{P}_1^2}\,, \quad u_{31} = 0\,, \quad u_{32} = 0\,, \quad u_{33} = \frac{l_2\bar{P}_1}{\alpha+\gamma\bar{P}_1^2} - (\delta_3 + d_2).$$

So, the characteristic equation of J_2 is

$$\left(\frac{a_1}{1+m_1\bar{P}_1} - (c_1\bar{P}_1 + h_1) - \lambda\right)\left[(a_2 - [2\bar{P}_1(b_2+\delta_2+h_2)+d_1] - \lambda)\left(\frac{l_2\bar{P}_1}{\alpha+\gamma\bar{P}_1^2} - (\delta_3 + d_2) - \lambda\right)\right] = 0. \qquad (15)$$

And hence the eigenvalues of J_2 are:

$$\lambda_{2P} = \frac{a_1}{1+m_1\bar{P}_1} - (c_1\bar{P}_1 + h_1)\,,$$
$$\lambda_{2P_1} = a_2 - [2\bar{P}_1(b_2+\delta_2+h_2)+d_1] \quad \text{and} \quad \lambda_{2P_2} = \frac{l_2\bar{P}_1}{\alpha+\gamma\bar{P}_1^2} - (\delta_3 + d_2).$$

Hence E_2 is locally asymptotically stable under the following conditions.

$$\frac{a_1}{1+m_1\bar{P}_1} < c_1\bar{P}_1 + h_1, \qquad (16)$$
$$a_2 < 2\bar{P}_1(b_2+\delta_2+h_2)+d_1, \qquad (17)$$
$$\frac{l_2\bar{P}_1}{\alpha+\gamma\bar{P}_1^2} < (\delta_3 + d_2). \qquad (18)$$

It is unstable otherwise.

- The (JM) of system (1) at $E_3 = (\ddot{P}, \ddot{P}_1\,, 0\,)$ can be written as

$$J_3 = J(E_3) = [\vartheta_{ij}]_{3\times 3} \qquad (19)$$

where:

$$\vartheta_{11} = \frac{a_1}{1+m_1\ddot{P}_1} - [\ddot{P}(2b_1 + 3\delta_1\ddot{P}) + c_1\ddot{P}_1 + h_1], \quad \vartheta_{12} = \frac{-a_1m_1\ddot{P}}{(1+m_1\ddot{P}_1)^2} - c_1\ddot{P}, \quad \vartheta_{13} = 0,$$

$$\vartheta_{21} = l_1\ddot{P}_1\,, \quad \vartheta_{22} = a_2 + l_1\ddot{P} - [2\ddot{P}_1(b_2+\delta_2+h_2)+d_1], \quad \vartheta_{23} = -a_2m_2\ddot{P}_1 - \frac{c_2\ddot{P}_1}{\alpha+\gamma\ddot{P}_1^2}\,,$$
$$\vartheta_{31} = 0, \quad \vartheta_{32} = 0\,, \quad \vartheta_{33} = \frac{l_2\ddot{P}_1}{\alpha+\gamma\ddot{P}_1^2} - (\delta_3 + d_2).$$

So, the characteristic equation of J_3 is:

$$(\vartheta_{33} - \lambda)[\lambda^2 - (\vartheta_{11} + \vartheta_{22})\lambda + \vartheta_{11}\vartheta_{22} - \vartheta_{12}\vartheta_{21}] = 0. \qquad (20)$$

So, either $(\vartheta_{33} - \lambda) = 0$,

which gives $\lambda_{3P_2} = \frac{l_2 \ddot{P}_1}{\alpha+\gamma \ddot{P}_1^2} - (\delta_3 + d_2)$,
or, $[\lambda^2 - (\vartheta_{11} + \vartheta_{22})\lambda + \vartheta_{11}\vartheta_{22} - \vartheta_{12}\vartheta_{21}] = 0$,
which gives

$$\lambda_{3P} + \lambda_{3P_1} = \left(\frac{a_1}{1+m_1\ddot{P}_1} - [\ddot{P}(2b_1 + 3\delta_1\ddot{P}) + c_1\ddot{P}_1 + h_1]\right) + \left(a_2 + l_1\ddot{P} - [2\ddot{P}_1(b_2 + \delta_2 + h_2) + d_1]\right),$$

$$\lambda_{3P} \cdot \lambda_{3P_1} = \left(\frac{a_1}{1+m_1\ddot{P}_1} - [\ddot{P}(2b_1 + 3\delta_1\ddot{P}) + c_1\ddot{P}_1 + h_1]\right) \left(a_2 + l_1\ddot{P} - [2\ddot{P}_1(b_2 + \delta_2 + h_2) + d_1]\right) - \left(\frac{-a_1 m_1 \ddot{P}}{(1+m_1\ddot{P}_1)^2} - c_1\ddot{P}\right)(l_1\ddot{P}_1).$$

Hence E_3 is locally asymptotically stable under the following conditions

$$\frac{l_2\ddot{P}_1}{\alpha+\gamma\ddot{P}_1^2} < (\delta_3 + d_2), \tag{21}$$

$$\frac{a_1}{1+m_1\ddot{P}_1} < \ddot{P}(2b_1 + 3\delta_1\ddot{P}) + c_1\ddot{P}_1 + h_1, \tag{22}$$

$$a_2 + l_1\ddot{P} < 2\ddot{P}_1(b_2 + \delta_2 + h_2) + d_1. \tag{23}$$

It is unstable otherwise.

- The (JM) of system (1) at $E_4 = (0, \tilde{P}_1, \tilde{P}_2)$ can be written as

$$J_4 = J(E_4) = [\varkappa_{ij}]_{3\times 3}, \tag{24}$$

where:

$$\varkappa_{11} = \frac{a_1}{1+m_1\tilde{P}_1} - c_1\tilde{P}_1 - h_1, \quad \varkappa_{12} = 0, \quad \varkappa_{13} = 0, \quad \varkappa_{21} = l_1\tilde{P}_1,$$

$$\varkappa_{22} = \frac{a_2}{1+m_2\tilde{P}_2} - \left(2\tilde{P}_1(b_2 + \delta_2 + h_2) + \frac{c_2\tilde{P}_2(\alpha-\gamma\tilde{P}_1^2)}{(\alpha+\gamma\tilde{P}_1^2)^2} + d_1\right), \quad \varkappa_{23} = \frac{-a_2 m_2\tilde{P}_1}{(1+m_2\tilde{P}_2)^2} - \frac{c_2\tilde{P}_1}{\alpha+\gamma\tilde{P}_1^2},$$

$$\varkappa_{31} = 0, \quad \varkappa_{32} = \frac{l_2\tilde{P}_2(\alpha-\gamma\tilde{P}_1^2)}{(\alpha+\gamma\tilde{P}_1^2)^2}, \quad \varkappa_{33} = \frac{l_2\tilde{P}_1}{\alpha+\gamma\tilde{P}_1^2} - 2h_3\tilde{P}_2 - (\delta_3 + d_2).$$

So, the characteristic equation of J_4 is:

$$(\varkappa_{11} - \lambda)[\lambda^2 - (\varkappa_{22} + \varkappa_{33})\lambda + \varkappa_{22}\varkappa_{33} - \varkappa_{23}\varkappa_{32}] = 0, \tag{25}$$

So, either $(\varkappa_{11} - \lambda) = 0$,
which gives $\lambda_{4P} = \frac{a_1}{1+m_1\tilde{P}_1} - (c_1\tilde{P}_1 + h_1)$,
or, $[\lambda^2 - (\varkappa_{22} + \varkappa_{33})\lambda + \varkappa_{22}\varkappa_{33} - \varkappa_{23}\varkappa_{32}] = 0$,
which gives

$$\lambda_{4P_1} + \lambda_{4P_2} = \left(\frac{a_2}{1+m_2\tilde{P}_2} - \left(2\tilde{P}_1(b_2 + \delta_2 + h_2) + \frac{c_2\tilde{P}_2(\alpha-\gamma\tilde{P}_1^2)}{(\alpha+\gamma\tilde{P}_1^2)^2} + d_1\right)\right) + \left(\frac{l_2\tilde{P}_1}{\alpha+\gamma\tilde{P}_1^2} - 2h_3\tilde{P}_2 - (\delta_3 + d_2)\right),$$

$$\lambda_{4P_1} \cdot \lambda_{4P_2} = \left(\frac{a_2}{1+m_2\tilde{P}_2} - (2\tilde{P}_1(b_2 + \delta_2 + h_2) + \frac{c_2\tilde{P}_2(\alpha-\gamma\tilde{P}_1^2)}{(\alpha+\gamma\tilde{P}_1^2)^2} + d_1)\right)\left(\frac{l_2\tilde{P}_1}{\alpha+\gamma\tilde{P}_1^2} - 2h_3\tilde{P}_2 - (\delta_3 + d_2)\right) - \left(\frac{-a_2 m_2\tilde{P}_1}{(1+m_2\tilde{P}_2)^2} - \frac{c_2\tilde{P}_1}{\alpha+\gamma\tilde{P}_1^2}\right)\left(\frac{l_2\tilde{P}_2(\alpha-\gamma\tilde{P}_1^2)}{(\alpha+\gamma\tilde{P}_1^2)^2}\right),$$

then E_4 is locally asymptotically stable under the following conditions

$$\frac{a_1}{1+m_1\tilde{P}_1} < (c_1\tilde{P}_1 + h_1), \tag{26}$$

$$\frac{l_2\tilde{P}_1}{\alpha+\gamma\tilde{P}_1^2} < (\delta_3 + d_2) + 2h_3\tilde{P}_2, \tag{27}$$
$$\gamma\tilde{P}_1^2 < \alpha, \tag{28}$$
$$\frac{a_2}{1+m_2\tilde{P}_2} < 2\tilde{P}_1(b_2+\delta_2+h_2) + \frac{c_2\tilde{P}_2(\alpha-\gamma\tilde{P}_1^2)}{(\alpha+\gamma\tilde{P}_1^2)^2} + d_1. \tag{29}$$
It is unstable otherwise.

- Finally, the (JM) of system (1) at $E_5 = (P^*, P_1^*, P_2^*)$ can be written as:

$$J_5 = J(E_5) = [\,\text{ȿ}_{ij}]_{3\times3}, \tag{30}$$
where:
$$\text{ȿ}_{11} = \frac{a_1}{1+m_1P_1^*} - [P^*(2b_1+3\delta_1P^*) + c_1P_1^* + h_1],$$
$$\text{ȿ}_{12} = \frac{-a_1m_1P^*}{(1+m_1P_1^*)^2} - c_1P^*, \quad \text{ȿ}_{13} = 0, \quad \text{ȿ}_{21} = l_1P_1^*,$$
$$\text{ȿ}_{22} = \frac{a_2}{1+m_2P_2^*} + l_1P^* - \left[2P_1^*(b_2+\delta_2+h_2) + \frac{c_2P_2{}^*(\alpha-\gamma P_1^{*2})}{(\alpha+\gamma P_1^{*2})^2} + d_1\right],$$
$$\text{ȿ}_{23} = \frac{-a_2m_2P_1^*}{(1+m_2P_2^*)^2} - \frac{c_2P_1^*}{\alpha+\gamma P_1^{*2}}, \quad \text{ȿ}_{31} = 0, \quad \text{ȿ}_{32} = \frac{l_2P_2^*(\alpha-\gamma P_1^{*2})}{(\alpha+\gamma P_1^{*2})^2},$$
$$\text{ȿ}_{33} = \frac{l_2P_1^*}{\alpha+\gamma P_1^{*2}} - \left((\delta_3+d_2) + 2h_3P_2^*\right).$$

Then the characteristic equation of J_5 is given by:
$$\lambda^3 + \beta_1\lambda^2 + \beta_2\lambda + \beta_3 = 0, \tag{31}$$
where:
$$\beta_1 = -(\text{ȿ}_{11} + \text{ȿ}_{22} + \text{ȿ}_{33}),$$
$$\beta_2 = \text{ȿ}_{11}(\text{ȿ}_{22} + \text{ȿ}_{33}) + \text{ȿ}_{22}\text{ȿ}_{33} - \text{ȿ}_{12}\text{ȿ}_{21} - \text{ȿ}_{23}\text{ȿ}_{32},$$
$$\beta_3 = \text{ȿ}_{11}(\text{ȿ}_{23}\text{ȿ}_{32} - \text{ȿ}_{22}\text{ȿ}_{33}) + \text{ȿ}_{12}\text{ȿ}_{21}\text{ȿ}_{33}.$$

So, by Routh-Hurwitz criteria, Equation (31) has real negative real parts, if $\beta_i > 0, i = 1,3$ and $\Delta = \Delta_1\beta_3 > 0$, where $\Delta_1 = \beta_1\beta_2 - \beta_3$. Now, $\beta_i > 0,\ i = 1,3$ if the next conditions hold:
$$\frac{a_1}{1+m_1P_1^*} < P^*(2b_1+3\delta_1P^*) + c_1P_1^* + h_1, \tag{32}$$
$$\frac{a_2}{1+m_2P_2^*} + l_1P^* < 2P_1^*(b_2+\delta_2+h_2) + \frac{c_2P_2{}^*(\alpha-\gamma P_1^{*2})}{(\alpha+\gamma P_1^{*2})^2} + d_1, \tag{33}$$
$$\frac{l_2P_1^*}{\alpha+\gamma P_1^{*2}} < (\delta_3+d_2) + 2h_3P_2^*, \tag{34}$$
Straight forward computation shows that:
$$\Delta_1 = \beta_1\beta_2 - \beta_3 = -\text{ȿ}_{22}(\text{ȿ}_{11}^2 + \text{ȿ}_{33}^2) - \text{ȿ}_{33}(\text{ȿ}_{11}^2 + \text{ȿ}_{22}^2) - \text{ȿ}_{22}(2\text{ȿ}_{11}\text{ȿ}_{33} - \text{ȿ}_{23}\text{ȿ}_{32}) - \text{ȿ}_{11}(\text{ȿ}_{22}^2 + \text{ȿ}_{33}^2) + \text{ȿ}_{12}\text{ȿ}_{21}(\text{ȿ}_{11} + \text{ȿ}_{22}) + \text{ȿ}_{23}\text{ȿ}_{32}\text{ȿ}_{33}.$$

Hence $\Delta = \Delta_1\beta_3 > 0$, if in addition to the conditions (32), (33), and (34) next condition is satisfied:
$$\alpha > \gamma P_1^{*2}. \tag{35}$$

So, eigenvalues of J_5 have a negative real part, which indicates that E_5 is locally stable, and unstable otherwise.

5. Global stability (GS)

In this section, the following theorems show that the (GS) of the equilibrium points, which are (LAS) of system (1), have been analytically studied with the suitable Lyapunov functions.

Theorem 5.1: The equilibrium point $E_0 = (0, 0, 0)$ is a globally asymptotically stable (GAS) in the subregion $\varphi_0 \subset R_+^3$ that satisfies the next condition:

$$\frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2} < \delta_1 P^3 + d_1 P_1 + (\delta_3 + d_2) P_2 \tag{36}$$

Proof: Consider the following function: $\dot{W}_0(P, P_1, P_2) = P + P_1 + P_2$.
Clearly, $\dot{W}_0 : \mathrm{R}_+^3 \to \mathrm{R}$ where $\dot{W}_0 \in \mathrm{C}^1$ is positive definite function,
Now, by deriving $\dot{W}_0$ W.R.T. time t, and simplifying the terms, we get

$$\frac{d\dot{W}_0}{dt} < \frac{a_1 P}{1 + m_1 P_1} - b_1 P^2 - \delta_1 P^3 - h_1 P - (c_1 - l_1) P P_1 + \frac{a_2 P_1}{1 + m_2 P_2} - b_2 P_1^2 - (c_2 - l_2) \frac{P_1 P_2}{\alpha + \gamma P_1^2} - \delta_2 P_1^2 - h_2 P_1^2 - d_1 P_1 - (\delta_3 + d_2) P_2 - h_3 P_2^2,$$

by the biological facts (food transfer rate less than attack rate), $l_i < c_i$, $i = 1, 2,$ so

$$\frac{d\dot{W}_0}{dt} < a_1 P \left(1 - \frac{b_1}{a_1} P\right) - \delta_1 P^3 + a_2 P_1 \left(1 - \frac{b_2}{a_2} P_1\right) - d_1 P_1 - (\delta_3 + d_2) P_2,$$

Now, since the logistic functions $f_1(P) = a_1 P \left(1 - \frac{b_1}{a_1} P\right)$ and $f_2(P_1) = a_2 P_1 \left(1 - \frac{b_2}{a_2} P_1\right)$ are bounded by constants $\left(\frac{a_1^2}{4b_1}\right), \left(\frac{a_2^2}{4b_2}\right)$ respectively. So

$$\frac{d\dot{W}_0}{dt} < \frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2} - [\delta_1 P^3 + d_1 P_1 + (\delta_3 + d_2) P_2],$$

So, by condition (36) $\frac{d\dot{W}_0}{dt} < 0$, and hence $\dot{W}_0$ is a Lyapunov function which gives a global asymptotic of E_0.

Theorem 5.2: The point $E_1 = (\breve{P}, 0, 0)$ is a globally asymptotically stable (GAS) in the subregion $\varphi_1 \subset R_+^3$ that satisfies the next conditions:

$$P > \breve{P}, \tag{37}$$

$$\breve{Z}_1 > \breve{Z}_2, \tag{38}$$

where:

$$\breve{Z}_1 = \frac{a_1 \breve{P}}{1 + m_1 P_1} + b_1 (P - \breve{P})^2 + a_1 P + (\delta_3 + d_2) P_2,$$

$$\breve{Z}_2 = \frac{a_2^2}{4b_2} + \frac{a_1 P}{1 + m_1 P_1} + \breve{P}(c_1 P_1 + a_1) + \delta_1 \breve{P}^2 (P - \breve{P}).$$

Proof: Consider the following function $\dot{W}_1(P, P_1, P_2) = \left(P - \breve{P} - \breve{P} ln \frac{P}{\breve{P}}\right) + P_1 + P_2$
Clearly, $\dot{W}_1 : \mathrm{R}_+^3 \to \mathrm{R}$ where $\dot{W}_1 \in \mathrm{C}^1$ is positive definite function.
Now, by deriving $\dot{W}_1$ W.R.T. time t, and simplifying the terms, we get

$$\frac{dW_1}{dt} < \frac{a_1 P}{1 + m_1 P_1} - \frac{a_1 \breve{P}}{1 + m_1 P_1} - b_1 (P - \breve{P})^2 - (c_1 - l_1) P P_1 - a_1 P - \delta_1 P^2 (P - \breve{P}) + \breve{P}(c_1 P_1 + a_1) + \delta_1 \breve{P}^2 (P - \breve{P}) + \frac{a_2 P_1}{1 + m_2 P_2} - b_2 P_1^2 - (c_2 - l_2) \frac{P_1 P_2}{\alpha + \gamma P_1^2} - (\delta_3 + d_2) P_2,$$

by the biological facts, $l_i < c_i$, $i = 1,2$ and condition (37), so

$$\frac{d\breve{W}_1}{dt} < \frac{a_1 P}{1+m_1 P_1} - \frac{a_1 \breve{P}}{1+m_1 P_1} - b_1 (P - \breve{P})^2 - a_1 P + \delta_1 \breve{P}^2 (P - \breve{P}) + \breve{P}(c_1 P_1 + a_1) + a_2 P_1 \left(1 - \frac{b_2}{a_2} P_1\right) - (\delta_3 + d_2) P_2$$

,

Now, since the logistic function $f(P_1) = a_2 P_1 \left(1 - \frac{b_2}{a_2} P_1\right)$ is bounded by the constant $\left(\frac{a_2^2}{4b_2}\right)$. So

$$\frac{d\breve{W}_1}{dt} < -\left[\frac{a_1 \breve{P}}{1+m_1 P_1} + b_1 (P - \breve{P})^2 + a_1 P + (\delta_3 + d_2) P_2\right] + \frac{a_2^2}{4b_2} + \frac{a_1 P}{1+m_1 P_1} + \breve{P}(c_1 P_1 + a_1) + \delta_1 \breve{P}^2 (P - \breve{P})$$

,

$\frac{d\breve{W}_1}{dt} < -\breve{Z}_1 + \breve{Z}_2$, where $\breve{Z}_1$ and $\breve{Z}_2$ are given in the state of theorem.

So, by condition (38) $\frac{d\breve{W}_1}{dt} < 0$,and hence $\dot{W}_1$ is a Lyapunov function which gives a global asymptotic of E_1.

Theorem 5.3: The point $E_2 = (0, \bar{P}_1, 0)$ is a globally asymptotically stable (GAS) in the subregion $\varphi_2 \subset R_+^3$ that satisfies the next conditions:

$$\bar{Z}_1 > \bar{Z}_2, \qquad (39)$$

where

$$\bar{Z}_1 = (b_2 + \delta_2 + h_2)(P_1 - \bar{P}_1)^2 + \frac{a_2 \bar{P}_1}{1+m_2 P_2} + a_2 P_1 + P(l_1 \bar{P} + h_1) + (\delta_3 + d_2) P_2,$$

$$\bar{Z}_2 = \frac{c_2 \bar{P}_1 P_2}{\alpha + \gamma P_1^2} + \frac{a_2 P_1}{1+m_2 P_2} + \frac{a_1^2}{4b_1} + a_2 \bar{P}_1.$$

Proof: Consider the following function $\dot{W}_2(P, P_1, P_2) = P + \left(P_1 - \bar{P}_1 - \bar{P}_1 ln\frac{P_1}{\bar{P}_1}\right) + P_2$

Clearly, $\dot{W}_2 : R_+^3 \to R$ where $\dot{W}_2 \in C^1$ is a positive definite function.

Now, by deriving $\dot{W}_2$ W.R.T. time t, and simplifying the terms, we get

$$\frac{d\bar{W}_2}{dt} < \frac{a_1 P}{1+m_1 P_1} - b_1 P^2 + \frac{a_2 P_1}{1+m_2 P_2} - (c_1 - l_1) P P_1 - P(l_1 \bar{P} + h_1) - \frac{a_2 \bar{P}_1}{1+m_2 P_2} - (b_2 + \delta_2 + h_2)(P_1 - \bar{P}_1)^2 + \frac{c_2 \bar{P}_1 P_2}{\alpha + \gamma P_1^2} - (c_2 - l_2)\frac{P_1 P_2}{\alpha + \gamma P_1^2} - a_2 P_1 - (\delta_3 + d_2) P_2 + a_2 \bar{P}_1,$$

by the biological facts $l_i < c_i$, $i = 1,2$, so

$$\frac{d\bar{W}_2}{dt} < a_1 P \left(1 - \frac{b_1}{a_1} P\right) + \frac{a_2 P_1}{1+m_2 P_2} - P(l_1 \bar{P} + h_1) - \frac{a_2 \bar{P}_1}{1+m_2 P_2} - (b_2 + \delta_2 + h_2)(P_1 - \bar{P}_1)^2 + \frac{c_2 \bar{P}_1 P_2}{\alpha + \gamma P_1^2} - a_2 P_1 + a_2 \bar{P}_1 - (\delta_3 + d_2) P_2,$$

Now, since the logistic function $f(P) = a_1 P \left(1 - \frac{b_1}{a_1} P\right)$ is bounded by the constant $\left(\frac{a_1^2}{4b_1}\right)$. So

$$\frac{d\dot{W}_2}{dt} < -\left[(b_2+\delta_2+h_2)(P_1-\bar{P}_1)^2+\frac{a_2\bar{P}_1}{1+m_2P_2}+a_2P_1+P(l_1\bar{P}+h_1)+(\delta_3+d_2)P_2\right]+ \frac{c_2\bar{P}_1P_2}{\alpha+\gamma P_1^2}+\frac{a_2P_1}{1+m_2P_2}+\left(\frac{a_1^2}{4b_1}\right)+a_2\bar{P}_1,$$

$\frac{d\dot{W}_2}{dt} < -\bar{Z}_1+\bar{Z}_2$, where $\bar{Z}_1$ and $\bar{Z}_2$ are given in the state of theorem.

So, by condition (39) $\frac{d\dot{W}_2}{dt} < 0$,and hence $\dot{W}_2$ is a Lyapunov function which gives a global asymptotic of E_2.

Theorem 5.4: The point $E_3=\left(\ddot{P},\ddot{P}_1,0\right)$ is a globally asymptotical stable (GAS) in the subregion $\varphi_3 \subset R_+^3$ that satisfies the next conditions:

$$P>\ddot{P}, P_1>\ddot{P}_1, \qquad (40)$$

$$\ddot{Z}_1>\ddot{Z}_2, \qquad (41)$$

where:

$$\ddot{Z}_1=b_1\left(P-\ddot{P}\right)^2+\frac{a_1}{1+m_1P_1}\left(P-\ddot{P}\right)+l_1\left(P\ddot{P}_1+\ddot{P}P_1\right)+(\delta_3+d_2)P_2,$$

$$\ddot{Z}_2=\frac{a_1}{1+m_1P_1}\left(P-\ddot{P}\right)+c_1\left(P\ddot{P}_1+\ddot{P}P_1\right)+\delta_1\ddot{P}^2\left(P-\ddot{P}\right)+\frac{a_2}{1+m_2P_2}\left(P_1-\ddot{P}_1\right)+\frac{c_2\ddot{P}_1P_2}{\alpha+\gamma P_1^2}.$$

Proof: Consider the following function

$$\dot{W}_3(P,P_1,P_2)=\left(P-\ddot{P}-\ddot{P}\ ln\frac{P}{\ddot{P}}\right)+\left(P_1-\ddot{P}_1-\ddot{P}_1\ ln\frac{P_1}{\ddot{P}_1}\right)+P_2$$

Clearly, $\dot{W}_3: \mathrm{R}_+^3 \to \mathrm{R}$ where $\dot{W}_3 \in \mathrm{C}^1$ is a positive definite function.
Now, by deriving $\dot{W}_3$ W.R.T. time t, and simplifying the terms, we get

$$\frac{d\dot{W}_3}{dt} < \frac{a_1}{1+m_1P_1}\left(P-\ddot{P}\right)-(c_1-l_1)PP_1-b_1\left(P-\ddot{P}\right)^2-(c_1-l_1)\ddot{P}\ddot{P}_1+c_1\left(P\ddot{P}_1+\ddot{P}P_1\right)- \delta_1P^2\left(P-\ddot{P}\right)-\frac{a_1}{1+m_1P_1}\left(P-\ddot{P}\right)+\delta_1\ddot{P}^2\left(P-\ddot{P}\right)+\frac{a_2}{1+m_2P_2}\left(P_1-\ddot{P}_1\right)- (b_2+\delta_2+h_2)\left(P_1-\ddot{P}_1\right)^2-l_1\left(P\ddot{P}_1+\ddot{P}P_1\right)-(c_2-l_2)\frac{P_1P_2}{\alpha+\gamma P_1^2}+\frac{c_2\ddot{P}_1P_2}{\alpha+\gamma P_1^2}- a_2\left(P_1-\ddot{P}_1\right)-(\delta_3+d_2)P_2,$$

by the biological facts $l_i < c_i$, $i=1,2$ and condition (40)

$$\frac{d\dot{W}_3}{dt} < -\left[b_1\left(P-\ddot{P}\right)^2+\frac{a_1}{1+m_1P_1}\left(P-\ddot{P}\right)+l_1\left(P\ddot{P}_1+\ddot{P}P_1\right)+(\delta_3+d_2)P_2\right]+ \frac{a_1}{1+m_1P_1}\left(P-\ddot{P}\right)+c_1\left(P\ddot{P}_1+\ddot{P}P_1\right)+\delta_1\ddot{P}^2\left(P-\ddot{P}\right)+\frac{a_2}{1+m_2P_2}\left(P_1-\ddot{P}_1\right)+\frac{c_2\ddot{P}_1P_2}{\alpha+\gamma P_1^2},$$

$\frac{d\dot{W}_3}{dt} < -\ddot{Z}_1+\ddot{Z}_2$, where $\ddot{Z}_1$ and $\ddot{Z}_2$ are given in the state of theorem.

So, by condition (41) $\frac{d\dot{W}_3}{dt} < 0$,and hence $\dot{W}_3$ is a Lyapunov function which gives a global asymptotic of E_3.

Theorem 5.5: The point $E_4=\left(0,\tilde{P}_1,\tilde{P}_2\right)$ is a globally asymptotically stable (GAS) in the subregion $\varphi_4 \subset R_+^3$ that satisfies the next conditions:

$$P_1>\tilde{P}_1, P_2>\tilde{P}_2, \qquad (42)$$

$$\tilde{Z}_1>\tilde{Z}_2, \qquad (43)$$

where:

$$\tilde{Z}_1 = P(l_1\tilde{P}_1 + h_1) + \frac{a_2}{1+m_2\tilde{P}_2}(P_1 - \tilde{P}_1) + l_2\left(\frac{P_1\tilde{P}_2}{\alpha+\gamma P_1^2} + \frac{\tilde{P}_1P_2}{\alpha+\gamma\tilde{P}_1^2}\right),$$
$$\tilde{Z}_2 = \frac{a_1^2}{4b_1} + \frac{a_2}{1+m_2P_2}(P_1 - \tilde{P}_1) + c_2\left(\frac{\tilde{P}_1P_2}{\alpha+\gamma P_1^2} + \frac{P_1\tilde{P}_2}{\alpha+\gamma\tilde{P}_1^2}\right).$$

Proof: Consider the following function

$$\dot{W}_4(P,P_1,P_2) = P + \left(P_1 - \tilde{P}_1 - \tilde{P}_1\ ln\frac{P_1}{\tilde{P}_1}\right) + \left(P_2 - \tilde{P}_2 - \tilde{P}_2\ ln\frac{P_2}{\tilde{P}_2}\right)$$

Clearly, $\dot{W}_4 : \mathrm{R}_+^3 \to \mathrm{R}$ where $\dot{W}_4 \in \mathrm{C}^1$ is a positive definite function.
Now, by deriving $\dot{W}_4$ W.R.T. time t, and simplifying the terms, we get

$$\frac{dW_4}{dt} < \frac{a_1P}{1+m_1P_1} - b_1P^2 - (c_1 - l_1)PP_1 + \frac{a_2}{1+m_2P_2}(P_1 - \tilde{P}_1) - (b_2 + \delta_2 + h_2)(P_1 - \tilde{P}_1)^2 - P(l_1\tilde{P}_1 + h_1) + \frac{c_2\tilde{P}_1P_2}{\alpha+\gamma P_1^2} - (c_2 - l_2)\frac{P_1P_2}{\alpha+\gamma P_1^2} - \frac{a_2}{1+m_2\tilde{P}_2}(P_1 - \tilde{P}_1) + \frac{c_2P_1\tilde{P}_2}{\alpha+\gamma\tilde{P}_1^2} - (c_2 - l_2)\frac{\tilde{P}_1\tilde{P}_2}{\alpha+\gamma\tilde{P}_1^2} - \frac{l_2P_1\tilde{P}_2}{\alpha+\gamma P_1^2} - h_3(P_2 - \tilde{P}_2)^2 - \frac{l_2\tilde{P}_1P_2}{\alpha+\gamma\tilde{P}_1^2},$$

by the biological facts $l_i < c_i$, $i = 1,2$ and condition (42), so

$$\frac{dW_4}{dt} < a_1P\left(1 - \frac{b_1}{a_1}P\right) + \frac{a_2}{1+m_2P_2}(P_1 - \tilde{P}_1) - P(l_1\tilde{P}_1 + h_1) - \frac{a_2}{1+m_2\tilde{P}_2}(P_1 - \tilde{P}_1) + c_2\left(\frac{\tilde{P}_1P_2}{\alpha+\gamma P_1^2} + \frac{P_1\tilde{P}_2}{\alpha+\gamma\tilde{P}_1^2}\right) - l_2\left(\frac{P_1\tilde{P}_2}{\alpha+\gamma P_1^2} + \frac{\tilde{P}_1P_2}{\alpha+\gamma\tilde{P}_1^2}\right).$$

Now, since the logistic function $f(P) = a_1P\left(1 - \frac{b_1}{a_1}P\right)$ is bounded by the constant $\left(\frac{a_1^2}{4b_1}\right)$. So,

$$\frac{dW_4}{dt} < -\left[P(l_1\tilde{P}_1 + h_1) + \frac{a_2}{1+m_2\tilde{P}_2}(P_1 - \tilde{P}_1) + l_2\left(\frac{P_1\tilde{P}_2}{\alpha+\gamma P_1^2} + \frac{\tilde{P}_1P_2}{\alpha+\gamma\tilde{P}_1^2}\right)\right] + \frac{a_1^2}{4b_1} + \frac{a_2}{1+m_2P_2}(P_1 - \tilde{P}_1) + c_2\left(\frac{\tilde{P}_1P_2}{\alpha+\gamma P_1^2} + \frac{P_1\tilde{P}_2}{\alpha+\gamma\tilde{P}_1^2}\right),$$

that is

$\frac{dW_4}{dt} < -\tilde{Z}_1 + \tilde{Z}_2$, where $\tilde{Z}_1$ and $\tilde{Z}_2$ are given in the state of theorem.

So, by conditions (43) $\frac{dW_4}{dt} < 0$,and hence $\dot{W}_4$ is a Lyapunov function which gives a global asymptotic of E_4.

Theorem 5.6: The point $E_5 = (P^*, P_1^*, P_2^*)$ is a globally asymptotically (GAS) in the subregion $\varphi_5 \subset R_+^3$ that satisfies the next conditions:

$$P > P^*, P_1 > P_1^*, P_2 > P_2^*, \quad (44)$$
$$Z_1^* > Z_2^*, \quad (45)$$

where:

$$Z_1^* = \frac{a_1}{1+m_1P_1^*}(P - P^*) + \frac{a_2}{1+m_2P_2^*}(P_1 - P_1^*) + l_1(PP_1^* + P^*P_1) + l_2\left(\frac{P_1P_2^*}{\alpha+\gamma P_1^2} + \frac{P_1^*P_2}{\alpha+\gamma P_1^{*2}}\right),$$

$$Z_2^* = \frac{a_1}{1+m_1P_1}(P-P^*) + \delta_1 P^{*2}(P-P^*) + c_2\left(\frac{P_1^*P_2}{\alpha+\gamma P_1^2} + \frac{P_1P_2^*}{\alpha+\gamma P_1^{*2}}\right) + \frac{a_2}{1+m_2P_2}(P_1-P_1^*) + c_1(PP_1^* + P^*P_1).$$

Proof: Consider the following function

$$\dot{W}_5(P,P_1,P_2) = \left(P - P^* - P^* \ln\frac{P}{P^*}\right) + \left(P_1 - P_1^* - P_1^* \ln\frac{P_1}{P_1^*}\right) + \left(P_2 - P_2^* - P_2^* \ln\frac{P}{P_2^*}\right)$$

Clearly, $\dot{W}_5 : \mathrm{R}_+^3 \rightarrow \mathrm{R}$ where $\dot{W}_5 \in \mathrm{C}^1$ is a positive definite function.
Now, by deriving $\dot{W}_5$ W.R.T. time t, and simplifying the terms, we get

$$\frac{dW_5}{dt} < \frac{a_1}{1+m_1P_1}(P-P^*) + c_1(PP_1^* + P^*P_1) - (c_1 - l_1)PP_1 - (c_1 - l_1)P^*P_1^* - \frac{a_1}{1+m_1P_1^*}(P-P^*) - \delta_1 P^2(P-P^*) + \delta_1 P^{*2}(P-P^*) - (b_2 + \delta_2 + h_2)(P_1 - P_1^*)^2 + \frac{a_2}{1+m_2P_2}(P_1 - P_1^*) - l_1(PP_1^* + P^*P_1) - (c_2 - l_2)\frac{P_1P_2}{\alpha+\gamma P_1^2} + c_2\left(\frac{P_1^*P_2}{\alpha+\gamma P_1^2} + \frac{P_1P_2^*}{\alpha+\gamma P_1^{*2}}\right) - \frac{a_2}{1+m_2P_2^*}(P_1 - P_1^*) - (c_2 - l_2)\frac{P_1^*P_2^*}{\alpha+\gamma P_1^{*2}} - l_2\left(\frac{P_1P_2^*}{\alpha+\gamma P_1^2} + \frac{P_1^*P_2}{\alpha+\gamma P_1^{*2}}\right) - h_3(P_2 - P_2^*)^2,$$

by the biological facts $l_i < c_i$, $i = 1,2$ and condition (44), so

$$\frac{dW_5}{dt} < -\left[\frac{a_1}{1+m_1P_1^*}(P-P^*) + \frac{a_2}{1+m_2P_2^*}(P_1 - P_1^*) + l_1(PP_1^* + P^*P_1) + l_2\left(\frac{P_1P_2^*}{\alpha+\gamma P_1^2} + \frac{P_1^*P_2}{\alpha+\gamma P_1^{*2}}\right)\right] + \frac{a_1}{1+m_1P_1}(P-P^*) + \delta_1 P^{*2}(P-P^*) + c_2\left(\frac{P_1^*P_2}{\alpha+\gamma P_1^2} + \frac{P_1P_2^*}{\alpha+\gamma P_1^{*2}}\right) + \frac{a_2}{1+m_2P_2}(P_1 - P_1^*) + c_1(PP_1^* + P^*P_1),$$

$\frac{dW_5}{dt} < -Z_1^* + Z_2^*$, where Z_1^* and Z_2^* are given in the state of theorem.

So, by condition (45) $\frac{dW_5}{dt} < 0$,and hence $\dot{W}_5$ is a Lyapunov function which gives a global asymptotic of E_5.

6.Numerical simulation

In this section, numerical simulations are used to investigate the dynamical behaviour of the system (1) dynamical behaviour. For a single collection of values and different initial points. The objectives of this study are: (I) Study the influence of parameters on the dynamics of our model. (II) Confirm the analytical results that we obtained. It seems that at the assumed parameter set (46) , the system has a globally positive equilibrium point, as shown in Figure (1) (a – e)

$$a_1 = 2, m_1 = 0.01, b_1 = 0.001, c_1 = 0.3, \delta_1 = 0.003, h_1 = 0.003, a_2 = 0.001, m_2 = 1, b_2 = 0.0001, l_1 = 0.3, c_2 = 0.5, \alpha = 0.1, \gamma = 0.4, \delta_2 = 0.03, h_2 = 0.03, d_1 = 0.00001, l_2 = 0.4, \delta_3 = 0.02, h_3 = 0.03, d_2 = 0.00001. \quad (46)$$

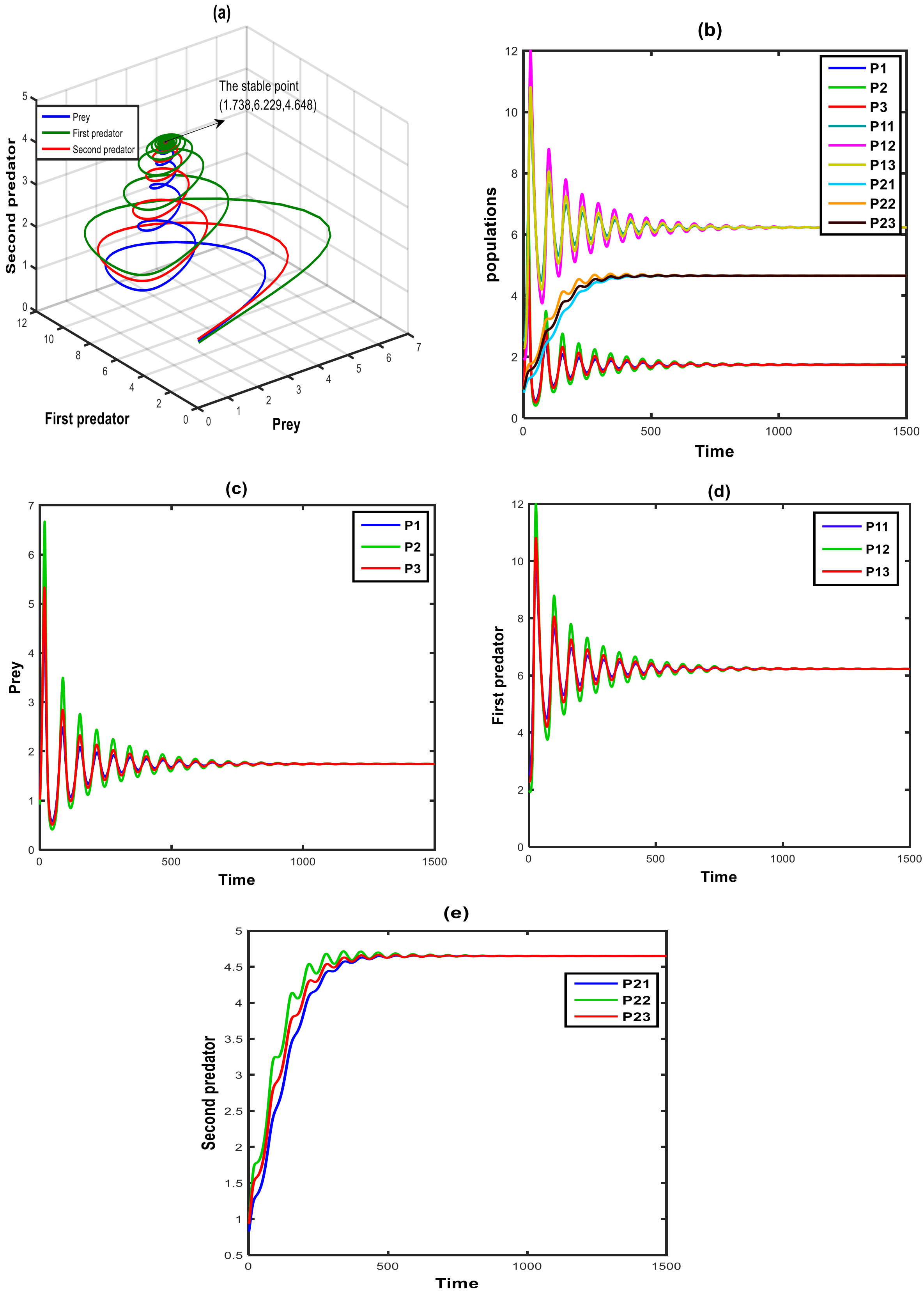

Figure 1: Time series of the system solution (T.S.) started with different initial points $(1, 2.5, 0.8)$, $(0.8, 2, 0.9)$ and $(0.9, 2.3, 0.9)$. (a) The solution asymptotically approaches to the positive equilibrium point $E_5 = (1.738, 6.229, 4.648)$, (b) trajectories of P, P_1, P_2 from different initial points, (c) trajectory of P as a function of time, (d) trajectory of P_1 as a function of time, (e) trajectory of P_2 as a function of time.

Now, to examine the influence of parameter values on the system's dynamical behaviour, by altering each parameter individually while keeping the other as specified in (46) and the results are presented in Table 2.

Table 2: The dynamical behaviour of system (1) at each parameter of the system

Range of Parameter	Stable Point
$1.86 \le a_1 < 5.$	E_5
$0.01 \le m_1 < 0.024.$	E_5
$0.001 \le b_1 < 0.26.$	E_5
$0.3 \le c_1 < 0.4.$	E_5
$0.003 \le \delta_1 < 0.5.$	E_5
$0.003 \le h_1 < 0.1.$	E_5
$0.001 \le a_2 < 0.8.$	E_5
$0.01 \le m_2 < 2.$	E_5
$0.0001 \le b_2 < 2.$	E_5
$0.07 \le l_1 < 0.3.$	E_5
$0.1 \le c_2 < 0.65.$	E_5
$0.1 \le \alpha < 1.$	E_5
$0.3610 \le \gamma < 1.$	E_5
$0.0167 \le \delta_2 < 0.8.$	E_5
$0.0171 \le h_2 < 0.7.$	E_5
$0.00001 \le d_1 < 0.5.$	E_5
$0.001 \le l_2 < 0.8.$	E_5
$0.01 \le \delta_3 < 0.230,$ $0.230 \le \delta_3 < 1.$	E_5 E_3
$0.022 \le h_3 < 2.$	E_5
$0.00001 \le d_2 < 0.17,$ $0.17 \le d_2 < 1.$	E_5 E_3

The influence of altering the toxin rate (δ_3) of second predator population as in $0.01 \le \delta_3 < 0.230$ the solution goes to E_5, as illustrated in Figure (2) $(a,\ a_1)$ with the typical value $\delta_3 = 0.01$ while, in the range $0.230 \le \delta_3 < 1$ the solution goes to the point E_3, as illustrated in Figure (2) (b, b_1) with the typical value $\delta_3 = 0.230$.

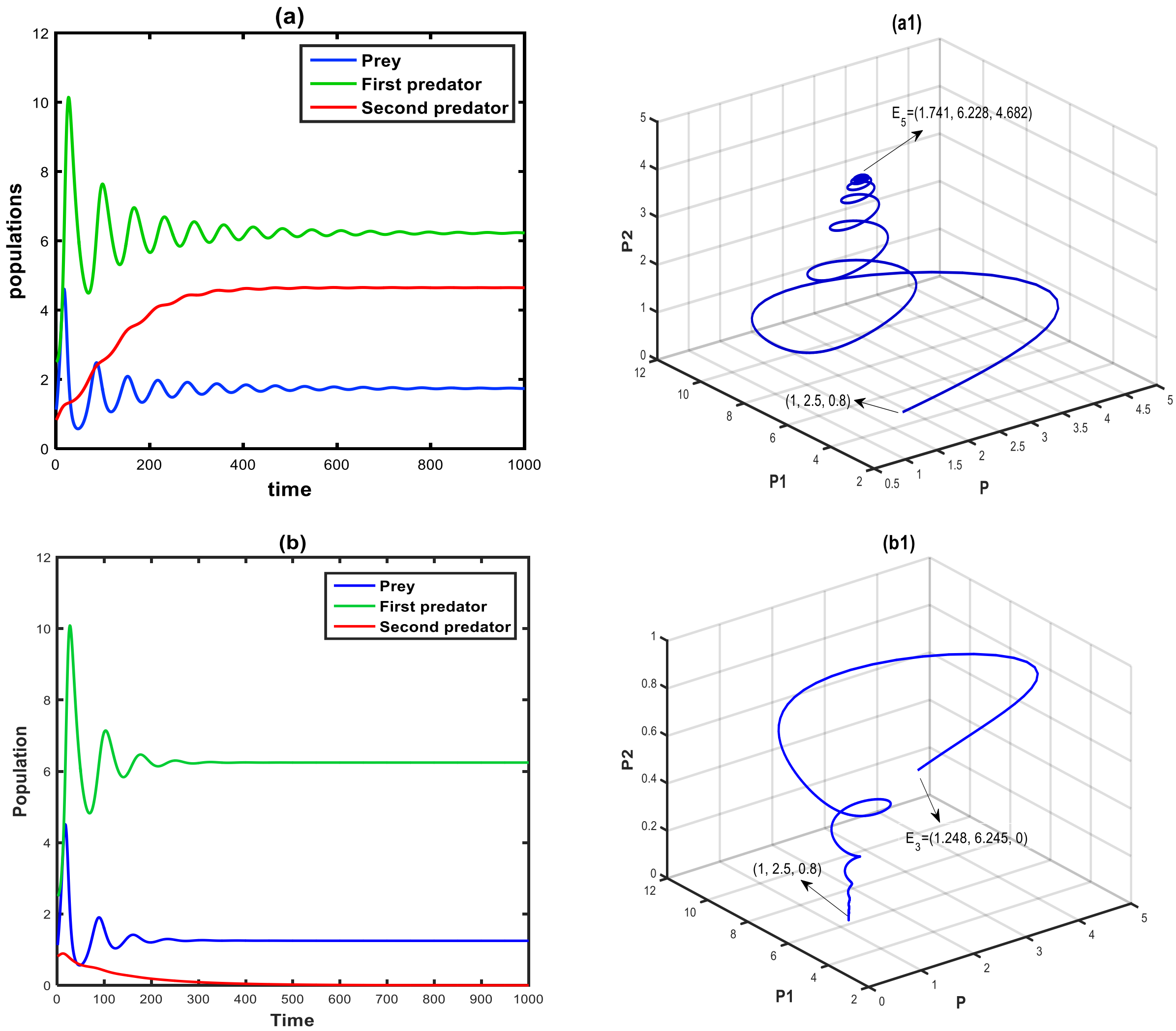

Figure 2: (a) (T.S.) of solution which goes to $E_5 = (1.741, 6.228, 4.682)$ when $\delta_3 = 0.01, (a1)$ 3D phase portrait of (a), (b) (T.S.) of the solution of system (1) which goes to $E_3 = (1.248, 6.245, 0)$ for the given values in equation (46) with δ_3 =0.230, $(b1)$ 3D phase portrait of (b).

Also, the influence of altering the natural death rate of second predator d_2 in the range $0.00001 \leq d_2 < 0.17$ solution goes to E_5 as it is illustrated in Figure (3) (a, a_1) with the typical value $d_2 = 0.00001$ while, in the range of $0.17 \leq d_2 < 1$ the solution goes to E_3, as it is illustrated in Figure (3) (b, b_1), with the typical value $d_2 = 0.17$.

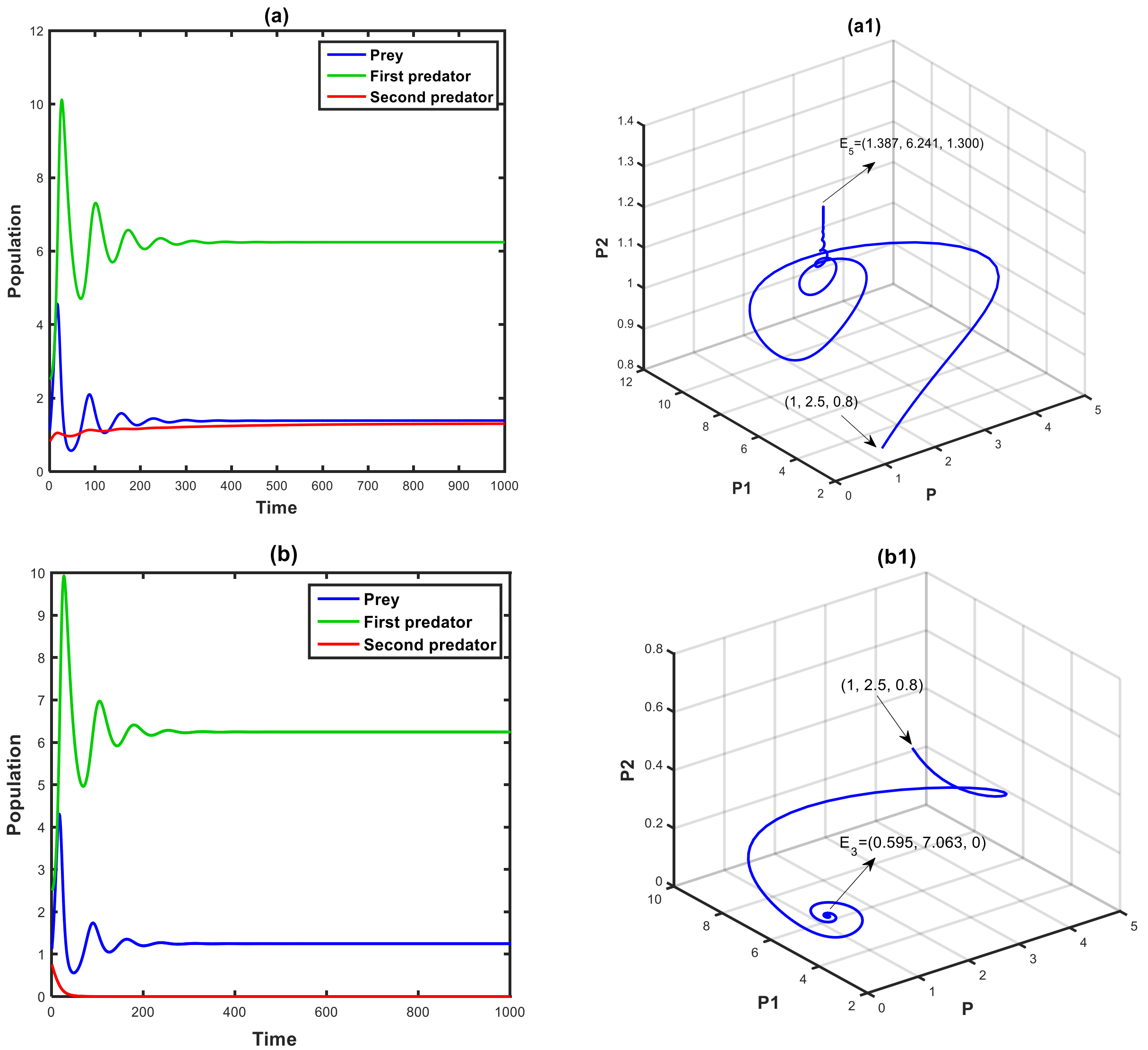

Figure 3: (a) (T.S.) of the solution which goes to $E_5 = (1.387, 6.241, 1.300)$ when $d_2 = 0.00001$, (a1) 3D phase portrait of (a), (b) (T.S.) of the solution which goes to $E_3 = (0.595, 7.063, 0)$ when $d_2 = 0.17$, (b1) 3D phase portrait of (b).

The influence of the growth rate, the internal competition rate of both the prey, the first predator, the harvesting rate of the prey, the toxin rate of both the prey, the second predator and the death rate of the predators $(a_1, a_2, b_1, b_2, h_1, \delta_1, \delta_3, d_1$ and $d_2)$ the solution goes to E_0, as it is illustrated in Figure (4) (a, b) with the values $a_1 = 0.1$, $a_2 = 0.1$, $b_1 = 0.2$, $b_2 = 0.2$, $h_1 = 0.7$, $\delta_1 = 0.3$, $\delta_3 = 0.2$, $d_1 = 0.7$ and $d_2 = 0.1$.

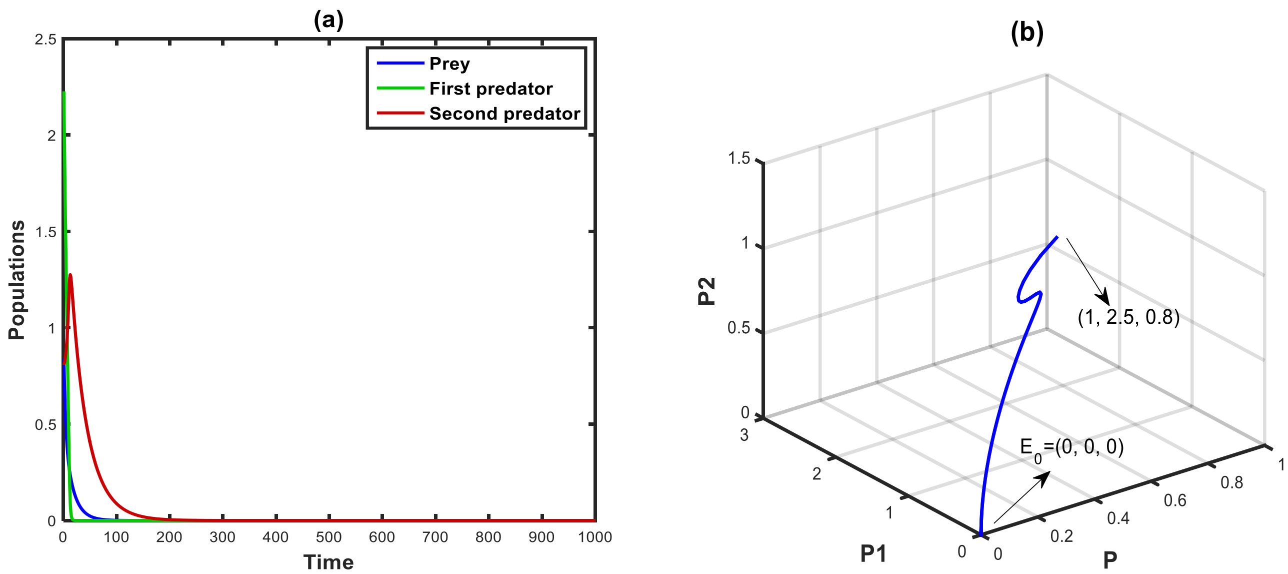

Figure 4: (a) (T.S.) of the solution which goes to $E_0 = (0,0,0\,)$ when values $a_1 = 0.1$, $a_2 = 0.1$, $b_1 = 0.2$, $b_2 = 0.2$, $h_1 = 0.7$, $\delta_1 = 0.3$, $\delta_3 = 0.2$, $d_1 = 0.7$ and $d_2 = 0.1$, (b) 3D phase portrait of (a).

The varying of the growth rate of prey population, the food transfer rate of the first predator population and the death rate of the predators $(a_1, l_1, d_1,$ and $d_2)$ in the range of $0.05 \leq a_1 < 0.2$, $0.01 \leq l_2 < 0.2$, $0.311 \leq d_1 < 0.99$ and $0.3 \leq d_2 < 0.9$ the solution goes to E_1, as it is illustrated in Figure (5) (a,b) with the typical values $a_1 = 0.1$, $l_1 = 0.1, d_1 = 0.9$ and $d_2 = 0.3$.

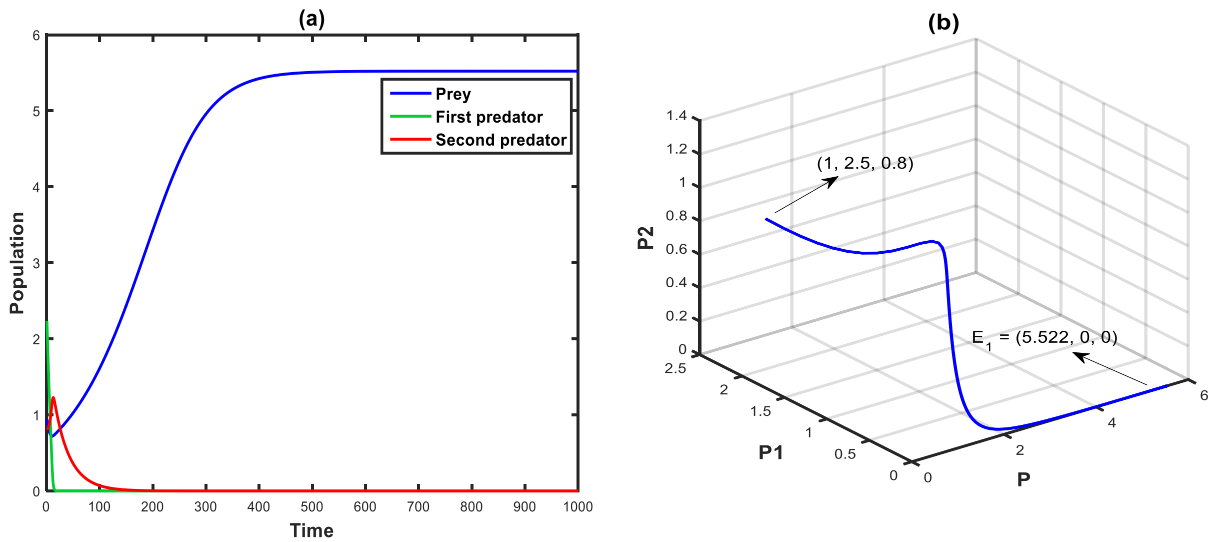

Figure 5: (a) (T.S.) of the solution which goes to $E_1 = (5.522, 0, 0)$ when $a_1 = 0.1$, $l_1 = 0.1$, $d_1 = 0.9$, and $d_2 = 0.3$, (b) 3D phase portrait of (a).

The varying of the growth rate of prey and first predator populations, the attack rate of prey, the toxin rate and the death rate of second predator population $(a_1, a_2, c_1, \delta_3,$ and $d_2)$ at the same time in the range $0.1 \leq a_1 < 1.6$, $0.1 \leq a_2 < 0.370$, $0.3 \leq c_1 < 1.290$,

$0.1 \le \delta_3 < 1$, and $0.1 \le d_2 < 0.999$ the solution goes to E_2, as it is illustrated in Figure (6), with

typical values $a_1 = 0.5$, $a_2 = 0.1$, $c_1 = 0.9$, $\delta_3 = 0.5$, and $d_2 = 0.5$.

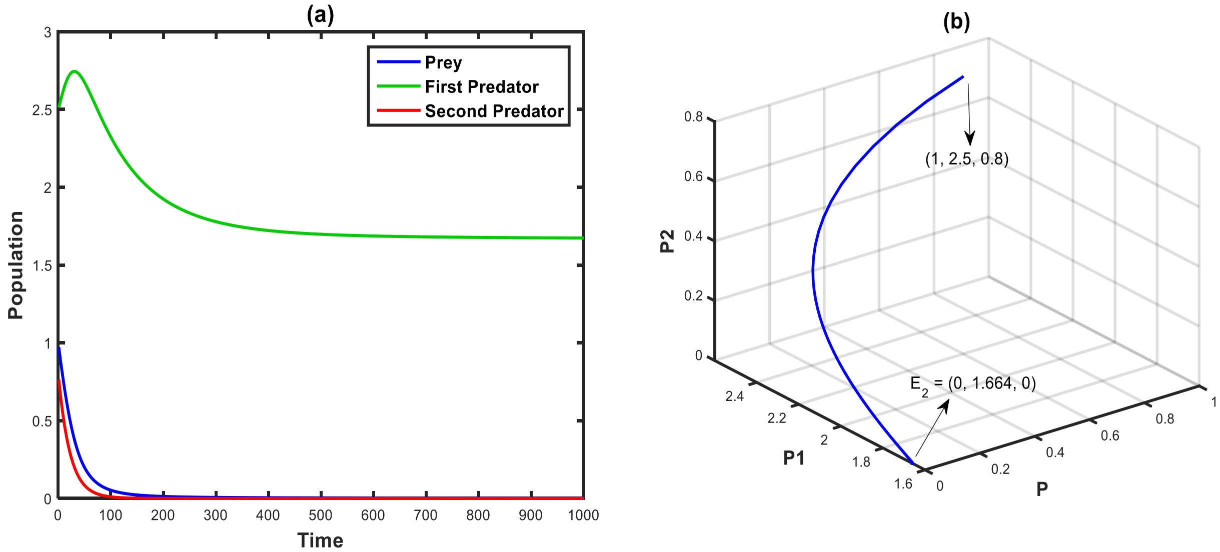

Figure 6: (a) (T.S.) of the solution which goes to $E_2 = (0, 1.664, 0)$ when $a_1 = 0.5$, $a_2 = 0.1$, $c_1 = 0.9$, $\delta_3 = 0.5$, and $d_2 = 0.5$, (b) 3D Phase portrait of (a).

Moreover, varying the growth rate of both prey and first predator, the fear rate of the first predator, the internal competition rate of the first predator, the half-saturation constant and the food transfer rate of second predator $(a_1, a_2, m_2, b_2, \alpha$ and $l_2)$ the solution goes to E_4, as it is illustrated in Figure (7) with the values $a_1 = 0.5$, $a_2 = 0.4$, $m_2 = 0.001$, $b_2 = 0.1$, $\alpha = 0.5$, and $l_2 = 0.01$.

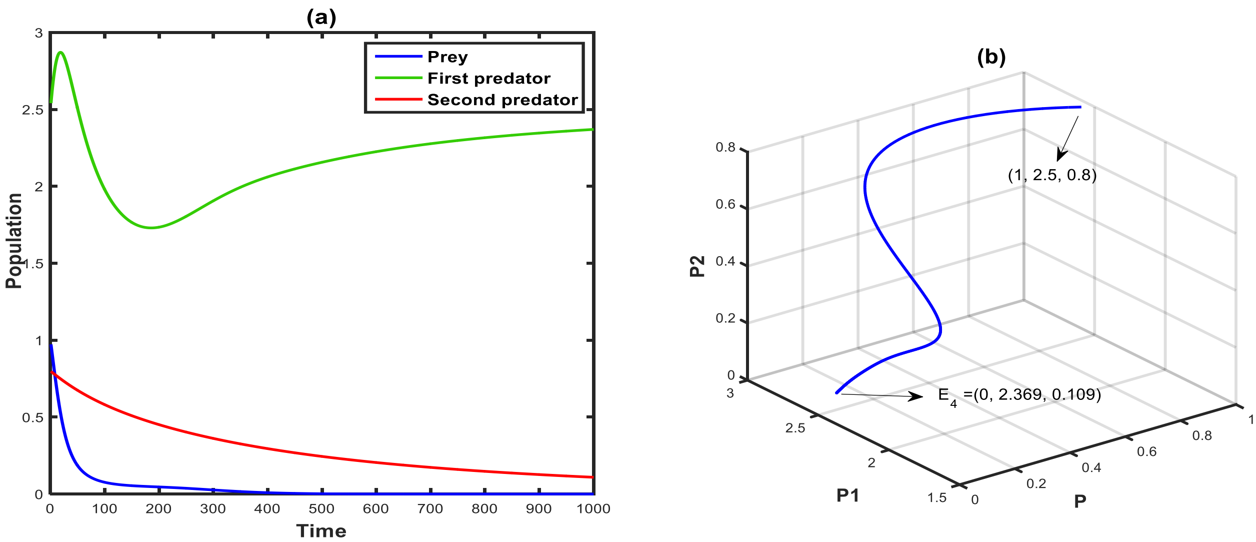

Figure 7: (a) (T.S.) of the solution which goes to $E_4 = (0, 2.369, 0.109)$ when $a_1 = 0.5, a_2 = 0.4,\ m_2 = 0.001, b_2 = 0.1, \alpha = 0.5,\ l_2 = 0.01$, (b)- 3D phase portrait of (a).

7.Conclutions and Discussions

In this study, we proposed a food chain prey-predator model that encompasses three species: the prey, the first predator, and the second predator, taking into account the effects of fear, toxins, and harvesting on the populations of all three species. We have theoretically established the uniqueness and boundedness of the system. The model features six equilibrium points, each of which is locally and globally asymptotically stable under appropriate conditions. Additionally, we have performed a numerical analysis of the model using the specified dataset in Equation (46) and various initial conditions. Identifying the dynamics of the system presented several challenges, as it necessitated the simultaneous adjustment of multiple parameters under specific conditions to locate all the equilibrium points. In contrast, changing each parameter individually while keeping others constant led to the following results which are summarized as follows:

1. System (1) does not have periodic solutions in Int. R^3_+.
2. For parameters (a_i, m_i, b_i, c_i, δ_i, l_i, $i = 1, 2,$ $h_i, i = 1, 2, 3,$, α, γ, d_1) the solution goes to the positive (EP) E_5 while for δ_3 and d_2 made extinction of second predator, that is the solution approaches to E_3 as illustrated in Table (2).
3. Varying the parameters $(a_1, a_2, b_1, b_2, h_1, \delta_1, \delta_3, d_1$ and $d_2)$ at the same time causes extinction of all species, that is the solution approaches to E_0 as illustrated in Figure (4).
4. Varying the parameters (a_1, l_1, d_1, and d_2) at the same time causes extinction of first and second predators, that is the solution approaches to E_1 as illustrated in Figure (5).
5. Varying the parameters $(a_1$, a_2, c_1, δ_3, and $d_2)$ at the same time causes extinction of prey and second predator, that the solution approaches to E_2 as illustrated in Figure (6)
6. Varying the parameters $(a_1, a_2, m_2, b_2, \alpha$ and $l_2)$ at the same time causes extinction of prey, that the solution approaches to E_4 as illustrated in Figure (7).

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