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Fully Small Dual Stable Modules

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Abstract
New types of modules named *Fully Small Dual Stable Modules* and *Principally Small Dual Stable* are studied and investigated. Both concepts are generalizations of *Fully Dual Stable Modules* and *Principally Dual Stable Modules* respectively. Our new concepts coincide when the module is *Small Quasi-Projective*, and by considering other kind of conditions. Characterizations and relations of these concepts and the concept of *Small Duo Modules* are investigated, where every fully small dual stable R-module M is small duo and the same for principally small dual stable.

Keywords: small dual stable, principally small dual stable, small duo.

المقاسات الصغيرة رديفة الأستقرارية التامة

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الخلاصة
انواع جديدة من المقاسات المسماة المقاسات صغيرة رديفة الاستقرارية التامة والمقاسات صغيرة رديفة الاستقرارية الرئيسية تمت دراستها وبحثها. حيث ان المفهومان هما في الحقيقة تعميمات للمقاسات رديفة الاستقرارية التامة والمقاسات رديفة الاستقرارية الرئيسية على التوالي. المفهومان الجديدان يتطابقان عندما يكون المقاس صغير شبه المقذوف. مبرهنات الوصف و العلاقات بين هذان المفهومان و مفهوم المقاس صغير الثنائية قد تم بحثه، حيث ان كلا المفهومين يقعان ضمن المقاسات صغيرة الثنائية.

1. Introduction

In our research S is an associative ring with non-zero identity and A a left S-module. We shall discuss the duality of fully small stable modules [1] that we call *Fully Small Dual Stable Modules* where we considered here the dual stability of a module regarding its small submodules. A submodule H of an S-module A is called *small* providing that for each submodule D with H+D=A then D=A [7]. This new concept is a generalization of the fully dual stable modules [2], which is already a dualization of full stability concept [3]. Where a submodule K of an S-module A is called *stable* providing that for each S-homomorphism $f: K \longrightarrow A$ we have $f(K) \subseteq K$ [3], this concept has been dualized in [2] such that a submodule K of an S-module A is called *dual stable* providing that for each R-homomorphism $f: A \longrightarrow A/K$ we have $K \subseteq \ker f$. Several characterizations and properties of this duality are mentioned, one of the characterizations considered the kernel of S-homomorphisms; that is, An S-module A is fully small dual stable if and only if $\ker g \subseteq \ker f$ for any S-homomorphism $f: A \longrightarrow M$ and small S-epimorphism $g: A \longrightarrow M$ where M is any module. Fully small dual stability is

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preserved in homomorphic images by small submodules of fully small dual stable modules. Fully small dual stability is not obtained by small cyclic submodules, and this motivated us to define the concept of *Principally Small Dual Stable Modules* as a generalization of the native one, this new concept has also been investigated and studied intensively. Where we proved that A is fully small dual stable when A is both principally small dual stable and small quasi-projective. Other conditions that coincide these concepts of fully small dual stable and principally small dual stable are considered, such as *The Small Quotient Embedding Property*. Moreover, we have proven that fully small dual stable modules are *Small Duo* [1] as well as principally small dual stable ones. Where an S-module A is called Small Duo providing that for each small submodule H of A, H is fully invariant; that is, each S-endomorphism $\alpha: A \to A$ implies that $\alpha(H) \subseteq H$ [4]. Recall that, $J(A)$ refers to the *Jacobson Radical* of A, where it has been defined as the intersection of all maximal submodules of A or as the sum of all small submodules of A [7].

2. Properties of Fully Small Dual Stable Modules

Definition (2.1)

Let A be an S-module, A is said to be *fully small dual stable*, briefly fully sd-stable if every small submodule of A is dual stable.

It is clear that the $\mathbb{Z}$ as $\mathbb{Z}$-module is fully small dual stable trivially. Moreover, it is clear that every fully d-stable module is fully sd-stable one, but the converse is not true in general. For example, if V is a field and R=$M_{2\times 2}(V)$ is the ring of all 2×2 matrices over V, then the only small submodule of R as R-module is the zero submodule which is trivially a subset of $\ker(\gamma)$ for each R-homomorphism γ. While R as R-module is not fully dual stable as it is shown in [2].

We have shown in [1] that every fully small stable module is small duo. The same result fits for fully sd- stable modules as the following proposition shows:

Proposition (2.2)

A fully small dual stable module is small duo.

Proof: Considering A as an S-module, K$\ll A$ and $\gamma: A \to A$ be any S-endomorphism of A, set $\alpha = \pi \circ \gamma = A \to A \to A/N$, where π is the natural epimorphism of A on*to* A/K. Then $K \subseteq \ker(\alpha)$, but $\ker(\alpha) = \ker(\pi \circ \gamma) = \gamma^{-1}(\ker(\pi)) = \gamma^{-1}(N)$, which implies $\gamma(N) \subseteq N$.

The converse of the above proposition is not right generally for example, $\mathbb{Z}_{p^\infty}$ *as* $\mathbb{Z}$-module is small duo while it is not fully small dual stable.

The next proposition provides us that fully sd- stability is closed under homomorphic images by small submodules.

Proposition (2.3)

A homomorphic image by small submodules of a fully sd-stable module is fully sd-stable.a

Proof: Let H be a small submodule of a fully sd-stable S-module A, and K a submodule of A containing H providing that $K/H \ll M/H$ which implies that $K \ll A$ [3]. Let $\alpha: A/H \to (A/H)/(K/H)$ be an S-homomorphism. Define φ as the composition of α *and* π, where $\pi: A \to A/H$ is the natural epimorphism and $\beta: (A/H)/(K/H) \to A/K$ as an isomorphism (third isomorphism theorem) such that $\varphi = \beta \circ \alpha \circ \pi: A \to A/K$ is a well-defined S-homomorphism. Now, $K \subseteq \ker(\varphi)$ by A being fully small dual stable, but $\ker(\varphi) = \ker(\beta \circ \alpha \circ \pi) = \pi^{-1}(\ker(\beta\alpha)) = \pi^{-1}(\alpha^{-1}(\ker(\beta)) = \pi^{-1}\big(\alpha^{-1}(0)\big) = \pi^{-1}(\ker(\alpha))$whichimplies

$K \subseteq \ker(\varphi) = \pi^{-1}(\ker(\alpha)) \Rightarrow \pi(k) \subseteq \pi(\pi^{-1}(\ker(\alpha)) \Rightarrow K/H \subseteq \ker(\alpha)$.

In the following, a sufficient condition for a small duo module to be fully sd-stable are considered. But first several definitions and propositions for this condition are mentioned.

Definition (2.4)

An S-module A is called *small quasi projective* if for each small submodule H of A and each S-homomorphism $\alpha: A \to A/H$, there exists an endomorphism f of A such that $\alpha = \pi \circ f$ where $\pi: A \to A/H$ is the natural epimorphism.

Proposition (2.5)

Every small, fully invariant submodule of a small quasi-projective module is small dual stable and hence every small quasi-projective, small duo module is fully small dual stable.

Proof: Let H be a small, fully invariant submodule of an S- module A and let $\alpha: A \to A/H$ be any R-homomorphism, then by small quasi-projectivity of A, there exists $f: A \to A$ an endomorphism of A such that $\alpha = \pi \circ f$, but $f(H) \subseteq H$ by assumption, thus $H \subseteq f^{-1}(H) = \ker(\alpha)$, which implies that H is small dual stable.

Corollary (2.6): Every small duo ring is fully small dual stable.

Proof: It is known that a ring (with identity) is projective which implies quasi-projective and hence small quasi-projective, so proposition (2.5) ends the proof.

Corollary (2.7): Every commutative ring is fully small dual stable.

Proof: The proof implies from the reality that every commutative ring is duo and hence small duo and corollary (2.6).

If $R = K[x, y]$ is the polynomial ring in two indeterminate over a field K. Then R (as a left module over itself) is fully d-stable R-module [2] and hence fully sd-stable while every submodule of R is small but not stable. Even though, the left ideal $A = Rx + Ry$ is a small submodule of R which is not fully sd-stable in fact it is not small duo. Consider the endomorphism f of A defined by $f(px + qy) = qx + py$, for some $p, q \in R$. Then $f(Rx) = Ry \not\subset Rx$, which shows that full sd-stability is not closed over small submodules.

In [1], a characterization for small duo modules with respect to their endomorphisms was investigated. In the following, a characterization for fully small dual stable modules due to its homomorphisms into A/H for each small submodule H of A is investigated, in the same manner for that of small duo modules.

Proposition (2.8)

Let A be a fully small dual stable S-module, H$\ll A$ and $\gamma: A \to A/H$ is any R-homomorphism, then for each $a \in J(A)$ there exists $s \in S$ such that $\gamma(a) = sa + H$ (s may depend on γ and a).

Proof: Define $\varphi: A/H \to A/H \ by \ \varphi(a + H) = \gamma(a)$. Full sd-stability of A implies that $H \subseteq \ker(\gamma)$, so if $a \in H$, then $\gamma(a) = \bar{0}$. this shows that φ is well-defined, clearly S-homomorphism. Proposition (2.3) implies that A/H is fully sd-stable S-module, and then small duo by proposition (2.2). Now, for each $a + H \in J(A/H)$, $\exists\, s \in S \ such \ that \ \varphi(a + H) = sa + H$ by [2]. Now, every small submodule of A is contained in $J(A)$ [5]; that is, H$\ll A$ implies that $H \subseteq J(A)$ and by using the correspondence theorem for modules implies that $a + H \in J(A)/H$ and hence $a \in J(A)$, thus $\varphi(a + H) = ra + H = \gamma(a)$. Then for each $a \in J(A)$ there exists $s \in S$ such that $\gamma(a) = sa + H$.

One can easily prove that, if $\gamma: A \to A/H$ has the property that for each $a \in J(A)$ there exists $s \in S$, such that $\gamma(a) = sa + H$, then $H \subseteq \ker(\gamma)$. Thus the following corollary is a characterization of fully small dual stable.

Corollary (2.9): An S-module A is fully small dual stable if and only if for each small submodule H, each R-homomorphism $\gamma: A \to A/H$ has the property that for each $a \in J(A)$ there exists $s \in S$, such that $\gamma(a) = sa + H$ (s depends on γ and a).

After this, the question that s might depend only on γ is asked. But first we need the following lemma. But first recall that an S-module A is called ***torsion-free*** if $t(A) = 0$, where $t(A) = \{a \in A | there \ exists \ \mathrm{s}(\neq 0) \in \mathrm{S} \ such \ that \ sa = 0\}$ [7, p.142].

Lemma (2.10): Let S be an integral domain, and A a torsion free S-module. Then A is small duo if and only if for each S-endomorphism f of A there exists s in S such that $f(a) = sa$ for all $a \in J(A)$.

Proof: let $f: A \to A$ be an S-endomorphism of a small duo module A. then by [1] we have that for each $a \in J(A)$ there exists $s \in S$ such that $f(a) = sa$. Now, if a and n are two distinct elements of $J(A)$ where $f(a) = sa$ and $f(n) = rn$; where $s, r \in S$, then the following two cases appear:

Case (1): If $\boldsymbol{Sa} \cap \boldsymbol{Sn} \neq (\mathbf{0})$

$Sa \cap Sn \neq (0); that \ is, there \ exist \ h(\neq 0) \in Sa \cap Sn, since \ a, n \in J(A), then \ Sa \ll A \ and \ so \ is \ Sn, but \ Sa \cap Sn \subseteq Sa \Rightarrow Sa \cap Sn \ll A$ [7]$and \ hence \ h \in J(A) which \ implies \ that \ \exists t \in S \ such \ that \ f(h) = th, then \ h = ua = vn \ for \ some \ u, v \in S\ , thus \ f(ua)tua = uan \ \ hence \ tua = rua \ and \ t = y. In \ the \ same \ way \ t = r, then \ s = r.$

Case (2): If $\boldsymbol{Sa} \cap \boldsymbol{Sn} = (\mathbf{0})$

$Sa \cap Sn = 0, since \ a, n \in J(A), then \ Sa, Sn \ll A$, and so is $Sa + Sn, which \ implies \ that \ a + n \in J(A), Thus \ f(a + n) = t(a + n), t \in S, but \ f(a + n) = f(a) + f(n) = sa + rn = ta + tn \to (s - t)m = (t - r)n \to s - t = t - r = 0 \ thus \ s = r \ in \ this \ case \ f(a) = sa \qquad for \ all \ a \ in \, J(A).$

The following proposition answers the above mentioned question

Proposition (2.11)

Let S be an integral domain, A be a fully small dual stable S-module and H be a small submodule of A such that A/H is torsion-free. Then for each S-homomorphism $\alpha: A \to A/H$ there is s in S such that $\alpha(a) = sa + H$ for all $a \in J(A)$.
Proof: In the same manner of the proof of proposition (2.8) that φ is an S-endomorphism of the torsion-free, small duo S-module A/H. Then by lemma (2.10) there exists $s \in S$ such that $\varphi(a + H) = s(a + H)$ for all $a + H \in J(A/H)$. Which yields into the same way of the proof of proposition (2.8) to that $\varphi(a) = sa + H$ for all $a \in J(A)$.

The following proposition gives another characterization for fully small dual-stable module. But first recall that an S-module A is called ***supplemented*** providing that each submodule H of A has a ***supplement***; that is, there exist a submodule V of A with the property that H+V=A and H∩V≪V [5].

Proposition (2.12)

Let A be a f S-module. Then:

1. A is fully small dual stable.

2. $\ker g \subseteq \ker f$ for any S-homomorphism $f: A \to M$ and small S-epimorphism $g: A \to M$ (where M is any S-module). Moreover, if A is supplemented then $\ker g \ll \ker f$.

Proof:(1)⟹(2) Let $N = kerg$. Then M is isomorphic to A/N; that is, there exists $\varphi: M \to A/N$ an isomorphism, hence $\alpha = \varphi \circ f: A \to A/N$ and $ker\alpha = \ker(\varphi \circ f) = f^{-1}(ker\varphi) = f^{-1}(0) = kerf$, but A is fully sd-stable and $N \ll A$ (since $N = \ker g \ll A$, since g is a small surjective R-homomorphism) implies that $N \subset ker\alpha = \ker f, thus$ $\ker g \subseteq \ker f$. Moreover, by A being supplemented we get that $\ker g \ll \ker f$.

(2) ⟹(1) Let H be a small submodule of A and $\alpha: A \to A/H$ an R-homomorphism. Then $\pi_H = A \longrightarrow A/H$, the natural epimorphism is surjective and small since $ker\pi_H = H$. Thus by hypothesis we get that $\ker \pi_H \subseteq \ker f$ which implies that $H \subseteq \ker f$, and hence A is fully small dual stable.

Proposition (2.13):

Let A be a fully small dual stable S-module and $\alpha: A \to A/H$ be a small S-epimorphism, where H is a small submodule of A. Then:

1. $H = \ker\ \alpha$
2. If $H \subseteq K, then\ \alpha(K) = K/H, where\ K\ is\ a\ small\ submodule\ of\ M.$

Proof:

1. Since $\alpha: A \longrightarrow A/H$ is an epimorphism, then there exists an isomorphism $\varphi: A/H \longrightarrow A/\ker\alpha$ (by the factorization theorem [7]) define $\beta: A \longrightarrow A/\ker\alpha$ by $\beta = \varphi \circ \pi$, where $\pi: A \longrightarrow A/H$ is the natural epimorphism. Then $ker\beta = \ker(\varphi \circ \pi) = \pi^{-1}(\ker(\varphi)) = \pi^{-1}(0) = \ker(\pi) = H,$ but by proposition (2.12) we have $\ker\alpha \subseteq \ker\beta,$ hence $\ker\alpha \subseteq H$. Full small dual stability of A implies that $H \subseteq \ker\alpha,$ thus $H = \ker(\alpha)$.

2. Define $\theta: A/H \longrightarrow A/K$ by $\theta(x + H) = x + K$, it is clear that θ is a well-defined R-homomorphism. Moreover, θ is surjective since $\subseteq K$. Now, $K/H \ll A/H$ since $K \ll A,$ but $\ker(\theta) = K/H$ implies that θ is a small S-epimorphism [3]. The fact that both θ and α are small S-epimorphisms implies that $(\theta \circ \alpha): A \longrightarrow A/K$ is a small S-epimorphism [6]. By (1) we get that $\ker(\theta \circ \alpha) = K \Longrightarrow \alpha^{-1}(\ker(\theta)) \Longrightarrow \alpha^{-1}(K/H) = K\ and\ hence\ \alpha(K) = K/H\ ,$ since α is an epimorphism.

Proposition (2.14)

Let H be a small, dual stable submodule of an S-module A and L a small dual stable submodule of H, then L is small dual stable in A.

Proof: Let $\alpha: A \longrightarrow A/L$ be an R-homomorphism. Define $\beta: A/L \to A/H$ by $\beta(x + L) = x + H$ for all x in A, hence $H \subseteq \ker(\beta \circ \alpha) = \alpha^{-1}(\ker(\beta)) = \alpha^{-1}(H/L)$ and so $(H) \subseteq H/L$. Now, if $\delta = \alpha|_H$, then $\delta: H \longrightarrow H/L$ and $L \subseteq \ker\delta\ , but\ \delta = \ker\alpha \cap H, thus\ L \subseteq \ker(\alpha)$; that is , L is small dual stable.`

Fully small dual stability of a module cannot depend on it is small cyclic submodules, thus an introduction to the concept of principally small dual stable module is considered in the following.

Definition (2.15):

An S-module A is called ***principally small dual stable*** if each small cyclic submodule of A is dual-stable.

Also a ring S is principally small dual stable if it is principally small dual stable S-module.

A sufficient condition for principally small dual stability to be fully small dual stable is small quasi-projectivity. However, the following proposition is needed first.

Proposition (2.16)

Let A be a small quasi projective S-module. Then A is fully small dual stable if and only if every small cyclic submodule of A is dual stable.

Proof: ⟹) Obvious.

⟸) Proposition (2.5) shows that it is enough to prove that A is small duo. Let f be an endomorphism of A and $x \in J(A)$ and π_x be the natural epimorphism of A onto A/Rx, put $\alpha = \pi_x \circ f$ then $R_x \subseteq \ker(\alpha)$ for each $x \in J(A)$, thus $\alpha(x) = 0$; that is, $(\pi_x \circ f)(x) = 0 \Longrightarrow \pi_x\big(f(x)\big) = 0$, so there exists $s \in S$ such that $f(x) = sx$. Hence A is small duo.

Corollary (2.17): Any small quasi-projective principally small dual stable module is fully small dual stable.

The following proposition is a generalization for proposition (2.2).

Proposition (2.18)

Every principally small dual stable module is small duo.

Proof: Let H be a small submodule of an S-module A and f an endomorphism of A. Define $\alpha: A \to A/Sx$ such that $\alpha = \pi_x \circ f$, where π_x is the natural epimorphism of A onto A/Sx, $x \in J(A)$. Thus $Sx \subseteq \ker(\alpha)$ by principally dual stability of A; that is, $\alpha(x) = 0$. Hence $f(x) \in Sx \subset H$ $\forall x \in H$, then $f(H) \subseteq H$.

Our rings are assumed to have an identity, which makes the concepts of small duo, fully small dual stable and principally small dual stable coincide for rings.

Proposition (2.19): A ring S is principally small dual stable if and only if it is fully small dual stable.

Proof: ⟸) obvious.

For the converse) Let I be a small left ideal of a left principally small dual stable ring S and let $\alpha: S \to S/I$ be an S-homomorphism, S is small duo by (2.18) and thus every small ideal is a two sided ideal. Now if $x \in I$, then $\alpha(x) = x\alpha(1) = xx_0 + I = \overline{0}$, since $xx_0 \in I$, for some $x_0 \in S$. Therefore, S is fully small dual stable.

As we introduced the condition that makes principally small dual stable coincide with the notion of fully small dual stable earlier, we see that it is convenient for us to look for another condition which is stated in the following.

Proposition (2.20)

Let A be an S-module (where S is a small duo ring), with the property that any submodule of A is contained a small cyclic submodule. If A is principally small dual stable, then it is fully small dual stable.

Proof: Let H be a small submodule of A contained in Sx (for some $x \in J(A)$). Then H is dual stable in Sx, since Sx is a cyclic module and hence fully small dual stable by (2.8 (b)), Sx is also a dual stable submodule of A by A being principally small dual stable S-module. Thus by transitivity of small dual stability see proposition (2.14), H is also a dual stable submodule of A. Hence A is fully small dual stable.

Another condition to deduce full small dual stability from principal small dual stability is regarded next.

Definition (2.21)

An S-module A is said to have the small quotient embedding property (sqe-property) if A/H can be embedded in to A/Sx for each small submodule H of A and each 0 ≠ x ∈ H.

Proposition (2.22)

let A be an S-module. If A/Sx is semi simple for each $0 \neq x \in A$, then A has the sqe-property. In particular, every semi simple module has the sqe-property.

Proof: Let $\in H$, where H is a small submodule of A. Then there is a natural epimorphism δ: A$/Sx \to A/H$ defined by $\delta(a + Sx) = a + H$ for each $a \in A$ with $\ker(\delta) = H/Sx$. Now, since A/Sx is semi simple then H/Sx is a direct summand of A/Sx, but $H/Sx \ll A/Sx$ and hence $H/Sx = (0)$, that is, $\ker(\delta) = (0)$. Thus δ is an isomorphism and hence A/H can be embedded in A/Sx.

Proposition (2.23)

Let A be a principally small dual stable S-module. If A has the sqe-property, then A is fully small dual stable.

Proof: Let H be a small submodule of a principally small dual stable S-module A and $\alpha: A \longrightarrow A/H$ is an S-homomorphism. Now, let $x \in H$ then there is a monomorphism $\beta: A/H \to A/Sx$ since A has the sqe-property, $\beta \circ \alpha: A \longrightarrow A/H \longrightarrow A/Sx$ is an S-homomorphism and $Sx \ll A, thus\, Sx \subseteq \ker(\beta \circ \alpha)$, $but \ker(\beta \circ \alpha) = \alpha^{-1}(ker\beta) = \alpha^{-1}(0) = \ker(\alpha)$ by β being a monomorphism. Hence $H \subseteq \ker(\alpha)$ since x is an arbitrary element of H.

The following corollary is immediate. But first, recall that an S-module A is called ***semisimple*** if every submodule of A is a direct summand of A [7, p. 191].

Corollary (2.24): Let A be a principally small dual stable S-module such that A/Sx is semi simple for each $x \in A$. Then A is fully small dual stable.

The following proposition gives a property for small cyclic submodules of principally small dual stable modules.

Proposition (2.25)

Let A be a principally small dual stable S-module. If $x, y \in J(A)$ with $/Sx \cong \mathrm{A/Sy}$, then $Sx = Sy$.

Proof: Let $\varphi: \frac{A}{Sx} \longrightarrow \frac{A}{Sy}$ be an isomorphism, $\pi_x\ and\ \pi_y$ be the natural epimorphisms of A onto $\frac{A}{Sx}$ and $\frac{A}{Sy}$ respectively. Set $\alpha = \varphi \circ \pi_x$, $\beta = \varphi^{-1} \circ \pi_y$. Then $Sx \subseteq \ker\beta\ and\ Sy \subseteq ker\alpha,$ by A being principally small dual stable. But $\ker\alpha = \ker(\varphi \circ \pi_x) = {\pi_x}^{-1}(\ker(\varphi)) = {\pi_x}^{-1}(0) = \ker\pi_x = Sx$ and $\ker\beta = \ker\left(\varphi^{-1} \circ \pi_y\right) = {\pi_y}^{-1}(\ker(\varphi^{-1}))$ $={\pi_y}^{-1}(0) = \ker\pi_y$=$Sy$; that is, $Sx \subseteq ker\beta = Sy$ and $Sy \subseteq ker\alpha = Sx$. Hence $Sx = Sy$.

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