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Fekete-Szegö Functional for q-Bazilevič-Type Close-to-Convex Functions

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Abstract

Schlicht function theory is a subfield of complex analysis and falls within the broader scope of geometric function theory, focusing on the relationship between geometric transformations and one-to-one complex-valued functions. Inspired by the promising applications of fractional q-calculus, we examine analytic functions in this paper and investigate various subfamilies of regular, schlicht, and bi-schlicht functions based on generalized conditions and fundamental geometric properties such as starlikeness, convexity, and close-to-convexity. Furthermore, we determine bounded values for Fekete-Szegö-type problems. Our findings contribute to the growing structure of research on subfamilies of regular functions, with implications for further studies in the geometric function theory of complex analysis.

Keywords: Regular function, Schlicht function, Bi-schlicht function, Bazilevič function, Close-to-convex function, Fekete-Szegö inequality.

الدالة الوظيفية فكيتي–سيغو لعائلة الدوال القريبة من التحدب من النوع q–بازيلفيتش

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الخلاصه

تعتبر نظرية الدوال لشليخت حقلا جزئيا من التحليل العقدى، وتندرج ضمن المجال الأوسع لنظرية الدوال الهندسية، حيث تُركز على العلاقة بين التحويلات البيانية والدوال ذات القيم المركبة و التقابلية الواحد لواحد. واستنادًا إلى الآفاق الواعدة التي تتيحها تطبيقات حساب التفاضل والتكامل الكسري من النمط (q)، ندرس في هذا البحث دوال في متغير واحد ، ونبحث في اصناف جزئية مختلفة من الدوال المنتظمة، ودوال شليخت، ودوال شليخت المزدوجة، استنادًا إلى شروط معممة وخصائص هندسية أساسية مثل التشابه النجمي، ودوال بازيليفيتش، والتحدّب، واشباه التحدب. علاوةً على ذلك، نُحدد قيمًا حدودية لمسائل من نمط فكيتي–سيغو حيث تُساهم نتائجنا التي توصلنا اليها في تطوير و تنمية البنى للابحاث حول الاصناف الجزئية للدوال المنتظمة، مع تداعياتها و تأثيراتها على الدراسات المستقبلية في نظرية الدوال الهندسية للتحليل العقدي.

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1. Introduction

A function φ that is regular in $\mathbb{D}:=\{z \in \mathbb{C}: |z|<1\}$ is considered schlicht if it does not take the same value more than once in $\mathbb{D}$. The domain $\mathbb{D}$ is commonly chosen for studying regular schlicht functions ($\mathbb{RSF}$) because it simplifies computations and results in elegant formulae. This choice does not restrict generality, as the Riemann Mapping Theorem states that any proper simply connected domain ($\mathbb{SCD}$) of $\mathbb{C}$ can be mapped onto $\mathbb{D}$ through a schlicht transformation.

A family of regular functions $\mathcal{P}$ is referred to as a normal family if, for any sequence of functions $\{\phi_n(z)\}_{n\in\mathbb{N}}$ in $\mathcal{P}$, there exists a subsequence that converges uniformly on every compact subset of $\mathbb{D}$. This concept is well established and can be found in standard references on complex analysis.

We denote by $\mathcal{A}$ the family of all regular functions in $\mathbb{D}$ that satisfy the normalization conditions $\varphi(0)=0$ and $\varphi'(0)=1$. Functions in this family are expressed in the form

$$\varphi(z)=z+\sum_{n=2}^{\infty} \varphi_n z^n. \tag{1.1}$$

From a geometric perspective, the condition $\varphi(0)=0$ corresponds to a simple translation of the image domain, while $\varphi'(0)=1$ ensures that the function preserves the local orientation and scale at the origin. The subset of $\mathcal{A}$ consisting of $\mathbb{RSF}$ with these normalization conditions is denoted by $\mathcal{S}$. A fundamental example of a function in $\mathcal{S}$ is the Koebe function, defined as $\varphi(z)=z/(1-z)^2$. This function maps $\mathbb{D}$ onto $\mathbb{C}$, excluding $-\infty$ to $-\frac{1}{4}$.
If a sequence of functions $\{\phi_n(z)\}_{n\in\mathbb{N}}$ in $\mathcal{S}$ converges uniformly to a function φ on every compact subset of $\mathbb{D}$, then φ must be either schlicht or constant in $\mathbb{D}$. Another significant result concerning normal families states that a family of regular functions is normal if and only if it is locally bounded. In the complex plane, most simply connected domains can be conformally mapped in a one-to-one manner onto $\mathbb{D}$. This property is particularly useful as it allows many mathematical problems to be transformed into equivalent problems within the unit disk. Let $\mathcal{V}$ be a $\mathbb{SCD}$ that does not cover the entire complex plane, and let $z_0 \in \mathcal{V}$. Then, there exists a unique conformal mapping $\varphi(z)$ that maps $\mho$ onto $\mathbb{D}$, satisfying the conditions $\varphi(z_0)=0$ and $\varphi'(z_0)>0$.

The origins of modern geometric function theory ($\mathbb{GFT}$) can be traced back to Göttingen in 1851, when Riemann, at an inaugural conference, presented his well-known theorem on conformal mappings of $\mathbb{SCD}$. Later, in 1916, Bieberbach examined $|\varphi_2|$ of a function $\varphi \in \mathcal{S}$ and established that $|\varphi_2| \leqslant 2$, with equality occurring if and only if φ is a rotated version of the function $\varphi(z)$. He further suggested that the inequality $|\varphi_n| \leqslant n$ holds for all $n \in \mathbb{N} \setminus \{1\}$. This assertion became known as the Bieberbach conjecture ($\mathbb{BC}$).

In 1923, Löwner verified the $\mathbb{BC}$ for $n=3$, and over time, numerous authors examined its validity for specific values of n. Ultimately, in 1985, de Branges provided a complete proof of the conjecture for all coefficients using hypergeometric functions. Due to the difficulty of proving the $\mathbb{BC}$, several researchers turned their attention to function subfamilies defined by geometric properties. Among the most significant are the subfamilies of starlike, convex, and close-to-convex functions [1]. These subfamilies are denoted by $\mathcal{S}^*, \mathcal{C}$, and $\mathcal{K}$, respectively, and are defined as follows:

$$\mathcal{S}^* = \left\{\varphi \in \mathcal{S}: \operatorname{Re}\left(\frac{z\varphi'(z)}{\varphi(z)}\right) > 0, z \in \mathbb{D}\right\},$$
$$\mathcal{C} = \left\{\varphi \in \mathcal{S}: \operatorname{Re}\left(\frac{\left(z\varphi'(z)\right)'}{\varphi'(z)}\right) > 0, z \in \mathbb{D}\right\},$$
$$\mathcal{K} = \left\{\varphi \in \mathcal{S}: \operatorname{Re}\left(\frac{z\varphi'(z)}{\psi(z)}\right) > 0, \psi \in \mathcal{S}^* \text{ and } z \in \mathbb{D}\right\}.$$
It is important to note that $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K}$. Consider a function
$$\psi(z) = z + \sum_{k=2}^{\infty} \psi_k z^k \in \mathcal{S}^*. \tag{1.2}$$
Then, as stated in [1], the following inequalities hold:
$$\left.\begin{aligned} &|\psi_k| \leqslant k, k \in \mathbb{N} \setminus \{1\}, \\ &|\psi_3 - \sigma\psi_2^2| \leqslant 2\max\{1, |4\sigma - 3|\}, \sigma \in \mathbb{C}. \end{aligned}\right\} \tag{1.3}$$

Various subfamilies of normalized regular functions within $\mathbb{GFT}$ have been analyzed from multiple perspectives. For example, Tayyah and Atshan [2] established several characteristic properties for certain Ma–Minda families associated with a novel Carathéodory function, including growth and distortion results, inclusion results, and radius estimates. Illafe et al., [3] investigated specific subfamilies of regular functions, distinguished by the variation in the arguments of their coefficients. Their work sheds light on the properties of these functions and highlights their potential applications in the field of complex analysis. Hadi [4] investigated the Toeplitz determinants associated with the inverses of analytic functions, examined their structural properties and derived relevant mathematical formulations. Karthikeyan et al. [5] examined geometric characteristics of close-to-convex functions, providing new insights into their structural properties and expanding the understanding of this subfamily. Additionally, Ali et al. [6] examined novel differential subordination and superordination relationships for a specific subfamily of meromorphic univalent functions, introduced a new operator to establish inclusion results, and derived necessary conditions for function properties within this regular framework.

For $0 \leqslant \alpha < 1$, let $\mathcal{H}_\alpha$ denotes the family of regular functions h that satisfies the condition $\operatorname{Re}\big(h(z)\big) > \alpha$. These functions can be expressed through the Taylor-Maclaurin series expansion as follows:
$$h(z) = 1 + \sum_{n=1}^{\infty} \hbar_n z^n. \tag{1.4}$$
In particular, when $\alpha = 0$, the family $\mathcal{H}_0$, denoted simply by $\mathcal{H}$, consists of functions whose real part is positive, commonly known as Carathéodory functions. Moreover, for functions $h(z) \in \mathcal{H}$ given by (1.4), the following bounds hold [7]:
$$\left.\begin{aligned} &|\hbar_n| \leqslant 2, n \in \mathbb{N}, \\ &\left|\hbar_2 - \sigma\frac{\hbar_1^2}{2}\right| \leqslant 2\max\{1, |\sigma - 1|\}, \sigma \in \mathbb{C}. \end{aligned}\right\} \tag{1.5}$$
The families of starlike and convex functions of order $\alpha (0 \leqslant \alpha < 1)$ are represented by $\mathcal{S}^*(\alpha)$ and $C(\alpha)$, respectively. These families are defined as follows:
$$\mathcal{S}^*(\alpha) = \left\{\varphi \in \mathcal{S}: \operatorname{Re}\left(\frac{z\varphi'(z)}{\varphi(z)}\right) > \alpha, z \in \mathbb{D}\right\},$$
$$\mathcal{C}(\alpha) = \left\{\varphi \in \mathcal{S}: \operatorname{Re}\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > \alpha, z \in \mathbb{D}\right\}.$$

Another notable and well-established subfamily of $\mathcal{S}$ is the Bazilevič function family as well as the subfamily of close-to-convex functions of order $\tau (0 \leqslant \tau \leqslant \pi)$, which are defined as follows, respectively:
$\mathcal{B}(\tau) = \left\{ \varphi \in \mathcal{S} : \operatorname{Re}\left(\frac{z^{1-\tau}\varphi'(z)}{(\varphi(z))^{1-\tau}}\right) > 0, \tau \geqslant 0, z \in \mathbb{D} \right\},$
$\mathcal{K}(\tau) = \left\{ \varphi \in \mathcal{S} : \operatorname{Re}\left(e^{i\tau}\frac{z\varphi'(z)}{\psi(z)}\right) > 0, \psi \in \mathcal{S}^*, 0 \leqslant \tau \leqslant \pi, \text{and } z \in \mathbb{D} \right\}.$
$\mathbb{BC}$ served as the impetus for investigating the Taylor-Maclaurin expansion coefficients of schlicht and later bi-schlicht functions ($\mathbb{BSF}$). A function $\varphi \in \mathcal{A}$ is defined as $\mathbb{BSF}$ if both φ and its inverse are schlicht in $\mathbb{D}$, and this family is denoted by Σ. In 1967, Lewin studied subfamilies of functions in Σ and established that for every $\varphi \in \Sigma, |\varphi_2| \leqslant 1.51$, which sparked further exploration into the bounds of the coefficients. Additionally, in 1969, Netanyahu proved that the maximum value of $|\varphi_2|$ is $4/3$, with the maximum taken over all functions in Σ. Later, in 1981, Styer and Wright demonstrated the existence of a function $\varphi \in \Sigma$ for which $\max|\varphi_2| > 4/3$.
The most precise estimate for the coefficients of functions was obtained by Tan [8], where it was shown that max $|\varphi_2| \leqslant 1.485$. Subsequently, Branan and Taha [9] introduced several subfamilies of functions within the family Σ and derived bounds for the second and third coefficients of functions belonging to each of these subfamilies. The quest to find bounds for the coefficients, particularly the second and third ones, gained further momentum following the work of Srivastava et al., [10]. Their publication reignited interest in the study of functions in Σ and laid the foundation for subsequent research. In general, little is known about the higher coefficients of functions in Σ, noting that finding bounds for the coefficients $|\varphi_n|$ where $n \geqslant 3$ remains an open problem, given that the $\mathbb{BS}$ condition complicates the behavior of higher coefficients.

Many other researchers have since investigated various function subfamilies within Σ, following the path established by [10]. For instance, Tayyah and Atshan [11] derived coefficient inequalities and Fekete–Szegö inequalities for a comprehensive family of bi-Bazilevič and bi-pseudo-starlike functions by employing the Tremblay fractional derivative operator. Hidan et al., [12] studied the coefficient bounds for specific families of bi-Bazilevič and bi-Ozaki-close-to-convex functions, contributing valuable insights to the field of mathematical analysis. Lagad et al., [13] introduced a generalized subfamily of regular and $\mathbb{BSF}$, established coefficient bounds, and investigated properties relevant to function behavior within $\mathbb{GFT}$. Another work by Amini and Al-Omari [14] investigated the subordination properties of $\mathbb{BSF}$ that incorporate Horadam polynomials, providing a significant contribution to the understanding of function theory within the context of complex analysis. Sabir and Ali [15] examined the subordination properties of $\mathbb{SF}$ using a generalized derivative operator, thereby expanding the application of subordination in this field. Al-Hawary et al., [16] introduced a broad subfamily of $\mathbb{BSF}$ defined by error function and subject to subordination conditions, providing a unified framework for their study. Sharma et al., [17] presented a new family of concave $\mathbb{BSF}$ with bounded boundary rotation, advancing the geometric understanding of these functions. Wanas et al., [18] applied (M, N)-Lucas polynomials to a subfamily of $\mathbb{BSF}$, linking τ-pseudo-starlike functions to Sakaguchi-type functions, thus bridging classical and modern approaches. Finally, Atshan et al., [19] analyzed the second Hankel determinant for a specific subfamily of $\mathbb{BSF}$, established new coefficient bounds, and provided analytical insights.

Quantum (briefly, q-) calculus provides a more expansive framework than traditional analysis, eliminating the necessity for limit notation. Since then, the q-calculus serves as a valuable tool for investigating various subfamilies within the families $\mathcal{S}$ and Σ in the context of $\mathbb{GFT}$, where properties like starlikeness and convexity are central. Srivastava [20] examines various operators in basic (q-) calculus and fractional q-calculus, highlighting their importance in $\mathbb{GFT}$ within complex analysis, while exploring their properties and applications to characterize and analyze specific function subfamilies, and integrating fractional q-calculus techniques to offer insights into function behavior, structural properties, and potential extensions in mathematical analysis. Alqahtani et al., [21] characterized generalized q-convex functions using q-calculus, contributing to the deeper understanding of convexity in quantum calculus. Çağlar et al., [22] examined coefficient bounds for q-starlike functions related to q-Bernoulli numbers, bridging the gap between classical starlike functions and q-calculus. Louati et al., [23] defined and analyzed new subfamilies linked to symmetrical functions and quantum calculus, enriching the theoretical foundations of q-based function analysis. Furthermore, Lupaş [24] introduced a convolution operator of the q-hypergeometric type and studied strongly differential subordination and superordination by analyzing distortion constraints, and coefficient bounds.

For a function $\varphi \in \mathcal{A}$ given by (1.1) and for $0 < q < 1$, the q-derivative of φ is defined as follows:

$$\mathcal{D}_q\varphi(z) = \begin{cases} \frac{\varphi(qz)-\varphi(z)}{(q-1)z}, & z \neq 0, \\ \varphi'(0), & z = 0. \end{cases}$$

Or, equivalently,

$$\mathcal{D}_q\varphi(z) = 1 + \sum_{n=2}^{\infty} [n]_q\varphi_n z^{n-1}, \tag{1.6}$$

where $[n]_q$ in (1.6) is given by the formula [25]:

$$[n]_q = \begin{cases} 1 + q + \cdots + q^{n-2} + q^{n-1}, & n \in \mathbb{N}, \\ \dfrac{1-q^n}{1-q}, & n \in \mathbb{C}. \end{cases} \tag{1.7}$$

It is noteworthy that $[0]_q = 0$ and $\lim_{q\to 1^-}[n]_q = n$.

Utilizing $\mathcal{D}_q\varphi(z)$, the subfamilies of q-starlike function of order α and q-convex function of order α are defined, respectively, as follows:

$$\mathcal{S}_q^*(\alpha) = \left\{\varphi \in \mathcal{S} : \operatorname{Re}\left(\frac{z\mathcal{D}_q\varphi(z)}{\varphi(z)}\right) > \alpha, z \in \mathbb{D}\right\},$$

$$\mathcal{C}_q(\alpha) = \left\{\varphi \in \mathcal{S} : \operatorname{Re}\left(\frac{\mathcal{D}_q\left(z\mathcal{D}_q\varphi(z)\right)}{\mathcal{D}_q\varphi(z)}\right) > \alpha, z \in \mathbb{D}\right\}.$$

As $q \to 1^-$, we have $\mathcal{S}_q^*(\alpha) \to \mathcal{S}^*(\alpha)$ and $\mathcal{C}_q(\alpha) \to \mathcal{C}(\alpha)$.

The Littlewood-Paley conjecture (see [26]), which suggested that the coefficients of odd $\mathbb{RSF}$ are bounded by one, was disproved by Fekete and Szegö, a result widely recognized in the field of $\mathbb{GFT}$. The Fekete–Szegö functional problems are vital tools in the study of schlicht functions and are employed in the analysis of power series to verify whether a function possesses certain bounded properties in $\mathbb{D}$. For $\varphi \in \mathcal{S}$, the expression $\varphi_3 - \varphi_2^2$ is recognized as the Fekete–Szegö functional problem, which was later generalized to the form $\varphi_3 - \mu\varphi_2^2$, where μ is a real or complex parameter, and has been extensively studied in various settings. For instance, Mostafa and El-Hawsh [27] examined two subfamilies of regular meromorphic functions associated with the q-derivative operator and studied the bounds for the Fekete–Szegö functional. Gagandeep Singh and Gurcharanjit Singh [28] used the q-derivative operator to introduce two novel and unified subfamilies of $\mathbb{BSF}$ in $\mathbb{D}$, and obtained bounds for the initial coefficients and the Fekete–Szegö inequality for these families. Al-Ityan et al., [29] introduced a new operator based on the Salagean q-differential approach to define a new family of regular functions, provided estimates for the first two Taylor–Maclaurin coefficients, and addressed the Fekete–Szegö problems. Furthermore, numerous studies have examined and established the Fekete-Szegö inequality for various subfamilies of functions (see, for example, [30-36]).

Over the years, the renowned $\mathbb{BC}$ has been central to coefficient problems, leading to numerous findings and significantly shaping the development of $\mathbb{GFT}$. Although $\mathbb{GFT}$ has its foundations in classical mathematics, it remains highly relevant across various fields, including fluid dynamics, modern mathematical physics, partial differential equations, and nonlinear integrable systems.
Motivated by the $\mathbb{BC}$, prior research, and the crucial roles of starlikeness and convexity in $\mathbb{GFT}$ within complex analysis, we integrate the concepts of Bazilevič and close-to-convex functions. Our focus is on establishing coefficient bounds and Fekete-Szegö inequalities for regular functions within the designated function subfamilies. The remainder of this paper is structured as follows: Section 2 explores the boundary values for schlicht and $\mathbb{BSF}$ within the defined subfamilies. In Section 3, these results are utilized to establish the Fekete-Szegö inequalities. Lastly, Section 4 provides the conclusions of our study.

2. Boundary values

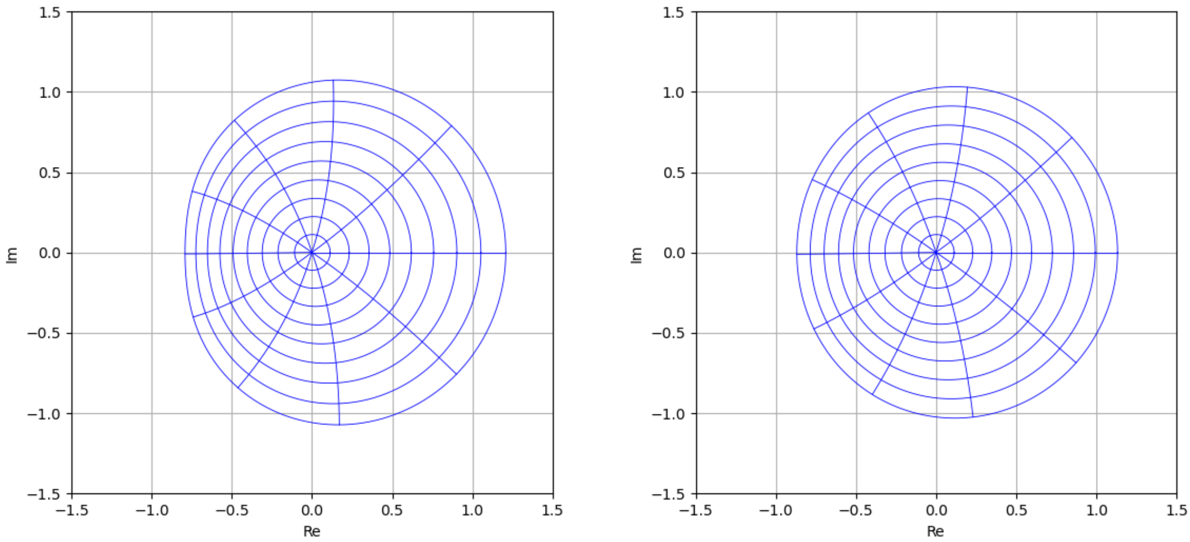
In this section, by employing the q-derivative operator given in (1.6), we introduce several unified subfamilies of Bazilevič and close-to-convex functions and derive coefficient bounds for analytic functions belonging to the defined families.

Definition 2.1. A function $\varphi(z)$ expressed in (1.1) is said to belong to the subfamily $\mathcal{K}^q(\alpha,\tau)(0 \leqslant \alpha < 1, \tau \geqslant 1, 0 < q < 1)$,
if there exists a function $\psi(z) \in \mathcal{S}^*$ such that the following condition holds:

$$\mathcal{K}^q(\alpha,\tau) = \left\{\varphi \in \mathcal{S} : \mathrm{Re}\left(\frac{z\mathcal{D}_q\varphi(z)(\varphi(z))^{\tau-1}}{(\psi(z))^\tau}\right) > \alpha, z \in \mathbb{D}\right\}. \tag{2.1}$$

Example 2.2. Let $\varphi(z) = z + az^2$, where $|a| < \frac{1-\alpha}{1+q}$, and let $\psi(z) = z$. Then, for $\tau = 1$,

$$\begin{aligned}\mathrm{Re}\left(\frac{z\mathcal{D}_q\varphi(z)(\varphi(z))^{\tau-1}}{z^\tau}\right) &= \mathrm{Re}(1 + a(q+1)z) \geqslant 1 - (q+1)|a||z| \\ &> 1 - (q+1)|a| > 1 - (q+1)\frac{1-\alpha}{1+q} = \alpha.\end{aligned}$$



(a) For $q = 0.2$ (b) For $q = 0.9$

Figure 1: Image of $\mathbb{D}$ under $\varphi(z)$ in Example 2.2 for various values of q.

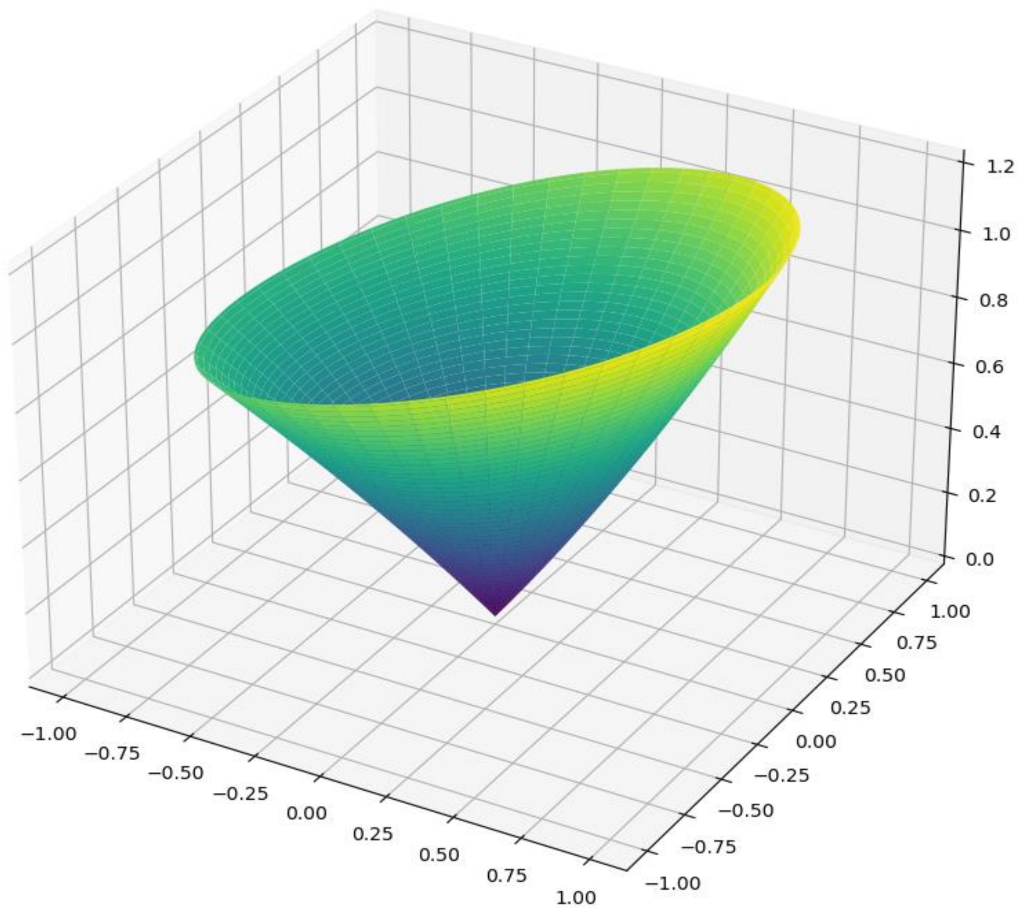


Figure 2: Image of $\mathbb{D}$ under $\varphi(z)$ in Example 2.2 for various values of q.

Theorem 2.3. Consider a function $\varphi(z)$ of the form (1.1). If $\varphi(z)$ belongs to the subfamily $\mathcal{K}^q(\alpha,\tau)$, then

$$|\varphi_2| \leqslant \frac{2(1-\alpha+\tau)}{[2]_q+\tau-1},$$

and

$$|\varphi_3| \leqslant \frac{2\left[(1-\alpha)\left(\left([2]_q+\tau-1\right)^2+2\tau\left(\tau\left|3-\tau-2[2]_q\right|+2[2]_q\tau+(\tau-1)^2-2\right)\right)+\tau\gamma\left([2]_q+\tau-1\right)^2\right]}{\left([2]_q+\tau-1\right)^2\left([3]_q+\tau-1\right)},$$

where

$$\gamma := \frac{|2\tau^2(\tau-3)+4\tau[2]q(\tau-1)+4\tau-(2\tau+1)([2]q+\tau-1)^2|}{\left([2]_q+\tau-1\right)^2}. \tag{2.2}$$

Proof. From (2.1), there exists a function $h(z)\in\mathcal{H}$ given by (1.4) such that

$$\frac{z\mathcal{D}_q\varphi(z)(\varphi(z)^{\tau-1})}{(\psi(z))^\tau} = \alpha+(1-\alpha)h(z), \tag{2.3}$$

where $\varphi(z)$ and $\psi(z)$ are defined by (1.1) and (1.2), respectively.
We now find that:

$$\begin{aligned}\frac{z\mathcal{D}_q\varphi(z)(\varphi(z)^{\tau-1})}{(\psi(z))^\tau} &= 1+\left(\varphi_2\tau+[2]_q\varphi_2-\varphi_2-\psi_2\tau\right)z \\ &\quad+\Big(\frac{1}{2}\varphi_2^2\tau^2+[2]_q\varphi_2^2\tau-\frac{3}{2}\varphi_2^2\tau-[2]_q\varphi_2^2+\varphi_2^2-\varphi_2\psi_2\tau^2-\varphi_2\psi_2\tau[2]_q \\ &\quad+\tau\varphi_2\psi_2+\tau\varphi_3+\Big[3]_q\varphi_3-\varphi_3+\frac{1}{2}\tau^2\psi_2^2+\frac{1}{2}\tau\psi_2^2-\tau\psi_3\Big)z^2+\cdots,\end{aligned}$$

and

$$\alpha+(1-\alpha)h(z) = 1+(1-\alpha)\hbar_1 z+(1-\alpha)\hbar_2 z^2+\cdots.$$

By equating the corresponding coefficients on both sides of (2.3), we derive

$$\varphi_2 = \frac{(1-\alpha)\hbar_1+\tau\psi_2}{[2]_q+\tau-1}. \tag{2.4}$$

Taking the absolute value of (2.4), we obtain

$$|\varphi_2| \leqslant \frac{(1-\alpha)|\hbar_1|+\tau|\psi_2|}{[2]_q+\tau-1}.$$

Thus, by applying the first part of both (1.3) and (1.5), we obtain the required bound for $|\varphi_2|$.
Further coefficient comparisons in (2.3) yield

$$\begin{aligned}&\frac{1}{2}\varphi_2^2\tau^2+[2]_q\varphi_2^2\tau-\frac{3}{2}\varphi_2^2\tau-[2]_q\varphi_2^2+\varphi_2^2-\varphi_2\psi_2\tau^2-\varphi_2\psi_2\tau[2]_q+\tau\varphi_2\psi_2+\tau\varphi_3 \\ &\quad+[3]_q\varphi_3-\varphi_3+\frac{1}{2}\psi_2^2+\frac{1}{2}\tau^2\psi_2^2+\frac{1}{2}\tau\psi_2^2-\tau\psi_3 = (1-\alpha)h_2.\end{aligned} \tag{2.5}$$

By substituting the expression for φ_2 from (2.4) into (2.5), we obtain

$$\begin{aligned}\left([3]_q+\tau-1\right)\varphi_3 &= (1-\alpha)\left(\hbar_2-\frac{2(1-\alpha)\delta}{\left([2]_q+\tau-1\right)^2}\frac{\hbar_1^2}{2}\right)+\tau\left(\psi_3-\frac{2\tau\delta+(1-\tau)([2]q+\tau-1)^2}{2([2]q+\tau-1)^2}\psi_2^2\right) \\ &\quad+\frac{\tau(1-\alpha)\left(\tau+[2]q^2-1\right)}{([2]q+\tau-1)^2}\hbar_1\psi_2,\end{aligned} \tag{2.6}$$

where

$$\delta := \frac{1}{2}\tau^2+\left([2]q-\frac{3}{2}\right)\tau-[2]q+1. \tag{2.7}$$

Since given that $\tau\geqslant 1$ and $0\leqslant\alpha<1$, by applying (1.3), (1.5), and (2.6), we get

$$\left([3]_q+\tau-1\right)|\varphi_3| \leqslant 2(1-\alpha)\max\left\{1, \frac{\left|2(1-\alpha)\delta-\left([2]_q+\tau-1\right)^2\right|}{\left([2]_q+\tau-1\right)^2}\right\}$$

$$+2\tau\max\left\{1, \frac{\left|4\tau\delta-(2\tau+1)\left([2]_q+\tau-1\right)^2\right|}{\left([2]_q+\tau-1\right)^2}\right\}+4\tau(1-\alpha)\left(\frac{\tau\left|3-\tau-2[2]_q\right|+2\left([2]_q-1\right)}{\left([2]_q+\tau-1\right)^2}+1\right). \tag{2.8}$$

Therefore, for $\tau \geqslant 1$, simplifying Equation (2.8) leads to

$$\left([3]_q+\tau-1\right)|\varphi_3| \leqslant 2(1-\alpha)+\frac{2\tau}{\left([2]_q+\tau-1\right)^2}\left|4\tau\delta+2(1-\tau)\left([2]_q+\tau-1\right)^2-3\left([2]_q+\tau-1\right)^2\right|$$

$$+\frac{4\tau(1-\alpha)}{\left([2]_q+\tau-1\right)^2}\left(\tau\left|3-\tau-2[2]_q\right|+\left([2]_q\right)^2+2[2]_q\tau+(\tau-1)^2-2\right).$$

This gives the required bound on $|\varphi_3|$, completing the proof.

Definition 2.4. A function $\varphi(z)$ defined by (1.1) is said to belong to the subfamily $\mathcal{K}_\Sigma^q(\alpha,\tau)(0 \leqslant \alpha < 1, \tau \geqslant 1, 0 < q < 1)$,
if it fulfils the following conditions:

$$\varphi \in \Sigma, \varphi \in \mathcal{K}^q(\alpha,\tau), \text{ and } \varphi^{-1} \in \mathcal{K}^q(\alpha,\tau). \tag{2.9}$$

Theorem 2.5. Let $\varphi(z)$ be a regular function in $\mathbb{D}$, expressed in the form of (1.1). If $\varphi(z) \in \mathcal{K}_\Sigma^q(\alpha,\tau)$, then

$$|\varphi_2| \leqslant \sqrt{\frac{4\left(3\tau^2+2\tau[2]_q-3\tau+1-\alpha\right)}{\tau^2+\left(2[2]_q-1\right)\tau-2[2]_q+2[3]_q}},$$

and

$$|\varphi_3| \leqslant \frac{2(1-\alpha)+7\tau}{[3]q+\tau-1}+\frac{4\left(3\tau^2+2\tau\left([2]_q-1\right)+(1-\alpha)^2\right)}{\left([2]_q+\tau-1\right)^2}.$$

Proof. Based on Expression (2.9), there exist functions $h(z)=1+\sum_{n=1}^{\infty} \hbar_n z^n$ and $s(w)=1+\sum_{n=1}^{\infty} s_n w^n$, both belonging to $\mathcal{H}$, such that

$$\frac{z\mathcal{D}_q\varphi(z)(\varphi(z))^{\tau-1}}{(\psi(z))^\tau}=\alpha+(1-\alpha)h(z), \tag{2.10}$$

and

$$\frac{w\mathcal{D}_q\phi(w)(\phi(w))^{\tau-1}}{(\Psi(w))^\tau}=\alpha+(1-\alpha)s(w), \tag{2.11}$$

where $\varphi(z)$ and $\psi(z)$ are defined by (1.1) and (1.2), respectively, and their corresponding inverses, $\phi(w)$ and $\Psi(w)$, are given by

$$\phi(w):=w-\varphi_2w^2+(2\varphi_2^2-\varphi_3)w^3+\cdots, \tag{2.12}$$

$$\Psi(w):=w-\psi_2w^2+(2\psi_2^2-\psi_3)w^3+\cdots. \tag{2.13}$$

Clearly, we have

$$1+(1-\alpha)\hbar_1z+(1-\alpha)\hbar_2z^2+\cdots,$$

and

$$1+(1-\alpha)s_1w+(1-\alpha)s_2w^2+\cdots.$$

We further find that

$$\frac{z\mathcal{D}_q\varphi(z)(\varphi(z)^{\tau-1})}{(\psi(z))^\tau}=1+\left(\varphi_2\tau+[2]_q\varphi_2-\varphi_2-\psi_2\tau\right)z$$

$$+(\frac{1}{2}\varphi_2^2\tau^2+[2]_q\varphi_2^2\tau-\frac{3}{2}\varphi_2^2\tau-[2]_q\varphi_2^2+\varphi_2^2-\varphi_2\psi_2\tau^2-\varphi_2\psi_2\tau[2]_q$$

$$+\tau\varphi_2\psi_2+\tau\varphi_3+\left[3\right]_q\varphi_3-\varphi_3+\frac{1}{2}\tau^2\psi_2^2+\frac{1}{2}\tau\psi_2^2-\tau\psi_3\Big)z^2+\cdots,$$

and

$$\frac{w\mathcal{D}_q\phi(w)(\phi(w))^{\tau-1}}{(\Psi(w))^\tau} = 1 + \left(-\varphi_2\tau - [2]_q\varphi_2 + \varphi_2 + \psi_2\tau\right)w$$
$$+\Big(\frac{1}{2}\varphi_2^2\tau^2 + [2]_q\varphi_2^2\tau - \frac{3}{2}\varphi_2^2\tau - [2]_q\varphi_2^2 - \varphi_2^2 - \varphi_2\psi_2\tau^2 - \varphi_2\psi_2\tau[2]_q$$
$$+\tau\varphi_2\psi_2 + \tau\varphi_3 + [3]_q\varphi_3 - \varphi_3 - 2[3]_q\varphi_2^2 + \frac{1}{2}\tau^2\psi_2^2 + \frac{1}{2}\tau\psi_2^2 + \tau\psi_3\Big)w^2 + \cdots.$$

By matching the corresponding coefficients in Equations (2.10) and (2.11), we obtain

$$\tau\varphi_2 + [2]_q\varphi_2 - \varphi_2 - \tau\psi_2 = (1-\alpha)\hbar_1, \tag{2.14}$$

$$\begin{aligned}\left([3]_q + \tau - 1\right)\varphi_3 = (1-\alpha)\hbar_2 - \frac{1}{2}\tau^2\varphi_2^2 - [2]_q\tau\varphi_2^2 + \frac{3}{2}\tau\varphi_2^2 + [2]_q\varphi_2^2 - \varphi_2^2 \\ +\tau^2\varphi_2\psi_2 + [2]_q\tau\varphi_2\psi_2 - \tau\varphi_2\psi_2 - \frac{1}{2}\tau^2\psi_2^2 - \frac{1}{2}\tau\psi_2^2 + \tau\psi_3,\end{aligned} \tag{2.15}$$

$$-\tau\varphi_2 - [2]_q\varphi_2 + \varphi_2 + \tau\psi_2 = (1-\alpha)s_1, \tag{2.16}$$

and

$$\begin{aligned}-\left([3]_q + \tau - 1\right)\varphi_3 = (1-\alpha)s_2 - \frac{1}{2}\tau^2\varphi_2^2 + \tau^2\varphi_2\psi_2 - \frac{1}{2}\tau^2\psi_2^2 - [2]_q\tau\varphi_2^2 - \frac{1}{2}\tau\varphi_2^2 \\ +[2]_q\tau\varphi_2\psi_2 - \tau\varphi_2\psi_2 + \frac{3}{2}\tau\varphi_2^2 - \tau\psi_3 + [2]_q\varphi_2^2 - 2[3]_q\varphi_2^2 + \varphi_2^2.\end{aligned} \tag{2.17}$$

Utilizing Equations (2.14) and (2.16), we derive

$$\hbar_1 = -s_1, \tag{2.18}$$

and

$$2\left([2]_q + \tau - 1\right)\left(\left([2]_q + \tau - 1\right)\varphi_2^2 - 2\tau\right)\varphi_2\psi_2 = (1-\alpha)^2(\hbar_1^2 + s_1^2) - 2\tau^2\psi_2^2. \tag{2.19}$$

By summing Equations (2.15) and (2.17), we arrive at

$$\varphi_2^2 = \frac{(1-\alpha)(\hbar_2 + s_2) - \tau(\tau-1)\psi_2^2 + 2\tau\left([2]_q + \tau - 1\right)\varphi_2\psi_2}{(\tau-1)\left(\tau + 2[2]_q\right) + 2[3]_q}. \tag{2.20}$$

By considering the absolute value of (2.20) and applying (1.3) and (1.5), we derive the first required bound.

Next, to establish the bound on $|\varphi_3|$, we subtract (2.17) from (2.15), yielding

$$2\left([3]_q + \tau - 1\right)\varphi_3 = (1-\alpha)(\hbar_2 - s_2) + 2\left([3]_q + \tau - 1\right)\varphi_2^2 + 2\tau(\psi_3 - \psi_2^2). \tag{2.21}$$

Upon substituting (2.19) into (2.21), we derive

$$\varphi_3 = \frac{(1-\alpha)(\hbar_2 - s_2) + 2\tau(\psi_3 - \psi_2^2)}{[3]_q + \tau - 1} + \frac{(1-\alpha)^2(\hbar_1^2 + s_1^2) + 2\tau\left(2\left([2]_q + \tau - 1\right)\varphi_2\psi_2 - \tau\psi_2^2\right)}{2\left([2]_q + \tau - 1\right)^2}. \tag{2.22}$$

Finally, by taking the absolute value of (2.22) and applying (1.3) and (1.5), we derive the second required bound.

Definition 2.6. A function $\varphi(z)$ defined by (1.1) is classified under $\mathcal{K}^q(\beta,\tau)(0 < \beta \leqslant 1, \tau \geqslant 1, 0 < q < 1)$,
if there exists a function $\psi(z) \in \mathcal{S}^*$ that satisfies the following condition:

$$\mathcal{K}^q(\beta,\tau) = \left\{\varphi \in \mathcal{S}: \left|\arg\left(\frac{z\mathcal{D}_q\varphi(z)(\varphi(z))^{\tau-1}}{(\psi(z))^\tau}\right)\right| < \frac{\beta\pi}{2}, z \in \mathbb{D}\right\}. \tag{2.23}$$

Theorem 2.7. Let $\varphi(z) \in \mathcal{K}^q(\beta,\tau)$ be represented in the form (1.1). Then, we have:

$$|\varphi_2| \leqslant \frac{2(\beta+\tau)}{[2]_q+\tau-1},$$

and

$$|\varphi_3| \leqslant \frac{2\left[(\beta+\tau\gamma)([2]q+\tau-1)^2+2\tau\beta\left(\tau|3-\tau-2[2]q|+\left([2]_q+\tau\right)^2-2\tau-1\right)\right]}{([2]q+\tau-1)^2([3]q+\tau-1)},$$

where γ is given by (2.2).

Proof. From (2.23), there exists a function $h(z)$ given by (1.4) such that

$$\frac{z\mathcal{D}_q\varphi(z)(\varphi(z))^{\tau-1}}{(\psi(z))^{\tau}} = 1+\beta\hbar_1 z+\left(\frac{1}{2}\beta(\beta-1)\hbar_1^2+\beta\hbar_2\right)z^2+\cdots. \tag{2.24}$$

By comparing the coefficients on both sides of Equation (2.24), we obtain:

$$\varphi_2 = \frac{\beta\hbar_1+\tau\psi_2}{[2]_q+\tau-1}. \tag{2.25}$$

Now, taking the absolute value of (2.25) and applying the first part of both (1.3) and (1.5), we obtain the required bound for $|\varphi_2|$.
Furthermore, by comparing additional coefficients, we derive:

$$\begin{aligned}\frac{1}{2}\varphi_2^2\tau^2+[2]_q\varphi_2^2\tau-\frac{3}{2}\varphi_2^2\tau-[2]_q\varphi_2^2+\varphi_2^2-\varphi_2\psi_2\tau^2-\varphi_2\psi_2\tau[2]_q+\tau\varphi_2\psi_2+\tau\varphi_3\\ +[3]_q\varphi_3-\varphi_3+\frac{1}{2}\tau^2\psi_2^2+\frac{1}{2}\tau\psi_2^2-\tau\psi_3 = \beta\hbar_2+\frac{1}{2}\beta(\beta-1)\hbar_1^2.\end{aligned} \tag{2.26}$$

Upon substituting φ_2 from (2.25) into (2.26) and simplifying, we obtain

$$\begin{aligned}\left([3]_q+\tau-1\right)\varphi_3 &= \beta\left(\hbar_2-\frac{2\delta\beta+(1-\beta)\left([2]_q+\tau-1\right)^2}{\left([2]_q+\tau-1\right)^2}\frac{\hbar_1^2}{2}\right)\\ &+\tau\left(\psi_3-\left(\frac{\tau\delta}{\left([2]_q+\tau-1\right)^2}-\frac{\tau-1}{2}\right)\psi_2^2\right)+\frac{\tau\beta([2]_q-\tau-1)}{[2]_q+\tau-1}\hbar_1\psi_2.\end{aligned} \tag{2.27}$$

where δ is defined by (2.7).
Considering that $\tau\beta \geqslant 0$, and applying (1.3) and (1.5) to (2.27), we get:

$$\begin{aligned}\left([3]_q+\tau-1\right)|\varphi_3| &\leqslant 2\beta\max\left\{1,\frac{\beta\left|\left(2\delta-\left([2]_q+\tau-1\right)^2\right)\right|}{\left([2]_q+\tau-1\right)^2}\right\}\\ &+2\tau\max\left\{1,\frac{\left|4\tau\delta-(2\tau+1)\left([2]_q+\tau-1\right)^2\right|}{\left([2]_q+\tau-1\right)^2}\right\}+\frac{4\tau\beta}{\left([2]_q+\tau-1\right)^2}(\tau|3-\tau-2[2]_q\tau+\tau^2-(2\tau+1)|.\end{aligned} \tag{2.28}$$

Thus, for $\tau \geqslant 1$, Equation (2.28) simplifies to:

$$\begin{aligned}\left([3]_q+\tau-1\right)|\varphi_3| &\leqslant \frac{2(\beta+\tau)}{\left([2]_q+\tau-1\right)^2}\left|4\tau\delta-(2\tau+1)\left([2]_q+\tau-1\right)^2\right|\\ &+\frac{4\tau\beta}{\left([2]_q+\tau-1\right)^2}\left(\tau\left|3-\tau-2[2]_q\right|+\left([2]_q+\tau\right)^2-(2\tau+1)\right).\end{aligned}$$

This gives the desired bound on $|\varphi_3|$ as required. The proof is now concluded.

Definition 2.8. A function $\varphi(z)$ defined by (1.1) belongs to the subfamily $\mathcal{K}_{\Sigma}^{q}(\beta,\tau)(0<\beta\leqslant 1,\tau\geqslant 1,0<q<1)$,
if it satisfies the following conditions:

$$\varphi\in\Sigma, \varphi\in\mathcal{K}^q(\beta,\tau) \text{ and } \varphi^{-1}\in\mathcal{K}^q(\beta,\tau). \tag{2.29}$$

Theorem 2.9. Let φ be a function belonging to the subfamily $\mathcal{K}_{\Sigma}^{q}(\beta,\tau)$, with the form given by (1.1). Then,

$$|\varphi_2| \leqslant \sqrt{\frac{4\left(\beta^2+3\tau^2+\tau\left(2[2]_q-3\right)\right)}{\tau^2+\tau\left(2[2]_q-\right)-2[2]_q+2[3]_q}},$$

and

$$|\varphi_3| \leqslant \frac{2\beta^2+7\tau}{[3]q+\tau-1}+\frac{4(\beta^2+3\tau^2+2\tau[2]q-2\tau)}{\left([2]_q+\tau-1\right)^2}.$$

Proof. Based on Expression (2.29), there exist functions $h(z)=1+\sum_{n=1}^{\infty}\hbar_n z^n$ and $s(w)=1+\sum_{n=1}^{\infty}s_n w^n$, both belonging to $\mathcal{H}$, such that

$$\frac{z\mathcal{D}_q\varphi(z)(\varphi(z))^{\tau-1}}{(\psi(z))^{\tau}}=[h(z)]^{\beta}, \tag{2.30}$$

and

$$\frac{w\mathcal{D}_q\phi(w)(\phi(w))^{\tau-1}}{(\Psi(w))^{\tau}}=[s(w)]^{\beta}, \tag{2.31}$$

where $\phi(w)$ and $\Psi(w)$ have the forms given in (2.12) and (2.13), respectively.
Clearly, we have

$$[h(z)]^{\beta}=1+\beta\hbar_1 z+\left(\frac{\beta(\beta-1)}{2}\hbar_1^2+\beta\hbar_2\right)z^2+\cdots,$$

and

$$[s(w)]^{\beta}=1+\beta s_1 w+\left(\frac{\beta(\beta-1)}{2}s_1^2+\beta s_2\right)w^2+\cdots.$$

By equating the corresponding coefficients in (2.30) and (2.31), we obtain:

$$\tau\varphi_2+[2]_q\varphi_2-\varphi_2-\tau\psi_2=\beta\hbar_1, \tag{2.32}$$

$$\begin{aligned}\left([3]_q+\tau-1\right)\varphi_3=\beta\hbar_2+\frac{1}{2}\beta(\beta-1)\hbar_1^2-\frac{1}{2}\tau^2\varphi_2^2-[2]_q\tau\varphi_2^2+\frac{3}{2}\tau\varphi_2^2+[2]_q\varphi_2^2-\varphi_2^2\\ +\tau^2\varphi_2\psi_2+[2]_q\tau\varphi_2\psi_2-\tau\varphi_2\psi_2-\frac{1}{2}\tau^2\psi_2^2-\frac{1}{2}\tau\psi_2^2+\tau\psi_3,\end{aligned} \tag{2.33}$$

$$-\tau\varphi_2-[2]_q\varphi_2+\varphi_2+\tau\psi_2=\beta s_1, \tag{2.34}$$

and

$$\begin{aligned}-\left([3]_q+\tau-1\right)\varphi_3=\beta s_2+\frac{1}{2}\beta(\beta-1)s_1^2-\frac{1}{2}\tau^2\varphi_2^2+\tau^2\varphi_2\psi_2-\frac{1}{2}\tau^2\psi_2^2-[2]_q\tau\varphi_2^2-\frac{1}{2}\tau\varphi_2^2\\ +[2]_q\tau\varphi_2\psi_2-\tau\varphi_2\psi_2+\frac{3}{2}\tau\,\psi_2^2-\tau\,\psi_3+[2]_q\varphi_2^2-2[3]_q\varphi_2^2+\varphi_2^2.\end{aligned} \tag{2.35}$$

By combining (2.32) and (2.34), we derive

$$\hbar_1=-s_1 \tag{2.36}$$

and

$$\begin{aligned}\left(2\left(\tau^2+[2]_q^2+1+2\tau[2]_q-2\tau-2[2]_q\right)\varphi_2^2-4\tau^2-4\tau\left([2]_q-1\right)\right)\varphi_2\psi_2\\ =\beta^2(\hbar_1^2+s_1^2)-2\tau^2\psi_2^2.\end{aligned} \tag{2.37}$$

By adding (2.33) to (2.35), we get

$$\varphi_2^2=\frac{\beta(\hbar_2+s_2)+\frac{1}{2}\beta(\beta-1)(\hbar_1^2+s_1^2)-(\tau^2-\tau)\psi_2^2+2\tau\left([2]_q+\tau-1\right)\varphi_2\psi_2}{(\tau-1)\left(\tau+2[2]_q\right)+2[3]_q}. \tag{2.38}$$

Now, taking the absolute value of (2.38), and applying (1.3) and (1.5), we obtain the desired first bound. Next, to find the bound on $|\varphi_3|$, we subtract Equation (2.35) from (2.33), resulting in

$$2\big([3]_q+\tau-1\big)\varphi_3=\beta(\hbar_2-s_2)+\frac{1}{2}\beta(\beta-1)(\hbar_1^2-s_1^2)+2\big([3]_q+\tau-1\big)\varphi_2^2+2\tau(\psi_3-\psi_2^2). \quad (2.39)$$

By substituting Equation (2.37) into (2.39), we obtain

$$\varphi_3=\frac{\beta(\hbar_2-s_2)+\frac{1}{2}\beta(\beta-1)(\hbar_1^2-s_1^2)}{2\big([3]_q+\tau-1\big)}+\frac{\beta^2(\hbar_1^2+s_1^2)-2\tau^2\psi_2^2+4\tau([2]q+\tau-1)\varphi_2\psi_2}{2([2]q+\tau-1)^2}+\frac{\tau(\psi_3-\psi_2^2)}{[3]_q+\tau-1}. \quad (2.40)$$

Finally, by taking the absolute value of Equation (2.40) and using (1.3) and (1.5), we obtain the second required result.

3. Fekete-Szegö inequalities

In this section, we present Fekete–Szegö inequalities for analytic functions within the previously defined subfamilies.

Theorem 3.1. Let $\varphi(z)\in\mathcal{K}^q(\alpha,\tau)$, where μ is an arbitrary complex number. Then,

$$|\varphi_3-\mu\varphi_2^2|\leqslant\frac{2}{[3]_q+\tau-1}[(1-\alpha)\eta+\tau\epsilon+\frac{2\tau(1-\alpha)}{\big([2]_q+\tau-1\big)^2}\times\big(\tau|3-\tau-2[2]q|+2[2]q\tau+(\tau-1)^2+2|\mu|\big([3]_q+\tau-1\big)-2\big),$$

where

$$\eta:=\max\left\{1,\frac{|(1-\alpha)(\tau(\tau-3)+2[2]q(\tau-1)+2+2\mu([3]q+\tau-1))-([2]q+\tau-1)^2|}{([2]q+\tau-1)^2}\right\},$$

$$\epsilon:=\max\left\{1,\frac{|(4\mu-3)\tau^2+(4\mu[3]q-4\mu+4-2[2]q)\tau-[2]q^2+2[2]q-1|}{([2]q+\tau-1)^2}\right\}, \quad (3.1)$$

Proof. From (2.4) and (2.6), it follows that

$$\varphi_3-\mu\varphi_2^2=\frac{1}{[3]_q+\tau-1}\Bigg[(1-\alpha)\left(\hbar_2-\frac{2(1-\alpha)\Big(\delta+\mu\big([3]_q+\tau-1\big)\Big)}{\big([2]_q+\tau-1\big)^2}\frac{\hbar_1^2}{2}\right)+\tau\left(\psi_3-\frac{\tau\Big(\delta+\mu\big([3]_q+\tau-1\big)\Big)+\frac{1}{2}(1-\tau)\big([2]_q+\tau-1\big)^2}{\big([2]_q+\tau-1\big)^2}\psi_2^2\right)+\frac{\tau(1-\alpha)}{\big([2]_q+\tau-1\big)^2}\Big(\tau\big(3-\tau-2[2]_q\big)+2\big([2]_q-1\big)-2\mu\big([3]_q+\tau-1\big)+\big([2]_q+\tau-1\big)^2\Big)\hbar_1\psi_2\Bigg]. \quad (3.2)$$

By taking the absolute value of (3.2) and applying (1.3) and (1.5), we obtain the desired result.

Theorem 3.2. Let $\varphi(z)$, defined by (1.1), belong to the class $\mathcal{K}_{\Sigma}^{q}(\alpha,\tau)$, and let $\mu \in \mathbb{C}$. Then,

$$|\varphi_3 - \mu\varphi_2^2| \leqslant \frac{(1-\alpha)\left(\chi_2 + 2|\mu|\left([3]_q + \tau - 1\right)\right)(\chi_4 + 1)}{\left([3]_q + \tau - 1\right)\chi_2} + \frac{2\tau\chi_3}{[3]_q + \tau - 1} + \frac{8\tau(\chi_2 + |\mu|([2]q + \tau - 1)^2)}{([2]q + \tau - 1)\chi_2} + \frac{4(1-\alpha)^2}{\chi_1},$$

where

$$\chi_1 := 2([2]q + \tau - 1)^2, \tag{3.3}$$

$$\chi_2 := (\tau - 1)\left(\tau + 2[2]_q\right) + 2[3]_q, \tag{3.4}$$

$$\chi_3 := \max\left\{1, \frac{4\tau([3]q + \tau - 1)}{([2]q + \tau - 1)^2} + \frac{4|\mu|(\tau - 1)\left([3]_q + \tau - 1\right)}{(\tau - 1)\left(\tau + 2[2]_q\right) + 2[3]_q}\right\}, \tag{3.5}$$

$$\chi_4 := \max\left\{1, \left|\frac{2\chi_2(1-\alpha)([3]q + \tau - 1)}{([2]q + \tau - 1)^2\left(\chi_2 + 2\mu\left([3]_q + \tau - 1\right)\right)} - 1\right|\right\}.$$

Proof. By combining Equations (2.20) and (2.22) in the functional $\varphi_3 - \mu\varphi_2^2$, we obtain

$$\varphi_3 - \mu\varphi_2^2 = \frac{(1-\alpha)\left(\chi_2 + 2\mu\left([3]_q + \tau - 1\right)\right)}{2\chi_2\left([3]_q + \tau - 1\right)}\left(\hbar_2 + \frac{4\chi_2(1-\alpha)\left([3]_q + \tau - 1\right)}{\chi_1\left(\chi_2 + 2\mu\left([3]_q + \tau - 1\right)\right)}\frac{\hbar_1^2}{2}\right) + \frac{\tau}{[3]_q + \tau - 1}\left(\psi_3 + \frac{\chi_1\chi_2 + \left(2\tau\chi_2 + \mu\chi_1(\tau - 1)\right)\left([3]_q + \tau - 1\right)}{\chi_1\chi_2}\psi_2^2\right) + \frac{(1-\alpha)\left(\chi_2 + 2\mu\left([3]_q + \tau - 1\right)\right)}{2\chi_2\left([3]_q + \tau - 1\right)}s_2 + \frac{(2\tau\left([2]_q + \tau - 1\right))(2\chi_2 + \mu\chi_1)}{\chi_1\chi_2}\varphi_1\psi_2 + \frac{(1-\alpha)^2}{\chi_1}s_1^2. \tag{3.6}$$

By applying the absolute value to Equation (3.6) and utilizing inequalities (1.3) and (1.5), we obtain the desired result.

Theorem 3.3. Let $\varphi(z)$ belong to the class $\mathcal{K}^{q}(\beta,\tau)$ and let $\mu \in \mathbb{C}$. Then,

$$|\varphi_3 - \mu\varphi_2^2| \leqslant \frac{2}{[3]q + \tau - 1}\Big[\beta\varsigma + \tau\epsilon + \frac{2\tau\beta}{([2]q + \tau - 1)^2} \times \left(\tau|3 - \tau - 2[2]q| + 2[2]q\tau + (\tau - 1)^2 + 2|\mu|\left([3]_q + \tau - 1\right) - 2\right)\Big],$$

where ϵ is given by (3.1) and

$$\varsigma := \max\left\{1, \frac{\beta\left|(\mu - 1)\tau + \mu\left([3]_q - 1\right) + 1\right|}{\left([2]_q + \tau - 1\right)^2}\right\}.$$

Proof. From Equations (2.25) and (2.27), we obtain

$$\varphi_3 - \mu\varphi_2^2 = \frac{1}{[3]_q + \tau - 1}\left[\beta\left(\hbar_2 - \left(\frac{2\beta\left(\delta + \mu\left([3]_q + \tau - 1\right)\right)}{\left([2]_q + \tau - 1\right)^2} - \beta + 1\right)\frac{\hbar_1^2}{2}\right) + \tau\left(\psi_3 - \left(\frac{\tau\delta + \tau\mu\left([3]_q + \tau - 1\right)}{\left([2]_q + \tau - 1\right)^2} + \frac{1}{2}(1 - \tau)\right)\psi_2^2\right) + \frac{\tau\beta}{\left([2]_q + \tau - 1\right)^2}\left([2]_q^2 + 2[2]_q\tau - 2\mu[3]_q + 2\mu - \tau^2 + 3\tau - 2\right)\hbar_1\psi_2\right]. \tag{3.7}$$

By taking the absolute value of (3.7) and applying (1.3) and (1.5), we obtain the desired result.

Theorem 3.4. Let $\varphi(z) \in \mathcal{K}_{\Sigma}^{q}(\beta,\tau)$ and $\mu \in \mathbb{C}$. Then

$$|\varphi_3 - \mu\varphi_2^2| \leqslant \frac{\beta\big(\chi_2 + 2|\mu|([3]q+\tau-1)\big)(\vartheta+1)}{([3]q+\tau-1)\chi_2} + \frac{2\tau\chi_3}{\big([3]_q+\tau-1\big)}$$
$$+\frac{2\beta(2\beta\chi_2 + |\mu\chi_1(\beta-1)|)([3]q+\tau-1) + \frac{1}{2}|\beta-1|\chi_1\chi_2}{([3]q+\tau-1)\chi_1\chi_2} + \frac{8\tau\big([2]_q+\tau-1\big)(2\chi_2+|\mu|\chi_1)}{\chi_1\chi_2},$$

where χ_1, χ_2, and χ_3 are given by (3.3), (3.4), and (3.5), respectively, and

$$\vartheta := \max\left\{1, \left|\frac{4\beta\chi_2([3]q+\tau-1)}{\chi_1\big(\chi_2+2\mu([3]q+\tau-1)\big)} + \beta - 2\right|\right\}.$$

Proof. From Equations (2.38) and (2.40), we obtain

$$\varphi_3 - \mu\varphi_2^2 = \frac{\beta\left(\chi_2 + 2\mu\big([3]_q+\tau-1\big)\right)}{2\chi_2\big([3]_q+\tau-1\big)}\left(\hbar_2 + \frac{2(\mu\chi_1(\beta-1)+2\beta\chi_2)\big([3]_q+\tau-1\big)+(\beta-1)\chi_1\chi_2}{\chi_1\left(\chi_2+2\mu\big([3]_q+\tau-1\big)\right)}\frac{\hbar_1^2}{2}\right)$$
$$+\frac{\tau}{\big([3]_q+\tau-1\big)}\left(\psi_3 + \frac{(2\tau\chi_2+\mu\chi_1(\tau-1))\big([3]_q+\tau-1\big)+\chi_1\chi_2}{\chi_1\chi_2}\psi_2^2\right) + \frac{\beta\left(\chi_2+2\mu\big([3]_q+\tau-1\big)\right)}{2\chi_2\big([3]_q+\tau-1\big)}s_2$$
$$+\frac{\beta\big((2\beta\chi_2+\mu\chi_1(\beta-1))\big([3]_q+\tau-1\big)+\frac{1}{2}(\beta-1)\chi_1\chi_2\big)}{2\chi_1\chi_2\big([3]_q+\tau-1\big)}s_1^2 + \frac{(2\chi_1+\mu\chi_2)}{\chi_1\chi_2}\varphi_2\psi_2. \quad (3.8)$$

By taking the absolute value of (3.8) and applying the inequalities (1.3) and (1.5), the desired result is obtained.

4. Corollaries and consequences

In this section, we derive the following corollaries by setting $\alpha = 0, \beta = 1$ and taking the limit as $q \to 1^-$.

Corollary 4.1. Let $\varphi(z)$ be given by the form (1.1), and assume that $\varphi(z)$ belongs to the class $\mathcal{K}(\tau)$, which is equivalent to $\mathcal{K}^{q\to1^-}(0,\tau)$ or $\mathcal{K}^{q\to1^-}(1,\tau)$. Then

$$|\varphi_2| \leqslant 2,$$
$$|\varphi_3| \leqslant \frac{2(7\tau+1)}{\tau+2},$$

and

$$|\varphi_3 - \mu\varphi_2^2| \leqslant \frac{2(\xi_1+\tau\xi_2)}{\tau+2} + \frac{2\tau(2\tau^2+\tau(3+2|\mu|)+4|\mu|-1)}{(\tau+1)^2},$$

where

$$\xi_1 := \max\left\{1, \frac{|\mu(\tau+2)-(\tau+3)|}{(\tau+1)^2}\right\},$$
$$\xi_2 := \max\left\{1, \frac{|3\tau^3-3\tau^2-19\tau+2\mu(\tau+2)-5|}{2(\tau+1)^2}\right\}.$$

Corollary 4.2. Let $\varphi(z)$ be given in the form (1.1), and assume that $\varphi(z)$ belongs to the class $\mathcal{K}_{\Sigma}(\tau) \equiv \mathcal{K}_{\Sigma}^{q\to1^-}(0,\tau) \equiv \mathcal{K}_{\Sigma}^{q\to1^-}(1,\tau)$. Then

$$|\varphi_2| \leqslant \sqrt{\frac{4(3\tau^2+\tau+1)}{\tau^2+3\tau+2}},$$

$$|\varphi_3| \leqslant \frac{2+7\tau}{\tau+2} + \frac{4(3\tau^2+2\tau+1)}{(\tau+1)^2},$$

and

$$|\varphi_3 - \mu\varphi_2^2| \leqslant \frac{\left(\tau^2+3\tau+2+2|\mu|(\tau+2)\right)(\zeta_1+1)}{(\tau^2+3\tau+2)(\tau+2)} + \frac{2\tau\zeta_2}{\tau+2} + \frac{8\tau(\tau^2+3\tau+2+|\mu|(\tau+1)^2)}{(\tau+1)(\tau^2+3\tau+2)} + \frac{2}{(\tau+1)^2},$$

where

$$\zeta_1 \coloneqq \max\left\{1, \left|\frac{2(\tau^2+3\tau+2)(\tau+2)}{(\tau+1)^2\left(\tau^2+3\tau+2+2\mu(\tau+2)\right)} - 1\right|\right\},$$

$$\zeta_2 \coloneqq \max\left\{1, \frac{4\tau(\tau+2)}{(\tau+1)^2} + \frac{4|\mu|(\tau-1)(\tau+2)}{\tau^2+3\tau+2}\right\}.$$

5. Conclusions

This research paper has delved into the realm of Schulich function theory, a vital aspect of complex analysis that examines the interplay between graphical transformations and one-to-one complex-valued functions. Building on the promising applications of fractional q-calculus, we have focused on functions of a single variable and examined various subfamilies of regular, schlicht, and $\mathbb{BSF}$ through generalized conditions and essential geometric properties, including starlikeness, Bazilevič functions, convexity, and close-to-convexity. Additionally, we have derived bounded values for Fekete-Szegö-type problems.

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