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A Prey-Predator Model with Michael Mentence Type of Predator Harvesting and Infectious Disease in Prey

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Abstract

A prey-predator model with Michael Mentence type of predator harvesting and infectious disease in prey is studied. The existence, uniqueness and boundedness of the solution of the model are investigated. The dynamical behavior of the system is studied locally as well as globally. The persistence conditions of the system are established. Local bifurcation near each of the equilibrium points is investigated. Finally, numerical simulations are given to show our obtained analytical results.

Keywords: Prey-predator; Predator harvesting; Disease; Stability, Bifurcation.

نموذج الفريسة – المفترس مع حصاد المفترس من نوع ميكائيل مينتين ومرض معد في الفريسة

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الخلاصة

تمت دراسة نموذج الفريسة – المفترس مع حصاد المفترس من نوع ميكائيل مينتينس ومرض معد في الفريسة. وجود ووحدانية وقيد الحل للنموذج تحققت. تمت دراسة السلوك الديناميكي للنظام محليًا وشاملاً. شروط الاصرار للنظام وجدت. التشعب المحلي بالقرب من كل نقطة من نقاط التوازن اثبتت. المحاكاة العددية تظهر نتائجنا التحليلية التي تم الحصول عليها.

1. Introduction:

There has been growing interest in the study of diseases in prey-predator models, due to the existence of many species in the environment which are in contact with each other continuously in different ways. This is helping the transition of disease between the species rapidly. On the other hand, the impact of harvesting on the community is very important from both ecological and economical points of view. In fact, the presence of disease in the prey, predator, or both is natural in the ecological environment.

Many researchers focused on the study of disease in the prey only [1-5], while others concentrated on the study of disease in the predator only [6-9]. However, there are some studies about the diseases in both prey and predator [10-13].

It is known that the harvesting of the species is required for the cohabitation of the species, and hence it attracted a lot of attention from the researchers in their suggested ecological models. Various kinds of harvesting have been suggested and studied including constant harvesting, density dependent proportional harvesting, and nonlinear harvesting [14-18].

In this paper, a prey-predator system with Michael Mentence type predator harvesting and infectious disease in prey is proposed and studied. In the next section we formulate the system

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mathematically. The existence, uniqueness, and boundedness of the solution are also discussed for the proposed model. Section three deals with the stability analysis of the proposed model and its persistence. Section four is concerned with the study of local bifurcation, while numerical simulation is carried out in section five. Finally, section six includes the discussion and conclusions of our obtained results.

2. Mathematical Model

In this section, the dynamics of a prey-predator model with Michaelis–Menten type of harvesting from predator and infectious disease in prey is proposed and studied. The following hypotheses are adopted to formulate the mathematical model.

(1) The prey population is divided into two classes: the susceptible individuals $S(T)$ and the infected individuals $I(T)$. Here $S(T)$ represents the density of the susceptible prey population at time , while $I(T)$ represents the density of the infected prey population at time T. Moreover, the density of the predator population at time T is represented by $Z(T)$.

(2) The prey population, in the absence of the predator, grows logistically with an intrinsic growth rate of $r > 0$ and an environmental carrying capacity of $k > 0$. It is assumed that the infected prey does not grow or reproduce, which is due to the fact that the disease makes the infected prey individuals weak. However, this population still competes with the susceptible one for food and space.

(3) The susceptible prey population becomes infected by contact according to a saturated incidence rate with an infection rate of $\beta > 0$, and the inhibition rate of disease is denoted by $\alpha > 0$. However the infected individuals cannot return to the susceptible state. Moreover, it is assumed that the disease causes death with a disease death rate denoted by $d_1 > 0$.

(4) The predator population consumes the infected prey according to Holling type-II functional response with a maximum attack rate of $\frac{a_1}{b_1} > 0$ and a half-saturation constant of $\frac{1}{b_1} > 0$. Moreover, the constant $e_1 \in (0,1)$ is the conversion rate from infected prey to predator.

(5) Finally the predator population is assumed to be harvested with the Michael Mentence type of harvesting function, where $E > 0$ represents hunting effort, $c > 0$ is the catchability coefficient of the predator, and $\ell_i, i = 1,2,$ are positive constants. Furthermore, in the absence of prey the predator decays exponentially with a natural death rate of $d_2 > 0$.

Keeping the above hypothesis in view, the dynamics of prey – predator model can be describe in the following set of differential equations :

$$\begin{aligned}
\frac{dS}{dT} &= rS\left(1-\frac{S+I}{k}\right)-\frac{\beta SI}{1+\alpha I}\\
\frac{dI}{dT} &= \frac{\beta SI}{1+\alpha I}-\frac{a_1 I Z}{1+b_1 I}-d_1 I\\
\frac{dZ}{dT} &= \frac{e_1 a_1 I Z}{1+b_1 I}-\frac{c E Z}{\ell_1 E+\ell_2 Z}-d_2 Z
\end{aligned} \tag{1}$$

where $S(0) \geq 0, I(0) \geq 0,$ and $Z(0) \geq 0$. The flow chart of the proposed system is shown in the following block diagram.

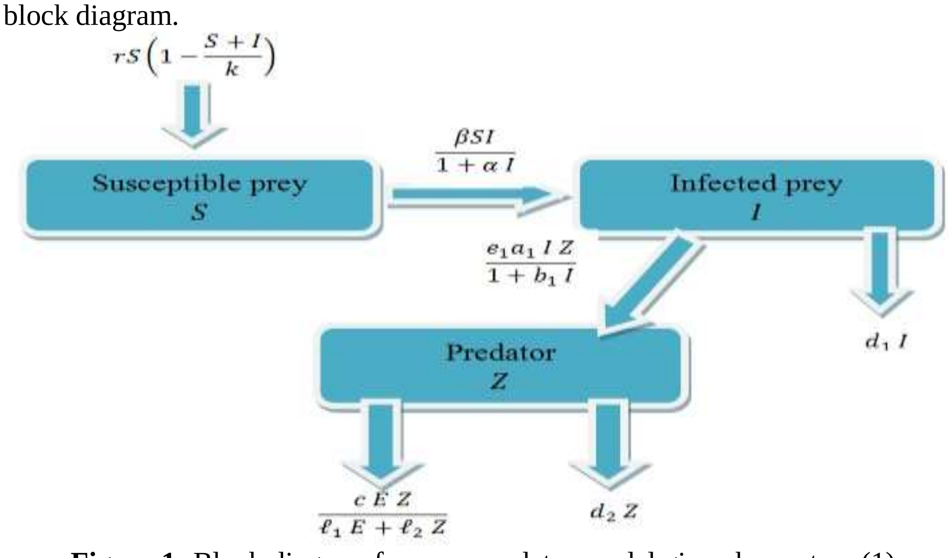


Figure 1- Block diagram for prey-predator model given by system (1).

Clearly, system (1) included 13 parameters, which makes the analysis difficult. Therefore, in order to simplify the system, the number of parameters is reduced to 8 by using the following dimensionless variables and parameters:

$$t = rT\,, s = \frac{S}{k}\,, i = \frac{I}{k}\,, z = \frac{a_1 Z}{k\, b_1\, r}\,, w_1 = \frac{\beta}{r\,\alpha}\,, w_2 = \frac{1}{\alpha\, k}\,, w_3 = \frac{d_1}{r}\,,$$
$$w_4 = \frac{1}{b_1\, k}, w_5 = \frac{e_1\, a_1}{b_1\, r}, w_6 = \frac{d_2}{r}, w_7 = \frac{c\, E\, a_1}{\ell_2 k\, b_1 r^2}\,, w_8 = \frac{\ell_1 E\, a_1}{\ell_2\, k\, b_1\, r} \tag{2}$$

Therefore, system (1) reduces to the following dimensionless system:

$$\begin{aligned}\frac{ds}{dt} &= s\left[1-(s+i)-\frac{w_1\, i}{w_2+i}\right] = sf_1(s,i,z)\\ \frac{di}{dt} &= i\left[\frac{w_1\, s}{w_2+i}-w_3-\frac{z}{w_4+i}\right] = if_2(s,i,z)\\ \frac{dz}{dt} &= z\left[\frac{w_5\, i}{w_4+i}-w_6-\frac{w_7}{w_8+z}\right] = zf_3(s,i,z)\end{aligned} \tag{3}$$

The interaction functions in the right hand side of system (3) are continuous and have continuous partial derivatives on $\mathbb{R}^3_+$. Therefore, these functions are Lipschitizian functions and hence system (3) has a unique solution. Further, in the following theorem, the uniformly boundedness of all the solutions of the system (3) in $\mathbb{R}^3_+$ is established.

Theorem 1. All solutions of system (3) are uniformly bounded.

Proof. According to the first equation of system (3), we get

$$\frac{ds}{dt} \le s[1-s]$$

By the usual comparison theorem, we have $s(t) \le \frac{s_0}{s_0+e^{-t}(1-s_0)}$, $s_0 = s(0)$ and then for $t \to \infty$, we get $s(t) \le 1$.

Now, define the function $\omega(t) = s(t)+i(t)+z(t)$; then the time derivative of $\omega(t)$ along the solution of system (3) is determined by $\frac{d\omega}{dt} \le 2s - \mu\,\omega$, where $\mu = min\{1, w_3\,,\ w_6\}$, and this gives that $\frac{d\omega}{dt} + \mu\,\omega \le 2$. Hence, due to the Granwall lemma [**19**], we obtain $\omega(t) \le \omega_0 e^{-\mu\, t} + \frac{2}{\mu}(1-e^{-\mu\, t})$. Thus, for $t \to \infty$, we have that $0 \le \omega(t) \le \frac{2}{\mu}$.

Hence, all solutions of system (3) are uniformly bounded and therefore we have finished the proof.

3. The stability analysis and persistence

In this section, the existence of the equilibrium points, stability analysis and persistence of system (3) are discussed. It is observed that system (3) has at most four equilibrium points, which can be stated as follows:

The trivial equilibrium point $x_0 = (0,0,0)$ always exists.

The axial equilibrium point (AEP) that is given by $x_1 = (1, 0, 0)$ always exists.

The predator free equilibrium point (PFEP) is given by $x_2 = (\bar{s}, \bar{i}, 0)$, where $\bar{s} = \frac{w_3\,(w_2+\bar{i})}{w_1}$ and $\bar{i}$ represents a unique positive root of the following second order polynomial equation:

$$D_1\, i^2 + D_2\, i + D_3 = 0 \tag{4}$$

where

$$\begin{aligned}D_1 &= -(w_1+w_3) < 0,\\ D_2 &= w_1 - 2w_2w_3 - w_1w_2 - {w_1}^2\\ D_3 &= w_1w_2 - w_3{w_2}^2\,.\end{aligned}$$

Thus, by *Descartes*' *rule* of sign [20], equation (4) has a unique positive root given by:

$$i = \frac{-D_2-\sqrt{{D_2}^2-4D_1D_3}}{2D_1} \tag{5}$$

provided that the following condition holds:

$$w_1 > w_2w_3 \tag{6}$$

The positive equilibrium point (PEP) of system (3) is denoted by $x_3 = (s^*, i^*, z^*)$ where

$$s^* = \frac{(w_2+i^*)-(w_2+i^*)i^*-w_1i^*}{(w_2+i^*)} \tag{7a}$$

$$z^* = \frac{(w_4\, w_6\, w_8+w_4\, w_7)-(w_5\, w_8-w_6\, w_8-w_7)i^*}{((w_5-w_6)i^*-w_4\, w_7)} \tag{7b}$$

while i^* is a unique positive root of the following fourth order polynomial equation:

$$B_1\, i^4 + B_2\, i^3 + B_3\, i^2 + B_4\, i + B_5 = 0 \tag{7c}$$

here

$$\begin{aligned}
B_1 &= -\sigma_1\, \sigma_2\,, \\
B_2 &= \sigma_1(w_1\sigma_3 + \sigma_4) + w_4 w_6\, \sigma_2 - w_7, \\
B_3 &= \sigma_1(w_1\sigma_5 - w_1 w_4\, \sigma_6 - w_2\, w_3\sigma_5) + w_4 w_6\, (w_1\sigma_7 + w_2\sigma_2 + w_3\sigma_8 + \sigma_9) \\
&\quad +2w_2(w_8\sigma_1 - w_7) - w_4\sigma_{10}\,, \\
B_4 &= w_2\sigma_1\, \sigma_{11} + w_2{}^2(w_3 w_4\sigma_1 + w_7) + w_1 w_4 w_6\sigma_{12} \\
&\quad +2w_2 w_4\sigma_{13} + w_2 w_4 w_6\sigma_{14}, \\
B_5 &= w_2 w_4(w_4 w_6\, \sigma_{15} - w_2\sigma_{10})\,,
\end{aligned}$$

with
$$\begin{aligned}
&\sigma_1 = w_5 - w_6,\ \sigma_2 = w_1 + w_3,\ \sigma_3 = 1 - w_2 - w_1 - w_4\,, \\
&\sigma_4 = w_8 - 2w_2 w_3 - w_3 w_4\,,\ \sigma_5 = w_2 + w_4\,,\ \sigma_6 = w_1 + w_2\,, \\
&\sigma_7 = w_1 + w_4,\ \sigma_8 = 1 + w_2,\ \sigma_9 = w_2 w_3 - 1,\ \sigma_{10} = w_6 w_8 + w_7\,, \\
&\sigma_{11} = w_1 w_4 + w_2\, w_8\,,\ \sigma_{12} = w_4(w_1 - 1) + w_2(w_4 - 1)\,, \\
&\sigma_{13} = w_3 w_4 w_6 - w_7,\ \sigma_{14} = w_2 w_3 - 2w_8\,,\ \sigma_{15} = w_2 w_3 - w_1\,.
\end{aligned}$$

Note that from the third equation of system (3), it is clear that the following condition is a necessary condition for growth and hence existence of the predator

$$w_5 > w_6 \tag{8}$$

This leads to $B_1 < 0$. Thus, by *Descartes' rule* of sign, equation (7c) has a unique positive root, provided that one set of the following sets of conditions holds:

$$B_2 < 0\,, B_4 < 0\,, B_5 > 0 \tag{9a}$$
$$B_2 < 0\,, B_4 > 0\,, B_5 > 0 \tag{9b}$$
$$B_2 > 0\,, B_4 > 0\,, B_5 > 0 \tag{9c}$$

Consequently, the positive equilibrium point $x_3 = (s^*, i^*, z^*)$ exists uniquely in the $Int.\mathbb{R}^3_+$, provided that, in addition to condition (8) with one of conditions (9a) or (9b) or (9c), the following conditions hold.

$$(w_1 + w_2 + i^*)i^* < (w_2 + i^*) \tag{10a}$$

$$\left.\begin{aligned}
&\frac{w_4 w_7}{w_5 - w_6} < i^* < \frac{w_4(w_6 w_8 + w_7)}{(w_5 - w_6)w_8 - w_7} \\
&\qquad\qquad or \\
&\frac{w_4(w_6 w_8 + w_7)}{(w_5 - w_6)w_8 - w_7} < i^* < \frac{w_4 w_7}{w_5 - w_6}
\end{aligned}\right\} \tag{10b}$$

Now, the local stability analysis of the above feasible equilibrium points of system (3) is studied using a linearization technique. Note that it is easy to verify that the Jacobian matrix of system (3) at the trivial equilibrium point $x_0 = (0,0,0)$ can be written in the form:

$$J(x_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -w_3 & 0 \\ 0 & 0 & -\left(w_6 + \frac{w_7}{w_8}\right) \end{bmatrix} \tag{11}$$

The eigenvalues of $J(x_0)$ are given by $\lambda_{01} = 1 > 0$, $\lambda_{02} = -w_3 < 0$, $\lambda_{03} = -\left(w_6 + \frac{w_7}{w_8}\right) < 0$. Therefore, the trivial equilibrium point is a saddle point.

The Jacobian matrix at the (AEP), $x_1 = (1, 0, 0)$, can be written in the form:

$$J(x_1) = \begin{bmatrix} -1 & -\left(1 + \frac{w_1}{w_2}\right) & 0 \\ 0 & \frac{w_1}{w_2} - w_3 & 0 \\ 0 & 0 & -\left(w_6 + \frac{w_7}{w_8}\right) \end{bmatrix} \tag{12}$$

Hence, the eigenvalues of $J(x_1)$ are given by $\lambda_{11} = -1 < 0$, $\lambda_{12} = \frac{w_1 - w_2 w_3}{w_2}$ and $\lambda_{13} = -\left(w_6 + \frac{w_7}{w_8}\right) < 0$. Clearly, the AEP is locally asymptotically stable, if the following condition holds:

$$w_1 < w_2 w_3 \tag{13}$$

Moreover, it is a saddle point if the condition (6) holds.

The Jacobian matrix at the (PFEP), $x_2 = (\bar{s}, \bar{i}, 0)$, can be written in the form:

$$J(x_2)=\begin{bmatrix} -\bar{s} & -\left(\bar{s}+\frac{w_1w_2\bar{s}}{(w_2+\bar{i})^2}\right) & 0 \\ \frac{w_1\bar{i}}{(w_2+\bar{i})} & -\frac{w_1\bar{s}\,\bar{i}}{(w_2+\bar{i})^2} & -\frac{\bar{i}}{(w_4+\bar{i})} \\ 0 & 0 & \frac{w_5\bar{i}}{(w_4+\bar{i})}-w_6-\frac{w_7}{w_8} \end{bmatrix} \tag{14}$$

The characteristic equation of $J(x_2)$ can be determined as follows:

$$(\lambda^2-T_2\lambda+D_2)\left(\frac{w_5\bar{\imath}}{(w_4+\bar{\imath})}-w_6-\frac{w_7}{w_8}-\lambda\right)=0 \tag{15}$$

where

$$T_2=-\bar{s}-\frac{w_1\bar{s}\,\bar{i}}{(w_2+\bar{i})^2}$$
$$D_2=(-\bar{s})\left(-\frac{w_1\bar{s}\,\bar{i}}{(w_2+\bar{i})^2}\right)+\left(\bar{s}+\frac{w_1w_2\bar{s}}{(w_2+\bar{i})^2}\right)\left(\frac{w_1\bar{i}}{(w_2+\bar{i})}\right)$$

Obviously, $T_2<0$ and $D_2>0$. Therefore, the two eigenvalues $\lambda_{21}=\frac{T_2}{2}+\frac{1}{2}\sqrt{T_2^{\,2}-4D_2}$, $\lambda_{22}=\frac{T_2}{2}-\frac{1}{2}\sqrt{T_2^{\,2}-4D_2}$ that are obtained from the quadratic term in Eq. (15) have negative real parts. While, the third eigenvalue that is given by $\lambda_{23}=\frac{w_5\bar{i}}{(w_4+\bar{i})}-w_6-\frac{w_7}{w_8}$ will be negative, provided that the following condition holds:

$$\frac{w_5\bar{i}}{(w_4+\bar{i})}<w_6+\frac{w_7}{w_8} \tag{16}$$

Accordingly, the (PFEP) is locally asymptotically stable provided that condition (16) holds.
The Jacobian matrix at the positive equilibrium point $x_3=(s^*,i^*,z^*)$ can be written in the form

$$J(x_3)=\left[a_{ij}\right]_{3\times3} \tag{17}$$

where

$$a_{11}=-s^*<0\,,\ a_{12}=-\left(s^*+\frac{w_1w_2s^*}{(w_2+i^*)^2}\right)<0,\ a_{13}=0$$
$$a_{21}=\frac{w_1i^*}{(w_2+i^*)}>0,\ a_{22}=-\frac{w_1s^*i^*}{(w_2+i^*)^2}+\frac{i^*z^*}{(w_4+i^*)^2}\,,\ a_{23}=-\frac{i^*}{(w_4+i^*)}<0$$
$$a_{31}=0,\ a_{32}=\frac{w_4\,w_5\,z^*}{(w_4+i^*)^2}>0,\ a_{33}=\frac{w_7\,z^*}{(w_8+z^*)^2}>0.$$

Then, the characteristic equation of $J(x_3)$ is

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0 \tag{18}$$

where

$$A_1=-(a_{11}+a_{22}+a_{33})$$
$$A_2=a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}-a_{23}a_{32}-a_{12}a_{21}$$
$$A_3=a_{11}a_{23}a_{32}+a_{12}a_{21}a_{33}-a_{11}a_{22}a_{33}$$

while

$$\begin{aligned}\Delta= A_1\,A_2-A_3 &= -(a_{11}+a_{22})[a_{11}a_{22}-a_{12}a_{21}] \\ &\quad -(a_{22}+a_{33})[a_{22}a_{33}-a_{23}a_{32}] \\ &\quad -a_{11}a_{33}[a_{11}+2a_{22}+a_{33}]=M_1+M_2+M_3\end{aligned}$$

here

$$M_1=-(a_{11}+a_{22})[a_{11}a_{22}-a_{12}a_{21}]>0$$
$$M_2=-(a_{22}+a_{33})[a_{22}a_{33}-a_{23}a_{32}]>0$$
$$M_3=-a_{11}a_{33}[a_{11}+2a_{22}+a_{33}]<0$$

Now, according to the Routh-Hawirtiz Criterion [21], the roots of the Jacobian matrix $J(x_3)$ have negative real parts, provided that $A_1>0$, $A_3>0$ and $\Delta>0$. Direct computation shows that these conditions hold provided that

$$z^*\left(\frac{i^*}{(w_4+i^*)^2}+\frac{w_7}{(w_8+z^*)^2}\right)<\frac{w_1\,s^*\,i^*}{(w_2+i^*)} \tag{19a}$$

$$\frac{w_7}{(w_8+z^*)^2}\left[\left(\frac{w_1s^*}{(w_2+i^*)^2}-\frac{z^*}{(w_4+i^*)^2}\right)+\frac{w_1}{w_2+i^*}\left(1+\frac{w_1\,w_2}{(w_2+i^*)^2}\right)\right]<\frac{w_4\,w_5}{(w_4+i^*)^3} \tag{19b}$$

$$M_1 + M_2 + M_3 > 0 \tag{19c}$$

Therefore, the positive equilibrium point is locally asymptotically stable.

Now, the persistence of the system (3) is studied. Biologically, the system is persisting , if and only if every species exists for all positive time. Moreover, from a mathematical point of view, the solution of system (3) is said to be persistent, if the solution do not have omega limit set in the boundary planes of positive cone. Accordingly, we will show at first that there is no possible omega limit set in the boundary planes, except the equilibrium points.

Clearly, system (3) has only one possible subsystem lying in the non-negative quadrant of si −plane. This subsystem can be written as:

$$\begin{aligned} s\left[1-(s+i)-\frac{w_1 i}{w_2+i}\right] &= g_1(s,i) \\ i\left[\frac{w_1 s}{w_2+i}-w_3\right] &= g_2(s,i) \end{aligned} \tag{20}$$

We define the Dulac function as $(s,i)=\frac{1}{s\,i}$. It is obvious that $B(s,i)>0$ and C^1 function in the $Int.\mathbb{R}_+^2$ of the si −plane . Now, we have

$$\Delta(s,i)=\frac{\partial(B\,g_1)}{\partial s}+\frac{\partial(B\,g_2)}{\partial i}=-\left(\frac{1}{i}+\frac{w_1}{(w_2+\,i)^2}\right)<0$$

Then $\Delta(s,i)$ does not identically zero in the $Int.\mathbb{R}_+^2$ of the si −plane and does not change sign. Thus, due to the Dulac-Bendixson criterion [22], there is no closed curve in the $Int.\mathbb{R}_+^2$ of the si −plane. Hence, according to the Poincare-Bendixon theorem [22], the unique equilibrium point in the $Int.\mathbb{R}_+^2$ of the si −plane, that is given by x_2, will be a globally asymptotically stable whenever it is locally asymptotically stable.

Theorem 2. System (3) is uniformly persistent provided that condition (6) and the following condition hold

$$\frac{w_5\bar{i}}{w_4+\bar{i}} > w_6+\frac{w_7}{w_8} \tag{21}$$

Proof. Consider the following function $(s,i,z)=s^{p_1}\,i^{p_2}\,z^{p_3}$, where p_j ,$\forall j=1,2,3$ are positive constants. Clearly, $\varphi(s,i,z)>0$ for all $(s,i,z)\in Int.\mathbb{R}_+^3$ and $\varphi(s,i,z)\to 0$ when $s\to 0$ or $i\to 0$ or $z\to 0$.

Consequently we obtain

$$\begin{aligned} \Omega(s,i,z)=\frac{\varphi'(s,i,z)}{\varphi(s,i,z)} &= p_1\left[1-(s+i)-\frac{w_1\,i}{w_2+\,i}\right] \\ &+p_2\left[\frac{w_1\,s}{w_2+\,i}-w_3-\frac{z}{w_4+\,i}\right] \\ &+p_3\left[\frac{w_5\,i}{w_4+\,i}-w_6-\frac{w_7}{w_8+z}\right] \end{aligned}$$

Now, the proof follows if $\Omega(E)>0$ for any boundary equilibrium point E, with suitable choice of constants $p_1>0$,$p_2>0$, and $p_3>0$.

$$\Omega(x_1)=p_2\left(\frac{w_1}{w_2}-w_3\right)+p_3\left(-w_6-\frac{w_7}{w_8}\right)$$

$$\Omega(x_2)=p_3\left(\frac{w_5\,\bar{i}}{w_4+\bar{i}}-w_6-\frac{w_7}{w_8}\right)$$

Clearly, $\Omega(x_1)>0$ under condition (6) with suitable choice of positive constants p_2 and p_3 , where p_2 is sufficiently large with respect to the constant p_3. While $\Omega(x_2)>0$ under condition (21). Hence, the proof is complete.

Now, the global stability of each equilibrium point of system (3) is studied using suitable Lyapunov function, as given in the following theorems.

Theorem 3. Assume that the AEP is locally asymptotically stable, then it is a globally asymptotically stable in the $Int.\mathbb{R}_+^3$ provided that the following condition holds.

$$w_3 > \left(1+\frac{w_1}{w_2+i}\right) \tag{22}$$

Proof. Recognize the following function

$$L_1(s,i,z)=\int_1^s \frac{u-1}{u}du\ +i+\frac{1}{w_5}z$$

Clearly, the function L_1 is a positive definite so that $L_1(1,0,0)=0$ and $L_1(s,i,z)>0$ for all $(s,i,z)\in\mathbb{R}_+^3$ with $(s,i,z)\neq(1,0,0)$.
Now, straightforward calculations give that

$$\frac{dL_1}{dt}\leq-(s-1)^2-i\left[-1-\frac{w_1}{w_2+i}+w_3\right]-z\left[\frac{w_6}{w_5}+\frac{w_7}{w_5(w_8+z)}\right]$$

Hence, under condition (22) we obtain that $\frac{dL_1}{dt}$ will be negative definite. Then, L_1 is a Lyapunov function. Therefore, *AEP* is a globally asymptotically stable.

Theorem 4. Assume that the PFEP is locally asymptotically stable, then it is a globally asymptotically stable in the Int. $\mathbb{R}_+^3$ provided that the following conditions hold.

$$q_{12}{}^2<4\,q_{11}q_{22} \tag{23a}$$
$$\frac{w_5\,i}{w_4+i}<w_6+\frac{w_7}{w_8+z} \tag{23b}$$
$$\frac{z}{(w_4+i)(w_4+\bar{i})}<\frac{w_1\,\bar{s}}{(w_2+i)(w_2+\bar{i})} \tag{23c}$$

Proof. Consider the following function

$$L_2(s,i,z)=\int_{\bar{s}}^s\frac{u-\bar{s}}{u}du+\int_{\bar{i}}^i\frac{v-\bar{i}}{v}dv+\ z$$

Clearly, the function $L_2(s,i,z)>0$ is a continuously differentiable real valued function for all $(s,i,z)\in\mathbb{R}_+^3$ with $(s,i,z)\neq(\bar{s},\bar{\imath},0)$ and $L_2(\bar{s},\bar{\imath},0)=0$.
Now, straightforward calculations give that

$$\frac{dL_2}{dt}=-q_{11}(s-\bar{s})^2-q_{12}(s-\bar{s})\left(i-\bar{i}\right)-q_{22}\left(i-\bar{i}\right)^2$$
$$-z\left[w_6+\frac{w_7}{w_8+z}-\frac{w_5\,i}{w_4+\ i}\right]$$

where $q_{11}=1,\ q_{12}=1-\frac{w_1\,\bar{i}}{(w_2+i)(w_2+\bar{i})}$, $q_{22}=\frac{w_1\,\bar{s}}{(w_2+i)(w_2+\bar{i})}-\frac{z}{(w_4+i)(w_4+\bar{i})}$.
Accordingly, using the given conditions (23a)–(23c), we obtain

$$\frac{dL_2}{dt}\leq-\left[\sqrt{q_{11}}(s-\bar{s})+\sqrt{q_{22}}\left(i-\bar{i}\right)\right]^2-z\ \left[w_6+\frac{w_7}{w_8+z}-\frac{w_5\,i}{w_4+i}\right]$$

Then $\frac{dL_2}{dt}$ will be negative definite and L_2 is a Lyapunov function. Therefore the *PFEP* is a globally asymptotically stable .

Theorem 5. Assume that the PEP, $x_3=(s^*,i^*,z^*)$ is locally asymptotically stable in the Int. $\mathbb{R}_+^3$, then it is a globally asymptotically stable provided that the following conditions hold :

$$q_{12}{}^2<4\,q_{11}q_{22} \tag{24a}$$
$$q_{33}(z-z^*)^2<\left[\sqrt{q_{11}}(s-s^*)+\sqrt{q_{22}}(i-i^*)\right]^2 \tag{24b}$$
$$\frac{z^*}{R_2R_2{}^*}<\frac{w_1\,s^*}{R_1R_1{}^*} \tag{24c}$$

Proof. Consider the positive definite function

$$L_3(s,i,z)\ =\int_{s^*}^s\frac{u-s^*}{u}du+\int_{i^*}^i\frac{v-i^*}{v}dv+\frac{(w_4+i^*)}{w_4\,w_5}\int_{z^*}^z\frac{w-z^*}{w}dw$$

Now, the derivative of this function with respect to time can be written as

$$\frac{dL_3}{dt}=-q_{11}(s-s^*)^2-q_{12}(s-s^*)(i-i^*)-q_{22}(i-i^*)^2+q_{33}(z-z^*)^2$$

here $q_{11}=1,q_{12}=1+\frac{w_1\,w_2}{R_1R_1{}^*}-\frac{w_1}{R_1}$, $q_{22}=\frac{w_1\,s^*}{R_1R_1{}^*}-\frac{z^*}{R_2R_2{}^*}$ and $q_{33}=\frac{(w_4+i^*)\,w_7}{w_4\,w_5\,R_3R_3{}^*}$.

with $R_1=(w_2+\ i),\ R_1{}^*=(w_2+\ i^*),$
$R_2=(w_4+\ i),\ R_2{}^*=(w_4+\ i^*),$

$R_3 = (w_8 + z), R_3^* = (w_8 + z^*)$.
Accordingly, using the given conditions (24a)–(24c) we obtain

$$\frac{dL_3}{dt} \le -\left[\sqrt{q_{11}}(s - s^*) + \sqrt{q_{22}}(i - i^*)\right]^2 + q_{33}(z - z^*)^2$$

Then $\frac{dL_3}{dt}$ will be negative definite and L_3 is a Lyapunov function. Therefore, the *PEP* is a globally asymptotically stable.

4. Local Bifurcation

In this section, the local bifurcation near the possible equilibrium points of system (3) is investigated using the Sotomayor's theorem [19]. It is well known that the existence of non-hyperbolic equilibrium point is a necessary but not a sufficient condition for bifurcation to occur. Therefore the candidate bifurcation parameter is selected so that the equilibrium point will be non-hyperbolic at a specific value of that parameter. Now rewrite system (3) in the form:

$$\frac{dX}{dt} = F(X) \tag{25}$$

where $X = (s, i, z)^T$ and $F = (sf_1, if_2, zf_3)^T$ with $f_i; i = 1,2,3$ represent the interaction functions in the right hand side of system (3). Then, according to Jacobian matrix of system (3), straightforward computation shows that for any non-zero vector $V = (v_1, v_2, v_3)^T$, we have the following second and third directional derivatives.

$$D^2F(s,i,z)(V,V) = \begin{pmatrix} -2{v_1}^2 - 2\left(1 + \frac{w_1 w_2}{(w_2+i)^2}\right) v_1 v_2 + 2\frac{w_1 w_2 s}{(w_2+i)^3}{v_2}^2 \\ 2\frac{w_1 w_2}{(w_2+i)^2} v_1 v_2 + 2\left(-\frac{w_1 w_2 s}{(w_2+i)^3} + \frac{w_4\, z}{(w_4+i)^3}\right){v_2}^2 - 2\frac{w_4}{(w_4+i)^2} v_2 v_3 \\ -2\frac{w_4 w_5\, z}{(w_4+i)^3}{v_2}^2 + 2\frac{w_4 w_5}{(w_4+i)^2} v_2 v_3 + 2\frac{w_7 w_8}{(w_8+z)^3}{v_3}^2 \end{pmatrix} \tag{26a}$$

$$D^3F(s,i,z)(V,V,V) = \begin{pmatrix} 6\frac{w_1 w_2}{(w_2+i)^3} v_1 {v_2}^2 - 6\frac{w_1 w_2\, s}{(w_2+i)^4}{v_2}^3 \\ -6\frac{w_1 w_2 v_1 {v_2}^2}{(w_2+i)^3} + 6\left(\frac{w_1 w_2\, s}{(w_2+i)^4} - \frac{w_4\, z}{(w_4+i)^4}\right){v_2}^3 + 6\frac{w_4\, {v_2}^2 v_3}{(w_4+i)^3} \\ 6\frac{w_4 w_5\, z}{(w_4+i)^4}{v_2}^3 - 6\frac{w_4 w_5}{(w_4+i)^3}{v_2}^2 v_3 - 6\frac{w_7 w_8}{(w_8+z)^4}{v_3}^3 \end{pmatrix} \tag{26b}$$

Theorem 6. System (3) undergoes a transcritical bifurcation at AEP when the parameter w_1 passes through the value $w_1^* = w_2 w_3$.

Proof. According to the Jacobian matrix $J(x_1)$ that is given in Eq. (12), system (3) at AEP and $w_1 = w_1^*$ has the following Jacobian matrix $J(x_1, w_1^*) = J_1$.

$$J_1 = \begin{bmatrix} -1 & -(1 + w_3) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\left(w_6 + \frac{w_7}{w_8}\right) \end{bmatrix}$$

Clearly, J_1 has a zero eigenvalue given by ${\lambda_{12}}^* = 0$ and, hence, AEP is a nonhyperbolic point. Now, let $U^{[1]} = \left(u_1^{[1]}, u_2^{[1]}, u_3^{[1]}\right)^T$ be the eigenvector corresponding to the eigenvalue ${\lambda_{12}}^* = 0$.

Thus, $J_1 U^{[1]} = \mathbf{0}$ gives that $U^{[1]} = \left(\beta u_2^{[1]}, u_2^{[1]}, 0\right)^T$, where $\beta = -(1 + w_3) < 0$ and $u_2^{[1]}$ represents any nonzero real number. Also, let $\psi^{[1]} = \left(\psi_1^{[1]}, \psi_2^{[1]}, \psi_3^{[1]}\right)^T$ represents the eigenvector corresponding to the eigenvalue ${\lambda_{12}}^* = 0$ of ${J_1}^T$.

Hence, ${J_1}^T \psi^{[1]} = \mathbf{0}$ gives that $\psi^{[1]} = \left(0, \psi_2^{[1]}, 0\right)^T$, where $\psi_2^{[1]}$ stands for any nonzero real number. Now because

$$\frac{\partial F}{\partial w_1} = F_{w_1}(X, w_1) = \left(-\frac{s\, i}{w_2 + i}, \frac{s\, i}{w_2 + i}, 0\right)^T$$

thus $F_{w_1}(x_1, w_1^*) = (0,0,0)^T$, which gives $\left(\psi^{[1]}\right)^T F_{w_1}(x_1, w_1^*) = 0$. So, according to Sotomayor's theorem for local bifurcation, system (3) has no saddle-node bifurcation at $w_1 = w_1^*$. Furthermore because we have

$$DF_{w_1}(x_1, w_1^*) = \begin{bmatrix} 0 & -\frac{1}{w_2} & 0 \\ 0 & \frac{1}{w_2} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we can show,

$$\left(\psi^{[1]}\right)^T \left(DF_{w_1}(x_1, w_1^*) U^{[1]}\right) = \left(0, \psi_2^{[1]}, 0\right)\left(-\frac{1}{w_2} u_2^{[1]}, \frac{1}{w_2} u_2^{[1]}, 0\right)^T = \frac{1}{w_2} \psi_2^{[1]} u_2^{[1]} \neq 0$$

Moreover, using Eq. (26a) with x_1, w_1^* and $U^{[1]}$ gives

$$D^2F(x_1, w_1^*)\left(U^{[1]}, U^{[1]}\right) = 2\left(u_2^{[1]}\right)^2 \left(-\beta^2 - \beta\,(1 + w_3) + w_3,\ \beta\, w_3 - \frac{w_3}{w_2}, 0\right)^T$$

Hence, it is obtained that

$$\left(\psi^{[1]}\right)^T D^2F(x_1, w_1^*)\left(U^{[1]}, U^{[1]}\right) = 2\, w_3 \left(\beta - \frac{1}{w_2}\right) \psi_2^{[1]} \left(u_2^{[1]}\right)^2 \neq 0.$$

Thus, based on Sotomayor's theorem, system (3) has a transcritical bifurcation at AEP as the parameter w_1 passes through the bifurcation value w_1^*, and that completes the proof.

Theorem 7. System (3) undergoes a transcritical bifurcation at PFEP when the parameter w_6 passes through the value $w_6^* = \frac{w_5 \bar{i}}{(w_4 + \bar{i})} - \frac{w_7}{w_8}$, provided that the following condition holds.

$$\frac{w_4 w_5}{(w_4 + i)^2}\, \alpha_2 + \frac{w_7}{w_8{}^2} \neq 0 \tag{27}$$

where α_2 is given in the proof. Otherwise it undergoes a pitchfork bifurcation while saddle node bifurcation cannot occur.

Proof. From the Jacobian matrix $J(x_2)$ that is given in Eq. (14), system (3) at PFEP and $w_6 = w_6^*$ has the following Jacobian matrix $J(x_2, w_6^*) = J_2$, which has zero eigenvalue, say $\lambda_{23}{}^* = 0$.

$$J_2 = \begin{bmatrix} -\bar{s} & -\left(\bar{s} + \dfrac{w_1 w_2 \bar{s}}{\left(w_2 + \bar{i}\right)^2}\right) & 0 \\ \dfrac{w_1 \bar{i}}{\left(w_2 + \bar{i}\right)} & -\dfrac{w_1 \bar{s}\, \bar{i}}{\left(w_2 + \bar{i}\right)^2} & -\dfrac{\bar{i}}{\left(w_4 + \bar{i}\right)} \\ 0 & 0 & 0 \end{bmatrix} = \left[b_{ij}\right]$$

Now, let $U^{[2]} = \left(u_1^{[2]}, u_2^{[2]}, u_3^{[2]}\right)^T$ represents the eigenvector corresponding to the eigenvalue $\lambda_{23}{}^* = 0$.

Therefore, $J_2 U^{[2]} = \mathbf{0}$ gives that $U^{[2]} = \left(\alpha_1 u_3^{[2]}, \alpha_2 u_3^{[2]}, u_3^{[2]}\right)^T$ where $\alpha_1 = \frac{b_{12}\, b_{23}}{b_{11} b_{22} - b_{12}\, b_{21}} > 0$, $\alpha_2 = -\frac{b_{23}\, b_{11}}{b_{11} b_{22} - b_{12}\, b_{21}} < 0$ and $u_3^{[2]}$ represents any nonzero real number. Also, let $\psi^{[2]} = \left(\psi_1^{[2]}, \psi_2^{[2]}, \psi_3^{[2]}\right)^T$ represents the eigenvector corresponding to the eigenvalue $\lambda_{23}{}^* = 0$ of $J_2{}^T$.

Hence $J_2{}^T \psi^{[2]} = \mathbf{0}$ gives that $\psi^{[2]} = \left(0, 0, \psi_3^{[2]}\right)^T$, where $\psi_3^{[2]}$ stands for any nonzero real number. Now since we have

$$\frac{\partial F}{\partial w_6} = F_{w_6}(X, w_6) = (0, 0, -z)^T$$

it follows that $F_{w_6}(x_2, w_6^*) = (0,0,0)^T$, which gives $\left(\psi^{[2]}\right)^T F_{w_6}(x_2, w_6^*) = 0$. So, according to Sotomayor's theorem for local bifurcation, system (3) has no saddle-node bifurcation at $w_6 = w_6^*$. Furthermore because we have

$$DF_{w_6}(x_2, w_6^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

we can show that

$$\left(\psi^{[2]}\right)^T \left(DF_{w_6}(x_2, w_6^*) U^{[2]}\right) = \left(0, 0, \psi_3^{[2]}\right)\left(0, 0, -u_3^{[2]}\right)^T = -\psi_3^{[2]} u_3^{[2]} \neq 0$$

Moreover, using Eq. (26a) with x_2, w_6^* and $U^{[2]}$ gives

$$D^2F(x_2,w_6^*)\left(U^{[2]},U^{[2]}\right) = 2\left(u_3^{[2]}\right)^2\begin{pmatrix} -\alpha_1{}^2-\left(1+\frac{w_1w_2}{\left(w_2+\bar{i}\right)^2}\right)\alpha_1\alpha_2+\frac{w_1w_2\,\bar{s}}{\left(w_2+\bar{i}\right)^3}\alpha_2{}^2 \\ \frac{w_1w_2}{\left(w_2+\bar{i}\right)^2}\alpha_1\alpha_2-\frac{w_1w_2\,\bar{s}}{\left(w_2+\bar{i}\right)^3}\alpha_2{}^2-\frac{w_4}{\left(w_4+\bar{i}\right)^2}\alpha_2 \\ \frac{w_4w_5}{\left(w_4+\bar{i}\right)^2}\,\alpha_2+\frac{w_7}{w_8{}^2} \end{pmatrix}$$

Hence, it is obtained that

$$\left(\psi^{[2]}\right)^TD^2F(x_2,w_6^*)\left(U^{[2]},U^{[2]}\right)=2\left(\frac{w_4w_5}{\left(w_4+\bar{i}\right)^2}\,\alpha_2+\frac{w_7}{w_8{}^2}\right)\psi_3^{[2]}\left(u_3^{[2]}\right)^2$$

Clearly, $\left(\psi^{[2]}\right)^TD^2F(x_2,w_6^*)\left(U^{[2]},U^{[2]}\right)\neq 0$, provided that condition (27) holds, and then by Sotomayor's theorem, system (3) has a transcritical bifurcation at PFEP as the parameter w_6 passes through the bifurcation value w_6^* . However, if the condition (27) is violating, then we get that $\left(\psi^{[2]}\right)^TD^2F(x_2,w_6^*)\left(U^{[2]},U^{[2]}\right)=0$, and hence further computation shows

$$\left(\psi^{[2]}\right)^TD^3F(x_2,w_6^*)\left(U^{[2]},U^{[2]},U^{[2]}\right)= -6\left(\frac{w_4w_5}{\left(w_4+\bar{i}\right)^3}\,\alpha_2{}^2+\frac{w_7}{w_8{}^3}\right)\psi_3^{[2]}\left(u_3^{[2]}\right)^3\neq 0$$

Therefore system (3) has a pitckfork bifurcation at PFEP as the parameter w_6 passes through the bifurcation value w_6^*, and hence the proof is complete. ■

Theorem 8. System (3) undergoes a saddle- node bifurcation at PEP when the parameter w_7 passes through the value $w_7{}^*=\frac{(w_8+z^*)^2}{z^*}\left(\frac{a_{11}\,a_{23}\,a_{32}}{a_{11}\,a_{22}-a_{12}a_{21}}\right)$, provided that condition (19a) with the following condition hold

$$\begin{aligned}\beta_1&\left[-\gamma_1{}^2-\left(1+\tfrac{w_1w_2}{(w_2+i^*)^2}\right)\gamma_1\gamma_2+\tfrac{w_1w_2\,s^*}{(w_2+i^*)^3}\gamma_2{}^2\right]\\ &+\beta_2\left[\tfrac{w_1w_2}{(w_2+i^*)^2}\gamma_1\gamma_2+\left(-\tfrac{w_1w_2\,s^*}{(w_2+i^*)^3}+\tfrac{w_4\,z^*}{(w_4+i^*)^3}\right)\gamma_2{}^2-\tfrac{w_4}{(w_4+i^*)^2}\gamma_2\right]\\ &\quad+\left[-\tfrac{w_4w_5\,z^*}{(w_4+i^*)^3}\gamma_2{}^2+\tfrac{w_4w_5}{(w_4+i^*)^2}\gamma_2+\tfrac{w_7{}^*\,w_8}{(w_8+z^*)^3}\right]\neq 0\end{aligned}\tag{28}$$

where a_{ij} for all $i,j=1,2,3$ are the elements of Jacobian matrix given by Eq. (17).

Proof. From the Jacobian matrix $J(x_3)$ that is given in Eq. (17), system (3) at PEP and $w_7=w_7{}^*$ has the following Jacobian matrix $J(x_3,w_7{}^*)=J_3=\left[a_{ij}{}^*\right]_{3\times 3}$, where $a_{ij}{}^*=a_{ij}$; $\forall\, i,j=1,2,3$ with $a_{33}{}^*=a_{33}(w_7{}^*)$. Straightforward computation shows that $A_3=0$ in the characteristic equation given by Eq. (18) and then x_3 becomes a nonhyperbolic equilibrium point with zero eigenvalue given by $\lambda^*=0$.

Now, let $U^{[*]}=\left(u_1^{[*]},u_2^{[*]},u_3^{[*]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda^*=0$ of J_3.

Thus $J_3U^{[*]}=\mathbf{0}$, which gives that $U^{[*]}=\left(\gamma_1\,u_3^{[*]},\gamma_2\,u_3^{[*]},u_3^{[*]}\right)^T$ where $\gamma_1=\frac{a_{12}\,a_{23}}{a_{11}\,a_{22}-a_{12}\,a_{21}}$ and $\gamma_2=-\frac{a_{23}\,a_{11}}{a_{11}\,a_{22}-a_{12}\,a_{21}}$ and $u_3^{[*]}$ represents any nonzero real numbers.

Now, let that $\psi^{[*]}=\left(\psi_1^{[*]},\psi_2^{[*]},\psi_3^{[*]}\right)^T$ represents the eigenvector corresponding to the eigenvalue $\lambda^*=0$ of $J_3{}^T$.

Hence $J_3{}^T\psi^{[*]}=\mathbf{0}$, which gives that $\psi^{[*]}=\left(\delta_1\ \psi_3^{[*]},\delta_2\,\psi_3^{[*]},\ \psi_3^{[*]}\right)^T$, where $\delta_1=\frac{a_{21}\,a_{32}}{a_{11}\,a_{22}-a_{12}\,a_{21}}$ and $\delta_2=-\frac{a_{11}\,a_{32}}{a_{11}\,a_{22}-a_{12}\,a_{21}}$, while $\psi_3^{[*]}$ represents any nonzero real number.

Now, since

$$\frac{\partial F}{\partial w_7}=F_{w_7}(X,w_7)=\left(0,0,-\frac{z}{w_8+z}\right)^T$$

thus $F_{w_7}(x_3, w_7^*) = \left(0,0,-\frac{z^*}{w_8+z^*}\right)^T$, which gives that

$$\left(\psi^{[*]}\right)^T F_{w_7}(x_3, w_7^*) = -\frac{z^*}{w_8+z^*}\ \psi_3^{[*]} \neq 0.$$

So, according to Sotomayor's theorem for local bifurcation, the transcritical and pitchfork bifurcation cannot occur, while the first condition of the saddle- node bifurcation is satisfied. Moreover, from Eq. (26a) with x_3, w_7^* and $U^{[*]}$ we obtain that:

$$D^2F(x_3, w_7^*)\left(U^{[*]}, U^{[*]}\right) =$$

$$2\left(u_3^{[*]}\right)^2 \begin{pmatrix} -\gamma_1^2 - \left(1+\frac{w_1 w_2}{(w_2+\ i^*)^2}\right)\gamma_1\gamma_2 + \frac{w_1 w_2\ s^*}{(w_2+\ i^*)^3}\gamma_2^2 \\ \frac{w_1 w_2}{(w_2+\ i^*)^2}\gamma_1\gamma_2 + \left(-\frac{w_1 w_2\ s^*}{(w_2+\ i^*)^3} + \frac{w_4\ z^*}{(w_4+\ i^*)^3}\right)\gamma_2^2 - \frac{w_4}{(w_4+\ i^*)^2}\gamma_2 \\ \left(-\frac{w_4 w_5\ z^*}{(w_4+\ i^*)^3}\gamma_2^2 + \frac{w_4 w_5}{(w_4+\ i^*)^2}\gamma_2 + \frac{w_7^*\ w_8}{(w_8+\ z^*)^3}\right) \end{pmatrix}$$

Hence we get that

$$\left(\psi^{[*]}\right)^T D^2F(x_3, w_7^*)\left(U^{[*]}, U^{[*]}\right) =$$

$$2\ \psi_3^{[*]}\left(u_3^{[*]}\right)^2 \Bigg[\beta_1\left[-\gamma_1^2 - \left(1+\frac{w_1 w_2}{(w_2+\ i^*)^2}\right)\gamma_1\gamma_2 + \frac{w_1 w_2\ s^*}{(w_2+\ i^*)^3}\gamma_2^2\right]$$

$$+\beta_2\ \left[\frac{w_1 w_2}{(w_2+\ i^*)^2}\gamma_1\gamma_2 + \left(-\frac{w_1 w_2\ s^*}{(w_2+\ i^*)^3} + \frac{w_4\ z^*}{(w_4+\ i^*)^3}\right)\gamma_2^2 - \frac{w_4}{(w_4+\ i^*)^2}\gamma_2\right]$$

$$+\left[-\frac{w_4 w_5\ z^*}{(w_4+\ i^*)^3}\gamma_2^2 + \frac{w_4 w_5}{(w_4+\ i^*)^2}\gamma_2 + \frac{w_7^*\ w_8}{(w_8+\ z^*)^3}\right]\Bigg] \neq 0$$

Hence, system (3) has saddle-node bifurcation at x_3 with parameter $w_7 = w_7^*$. Otherwise, when condition (28) is not satisfied the system (3) has no any type of bifurcation.

5. Numerical Simulation

Numerical simulation results are equally important to those obtained from analysis. The objective is to confirm the analytical findings and study the effects of varying the parameters values on the dynamical behavior of the system (3). All the numerical simulation results for system (3) are represented in some figures using MATLAB Now, for the following set of hypothetical parameters set:

$$w_1 = 0.5, w_2 = 0.1, w_3 = 0.1, w_4 = 0.5, w_5 = 0.5, w_6 = 0.1, w_7 = 0.1, w_8 = 1. \qquad (29)$$

We obtained that the trajectories of system (3) with three different sets of positive initial conditions approach asymptotically to the PEP, $x_3 = (0.35, 0.27, 0.29)$, as shown in Figure-2.

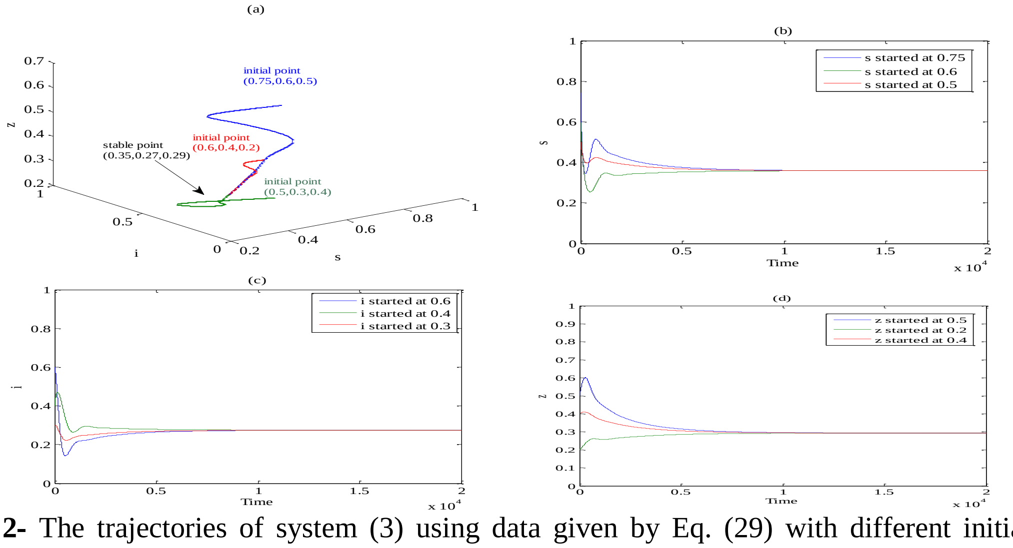


Figure 2- The trajectories of system (3) using data given by Eq. (29) with different initial points approach asymptotically to PEP, represented by $x_3 = (0.35, 0.27, 0.29)$. (a) 3D Phase plot of system (3). (b) Time series of the trajectories of s. (c) Time series of the trajectories of i. (d) Time series of the trajectories of z.

Clearly, Figure-2 shows that PEP is a globally asymptotically stable. Now, we investigate the effect of varying the parameter w_1 on the dynamical behavior of system (3), with the rest of parameters fixed, as in Eq. (29), in the ranges of $0 < w_1 < 0.02$, $0.02 \leq w_1 < 0.08$, $0.08 \leq w_1 \leq 0.9$ and $w_1 > 0.9$ respectively. It is observed that the trajectory of system (3) approaches asymptotically to AEP, PFEP in the interior of si −plane, PEP in the $Int.\mathbb{R}^3_+$ and again to PFEP in the interior of si −plane at the typical values of w_1, as shown in Figure-3.

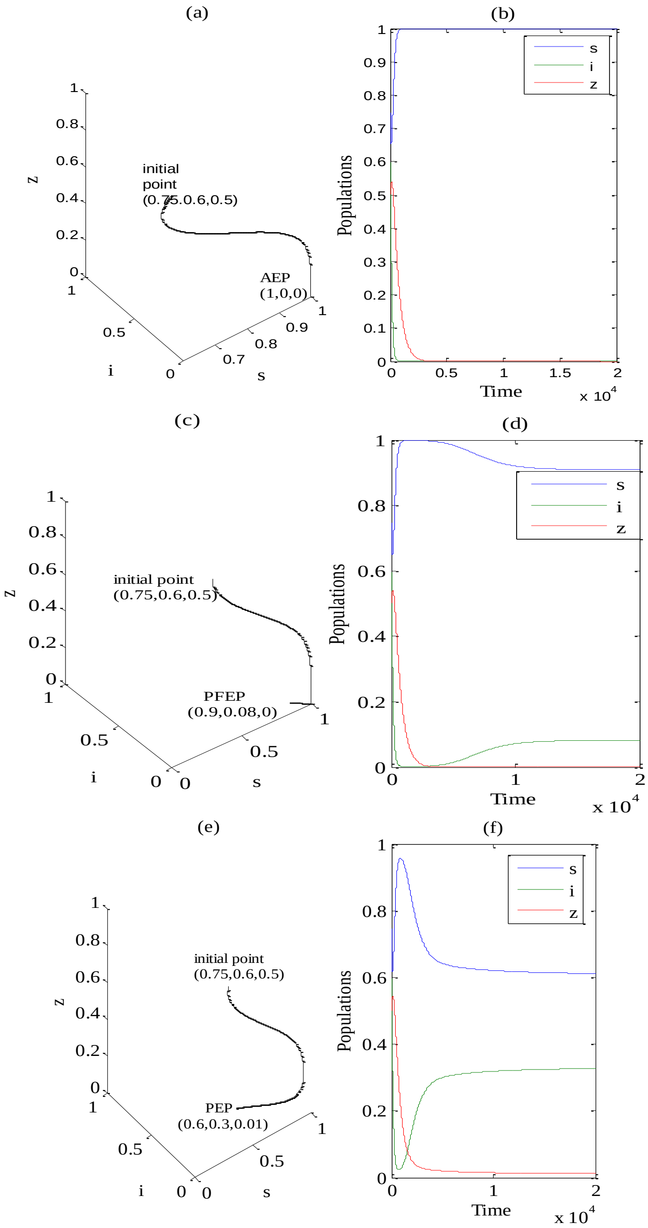


Figure 3-The trajectories of system (3) using data given by Eq. (29) with typical values of w_1. (a) System (3) approaches to AEP for $w_1 = 0.01$. (b) Time series of the trajectory given in (a). (c) System (3) approaches to PFEP in the interior of si −plane for $w_1 = 0.02$. (d) Time series of the trajectory given in (c). (e) System (3) approaches to PEP for $w_1 = 0.08$. (f) Time series of the trajectory given in (e).

The effect of varying w_2 on the dynamics of system (3) is studied. It is observed, for the data given by Eq. (29), that with $0 < w_2 < 2.7$, $2.7 \leq w_2 < 5.1$ and $w_2 \geq 5.1$, the trajectory of system (3) approaches asymptotically to PEP in the $Int.\mathbb{R}^3_+$, PFEP in the interior of si −plane, and AEP, respectively, as illustrated in Figure-4 for the typical values of w_2.

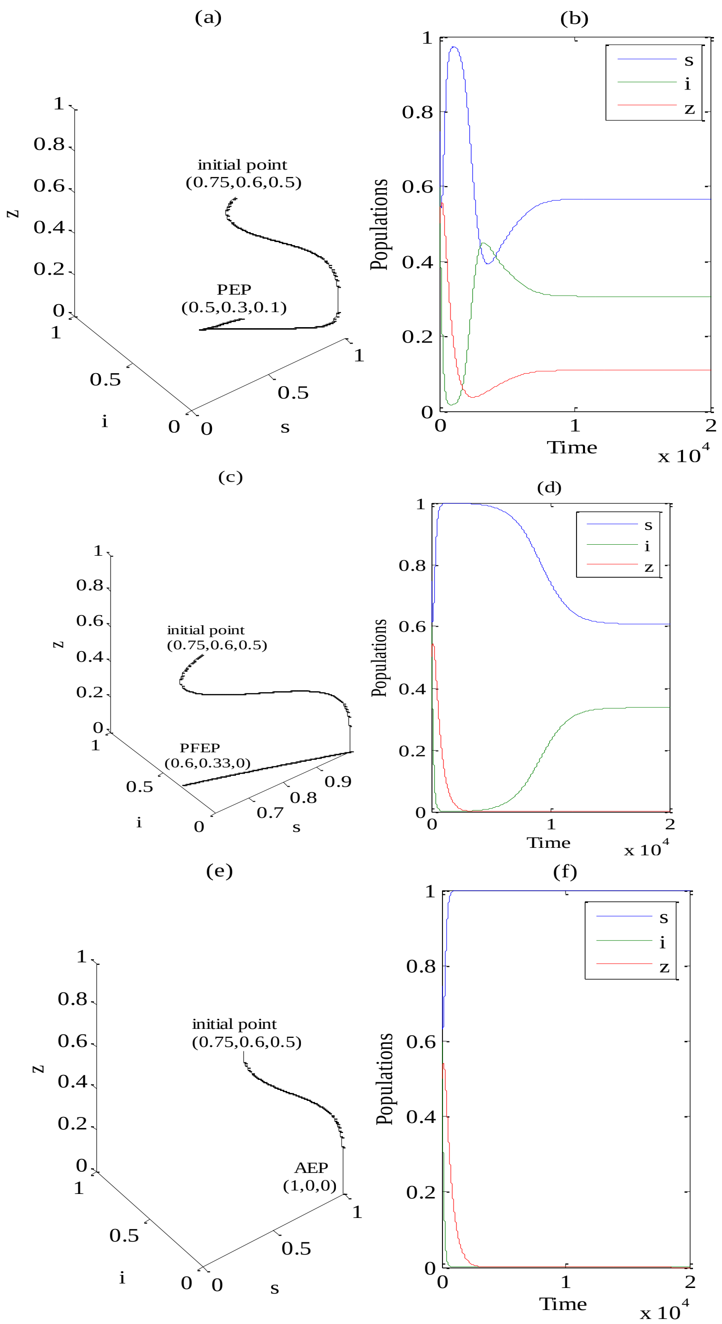


Figure 4- The trajectories of system (3) using data given by Eq. (29) with typical values of w_2. (a) System (3) approaches to PEP for $w_2 = 0.9$. (b) Time series of the trajectory given in (a). (c) System (3) approaches to PFEP in the interior of si −plane for $w_2 = 2.7$. (d) Time series of the trajectory given in (c). (e) System (3) approaches to AEP for $w_2 = 5.1$. (f) Time series of the trajectory given in (e).

According to Figures 3 and 4, the dynamics of system (3) is sensitive to varying in the value of w_1 or w_2. Now, the effect of varying the parameter w_3 in the ranges $0 < w_3 < 0.34$, $0.34 \leq w_3 < 5$ and $w_3 \geq 5$, while keeping the rest of parameters as in Eq. (29). is studied. It is observed that the trajectory of system (3) approaches asymptotically to PEPand PFEP in the interior of si −plane, and to AEP, respectively, as illustrated in Figure-5 for some the typical values.

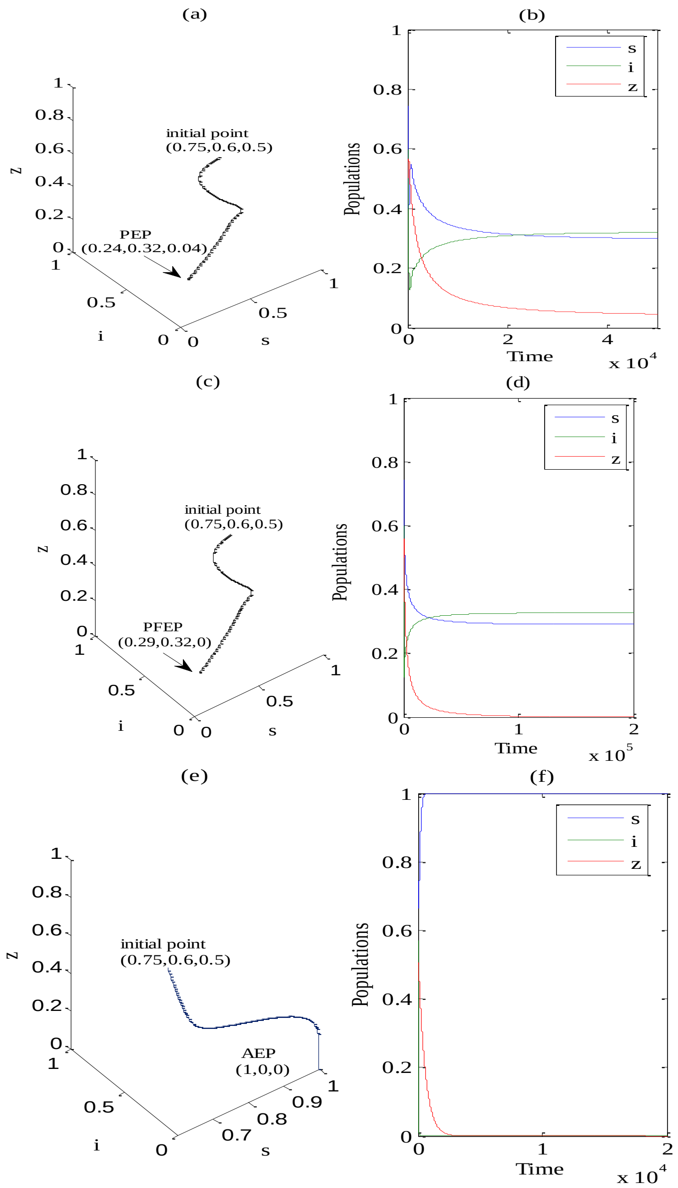


Figure 5- The trajectories of system (3) using data given by Eq. (29) with typical values of w_3. (a) System (3) approaches to PEP for $w_3 = 0.3$. (b) Time series of the trajectory given in (a). (c) System (3) approaches to PFEP in the interior of si −plane for $w_3 = 0.34$. (d) Time series of the trajectory given in (c). (e) System (3) approaches to AEP for $w_3 = 5$. (f) Time series of the trajectory given in (e).

On the other hand, the effect of varying w_4 on the dynamical behavior of system (3) is studied by solving the system numerically using the set of parameters given in Eq. (29) with $0.04 \le w_4 < 0.07$, $0.07 \le w_4 < 0.72$ and $w_4 \ge 0.72$, respectively. It is observed that the trajectory of system (3) approaches asymptotically to a periodic dynamics in $Int.\mathbb{R}^3_+$, PEP and PFEP in the interior of si −plane, as shown in Figure-6 for the typical values $w_4 = 0.05$, $w_4 = 0.07$ and $w_4 = 0.72$, respectively.

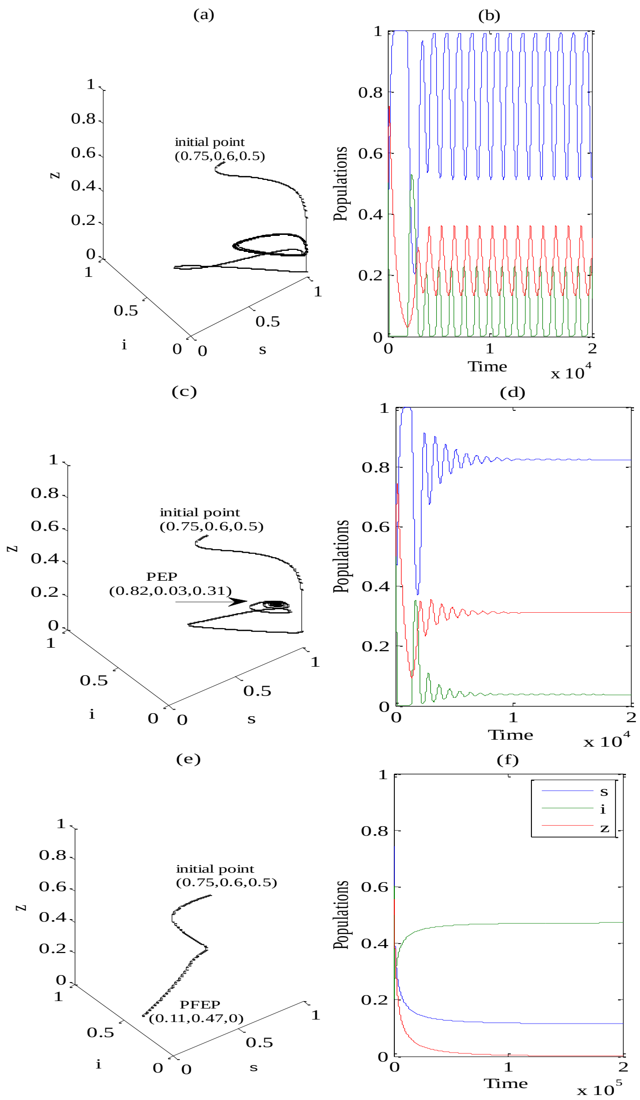
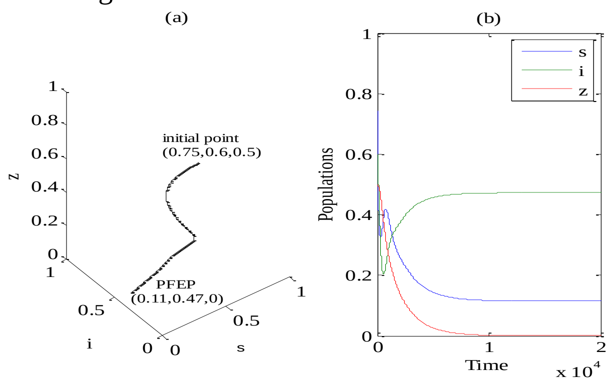


Figure 6- The trajectories of system (3) using data given by Eq. (29) with typical values of w_4. (a) Periodic dynamics in the $Int.\mathbb{R}^3_+$ for $w_4 = 0.05$. (b) Time series of the trajectory given in (a). (c) System (3) approaches to PEP for $w_4 = 0.07$. (d) Time series of the trajectory given in (c). (e) System (3) approaches to PFEP in the interior of si −plane for $w_4 = 0.72$. (f) Time series of the trajectory given in (e).

The effect of varying w_5 on the dynamical behavior of system (3) is investigated by solving the system numerically using the data given in Eq. (29) with different values of w_5. It is observed that for $0 < w_5 < 0.42$, the trajectory of system (3) approaches asymptotically to PFEP in the interior of si −plane. However, for $w_5 \geq 0.42$, the trajectory of system (3) approaches to the PEP, as explained for the typical values given in Figure-7.



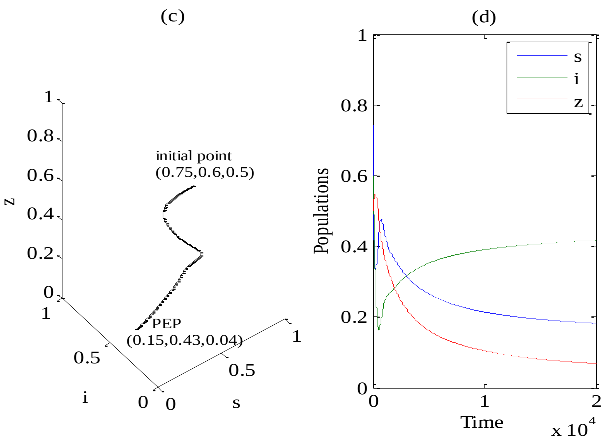


Figure 7- The trajectories of system (3) using data given by Eq. (29) with typical values of w_5. (a) System (3) approaches to PFEP in the interior of si −plane for $w_5 = 0.3$. (b) Time series of the trajectory given in (a). (c) System (3) approaches to PEP for $w_5 = 0.42$. (d) Time series of the trajectory given in (c).

The effect of varying w_6 on the dynamical behavior of system (3) is studied numerically. It is observed that for $0 < w_6 < 0.15$ and $w_6 \geq 0.15$, with the rest of parameters are as given in Eq. (29), the trajectory of system (3) approaches asymptotically to the PEP and PFEP in the interior of si −plane, respectively, as shown in Figure-8 for some typical values of w_6.

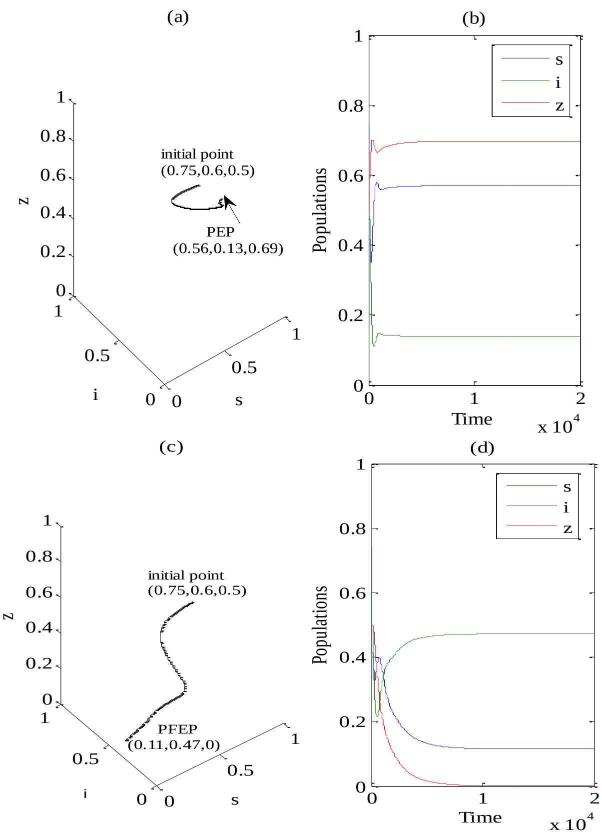


Figure 8- The trajectories of system (3) using data given by Eq. (29) with typical values of w_6. (a) System (3) approaches to PEP for $w_6 = 0.05$. (b) Time series of the trajectory given in (a). (c) System (3) approaches to PFEP in the interior of si −plane for $w_6 = 0.2$. (d) Time series of the trajectory given in (c).

The effect of varying w_7 on the dynamical behavior of system (3) is also studied numerically. It is observed that varying w_7 while keeping the rest of parameters as in Eq. (29) has similar effects on the dynamical behavior of system (3) as those shown with varying w_6. However, varying the parameter w_8 in the ranges of $0 < w_8 < 0.7$ and $w_8 \geq 0.7$, while keeping the rest of parameters as in Eq.(29), has similar effects on the dynamical behavior of system (3) as those occurred when varying w_5 in the ranges $0 < w_5 < 0.42$ and $w_5 \geq 0.42$, respectivelly.

6. Discussion and Conclusions

In this paper, a prey-predator model with infectious disease in prey and harvesting of predator is formulated mathematically and investigated analytically as well as numerically. The dynamical behavior of the proposed model is investigated locally as well as globally using the concepts of stability theory. The persistence and local bifurcation of the model, which are given by system (3), are also investigated. Finally, to complete our understanding of the global dynamical behavior of system (3), numerical simulation is used using hypothetical set of parameters values given by Eq. (29). In the following, the obtained numerical simulation results using data given by Eq. (29) are summarized.

1. The trajectory of system (3) approaches asymptotically to PEP starting from different initial points using the data Eq. (29), which indicates the existence of globally asymptotically stable PEP.
2. Decreasing the infection rate w_1 below a specific value causes a loss of persistence in system (3), while the trajectory approaches asymptotically to PFEP in the interior of si −plane. Further decreasing this parameter leads to extinction in the infected species and then the trajectory approaches to AEP. However, increasing the infection rate above a specific value leads to extinction in predator species again and the trajectory approaches asymptotically to PFEP in the interior of si −plane. Otherwise, the system still persists at a PEP.
3. Increasing the inhibition rate of disease w_2 or disease death rate w_3 above a specific value leads to extinction in predator species due to the lack in their food. Further increasing at least one of these parameters causes extinction in the infected prey specie, and the trajectory of system (3) approaches asymptotically to AEP. Otherwise, the system still persists at a PEP.
4. Decreasing the half saturation constant w_4 below a specific value leads to a destabilized PEP, but the system still persists in the form of periodic dynamics in the $Int. \mathbb{R}^3_+$. However, increasing this parameter above a specific value leads to extinction in predator species, and the trajectory of system (3) approaches asymptotically to PFEP in the interior of si −plane. Otherwise, the system still persists at a PEP.
5. Decreasing the conversion rate w_5 of the infected prey to a predator or the hunting effort w_8 from a predator below a specific value lead to extinction in predator species, and the trajectory of system (3) approaches asymptotically to PFEP in the interior of si −plane. Otherwise, the system still persists at a PEP.
6. Increasing the death rate w_6 of predator species or the catchability coefficient w_7 above a specific value leads to extinction in predator species, and the trajectory of system (3) approaches asymptotically to PFEP in the interior of si −plane. Otherwise, the system still persists at a PEP.

Keeping the above in view, system (3) is sensitive to varying in their parameters and the bifurcation occurs at all the parameters of the system, especially the infection rate.

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