Iraqi Journal of Science, 2018, Vol. 59, No.1B, pp: 404-407
DOI: 10.24996/ijs.2018.59.1B.20

ISSN: 0067-2904

NS-Primary Submodules

Iman A. Athab
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.

Abstract
Let R be a commutative ring with identity and let Mbe a unitary R-module. We shall say that a proper submodule N of M is nearly S-primary (for short *N*S-primary), if whenever$f \in S = \mathrm{End}(\mathrm{M})$, $x \in \mathrm{M}$, with $f(x) \in \mathrm{N}$ implies that either $x \in \mathrm{N} + J(\mathrm{M})$ or there exists a positive integer *n*, such that $f^n(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$, where $J(\mathrm{M})$ is the Jacobson radical of M. In this paper we give some new results of *N*S-primary submodule. Moreover some characterizations of these classes of submodules are obtained.

Keywords: nearly S-primary, nearly primary submodule, M*N*-primary submodule.

المقاسات الجزئية من النمط – *N*S

ايمان علي عذاب
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق.

الخلاصة
لتكن R حلقة ابدالية ذات عنصر محايد و M مقاسا معرفا على R يقال للمقاس الجزئي الفعلي N من M بأنه ابتدائي من النمط S تقريبا (NS− Primary) ، اذا كان لكل f ∈S =End(M) ، x ∈ M بحيث انf(x) ∈ N ، يؤدي الى $x \in \mathrm{N} + J(\mathrm{M})$ او $f^n(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$ ، $n \in \mathbb{Z}^+$ حيث ال$J(\mathrm{M})$ هو جذر جاكوبسون لقد درسنا هذا المفهوم واعطينا بعض النتائج الجديدة والتشخيصات لهذا النوع من المقاسات الجزئيه

1. Introduction

Throughout this paper all rings will be commutative with identity and all modules are unital. A proper submodule N of an R-module M is called primary if for any $r \in \mathrm{R}$ and $x \in \mathrm{M}$ such that $rx \in \mathrm{N}$ implies that either $x \in \mathrm{N}$or $r^n \in [\mathrm{N{:}\,M}] = \{s \in \mathrm{R}; s\mathrm{M} \subseteq \mathrm{N}\}$, for some $n \in \mathbb{Z}^+$,[5]. The termof S-primary submodule was introduced, in [7] as follows: A proper submodule, N of an R-module M is called S-primary,if for $f \in \mathrm{S} = \mathrm{End(M)}$and $m \in \mathrm{M}$ with $f(m) \in \mathrm{N}$,implies that either$m \in \mathrm{N}$ or$f^n(\mathrm{M}) \subseteq \mathrm{N}$ for some $n \in \mathbb{Z}^+$. The notion of nearly primary submodule (for short *N*-primary submodule) was given in [6], we say a proper submodule N of an R-module M*N*-primary submodule, if whenever $r \in \mathrm{R}, m \in \mathrm{M}$, such that $rm \in \mathrm{N}$, then either $m \in \mathrm{N} + J(\mathrm{M})$ or$r^{\Re} \in [\mathrm{N} + J(\mathrm{M})]$,for some $\Re \in \mathbb{Z}^+$. This paper contains a new class of submodules, which is called *N*S-primary submodules. This type of submodules defined as follows: if N is a proper submodule of an R-module M, we say that N is*N*S-primary submodule if whenever$f \in \mathrm{S} = \mathrm{End(M)}$, $x \in \mathrm{M}$such that $f(x) \in \mathrm{N}$, implies that either $x \in \mathrm{N} + J(\mathrm{M})orf^n(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$for some $n \in \mathbb{Z}^+$. Various properties of *N*S-primary submodules are introduced, as well as we prove a new characterization for this type of submodules.

Email: imanaliathab971@gmail.com

2. *N*S-primary submodules

Recall that a proper submodule N of an R-module M, is said to be S-primary submodule, if whenever $(m) \in$ N, for some $f \in \mathrm{End}(\mathrm{M})$and $m \in$ M, then either $m \in$ N or $f^n(\mathrm{M}) \subseteq$ N for some $n \in \mathbb{Z}^+$,[7]. We introduce the following definition.

Definition(2.1)

A proper submodule N of an R-module M is called nearly S-primary submodule (for short *N*S-primary submodule), if whenever $f(m) \in$ N, for some $f \in S = \mathrm{End}(\mathrm{M})$and $m \in$ M, implies that either $m \in \mathrm{N} + J(\mathrm{M})$or $f^n(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$, for some $n \in \mathbb{Z}^+$.

Remarks and examples (2.2)

1) Every S-primary submodule N of an R-module M is *N*S-primary submodule of M.

Proof:

Suppose that, $f(m) \in$ N where$f \in \mathrm{End}(\mathrm{M})andm \in$ M. But N isS-primary submodule of M, then either $m \in$ N or $f^n(\mathrm{M}) \subseteq$ N, for some$n \in \mathbb{Z}^+$, from this we get that either $m \in \mathrm{N} + J(\mathrm{M})$ or $f^n(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$, this complete the proof.

The converse of the previous remark is not true in general for example, let $\mathrm{N} = \langle \frac{1}{p^i} + \mathbb{Z} \rangle$ be a submodule of $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module, where *p* is a prime number and *i*is a non-negative integer, $J(\mathbb{Z}_{p^\infty}) = \mathbb{Z}_{p^\infty}$. Then N is *N*S-primary but it is not S-primary.

2) Every *N*S-primary submodule of an R-module M is *N*-primary.

Proof:

Let N be an *N*S-primary submodule of M, suppose that $rm \in$ Nfor$r \in$ R, $m \in$ M. Assume that $m \notin \mathrm{N} + J(\mathrm{M})$. Define$f\colon \mathrm{M} \longrightarrow \mathrm{M}$,by $f(x) = rx$, $x \in$ M. Now, $rm = f(m) \in$ N, but N is *N*S-primary submodule of M and $m \notin \mathrm{N} + J(\mathrm{M})$, therefore there exists a positive integer *n* with$f^n(\mathrm{M}) \subseteq \mathrm{N} + J(\mathrm{M})$, and hence $r^n\mathrm{M} \subseteq \mathrm{N} + J(\mathrm{M})$, this implies that a submodule N is *N*-primary.

The following example shows the converse of the previous remark, is not true in general.

Let $\mathrm{M} = \mathbb{Z}_p + \mathbb{Z}$ as $\mathbb{Z}$-module, where *p* is a prime number and let $\mathrm{N} = \{\bar{0}\} \oplus p\mathbb{Z}$. $J(\mathrm{M}) = 0$ one can show that N is a primary submodule , and hence by [6] N is *N*-primary. Now, define $f\colon \mathrm{M} \longrightarrow \mathrm{M}$by $f(\bar{n}, m) = (\bar{0}, m)$ for all$(\bar{n}, m) \in \mathbb{Z}_p + \mathbb{Z}$. Clearly $f \in \mathrm{End}(\mathrm{M})$, $f(\bar{1}, p) = (\bar{0}, p) \in \{\bar{0}\} \oplus p\mathbb{Z}$. But $(\bar{1}, p) \notin \mathrm{N} + J(\mathrm{M})$ and for all positive integer *n*, $f^n(\mathbb{Z}_p + \mathbb{Z}) = \{\bar{0}\} \oplus \mathbb{Z} \nsubseteq \{\bar{0}\} \oplus p\mathbb{Z}$. Therefore $\{\bar{0}\} \oplus p\mathbb{Z}$ is not *N*S-primary submodule of M.

3) The intersection of any two *N*S-primary submodules of an R-module M, is not necessary *N*S-primary submodule of M. For example, let M be the module $\mathbb{Z}_6$ as $\mathbb{Z}$-module, where $J(\mathbb{Z}_6) = 0$. Let $\mathrm{N}_1 = \langle \bar{2} \rangle$ and $\mathrm{N}_2 = \langle \bar{3} \rangle$ be submodules of $\mathbb{Z}_6$, we see that both N_1 and N_2 are *N*S-primary submodules of $\mathbb{Z}_6$ as $\mathbb{Z}$-module, but $\mathrm{N}_1 \cap \mathrm{N}_2 = \{0\}$ is not *N*S-primary since it is not *N*-primary submodule.

Let us prove the following proposition.

Proposition (2.3)

Let K be an *N*S-primary submodule of a module M, and let N be a submodule of M with $J(\mathrm{N}) = J(\mathrm{M})$, if N is M-injective then either $\mathrm{N} \subseteq \mathrm{K}$ or $\mathrm{K} \cap \mathrm{N}$is *N*S-primary submodule of N.

Proof:

Suppose that $\mathrm{N} \nsubseteq \mathrm{K}$, then $\mathrm{K} \cap \mathrm{N}$ is a proper submodule of N. Let $f(x) \in \mathrm{K} \cap \mathrm{N}$ where $f \in \mathrm{End}(\mathrm{N})andx \in$ N. Suppose that $x \notin (\mathrm{K} \cap \mathrm{N}) + J(\mathrm{N}) = (\mathrm{K} + J(\mathrm{N})) \cap \mathrm{N}$. Thus $x \notin \mathrm{K} + J(\mathrm{N})$, we have to show that $f^n(\mathrm{N}) \subseteq (\mathrm{K} \cap \mathrm{N}) + J(\mathrm{N})$ for some $n \in \mathbb{Z}^+$.

Now, Consider the following diagram

$$\begin{array}{ccc} O \rightarrow & \mathrm{N} & \xrightarrow{i} \mathrm{M} \\ f & \downarrow \swarrow & h \\ & \mathrm{N} & \end{array}$$

Where *i* is the inclusion map.

Since N is M-injective, then there exists $h\colon \mathrm{M} \rightarrow \mathrm{N}$such that $h \circ i = f$. Clearly that $h \in \mathrm{End}(\mathrm{M})$.

But $f(x) = h \circ i(x) = h(x) \in$ K. Since K is *N*S-primary submodule of M and $x \notin \mathrm{K} + J(\mathrm{M})$, therefore there exists a positive integer *n*such that $h^n(\mathrm{M}) \subseteq \mathrm{K} + J(\mathrm{M})$.Also$f^n(\mathrm{N}) = (h \circ i)^n(\mathrm{N}) = h^n(\mathrm{N}) \subseteq$ N and $f^n(\mathrm{N}) = h^n(\mathrm{N}) \subseteq h^n(\mathrm{M}) \subseteq \mathrm{K} + J(\mathrm{N}) \cap \mathrm{N}$. Therefore $f^n(\mathrm{N}) \subseteq (\mathrm{K} \cap \mathrm{N}) + J(\mathrm{N})$ and hence $\mathrm{K} \cap \mathrm{N}$ is *N*S-primary submodule of N.

Now, we give the following characterization.

Proposition (2.4)

Let M be a nonzero R-module, then $\{0_{\mathrm{M}}\}$ is *N*-primary submodule of M, if and only,if $\mathrm{Ann(N)} \subseteq \sqrt{[J(\mathrm{M}):\mathrm{M}]}$, for all nonzero submodule N of M.

Proof:

Suppose that N is a nonzero submodule of an R-module M and $\{0_{\mathrm{M}}\}$ is *N*-primary submodule of M.

Let $r \in \mathrm{Ann(N)}$, since $\mathrm{N} \neq 0$, so there exists $x \in \mathrm{N}$ with $x \neq 0$. Now $rx = 0$, But $\{0_{\mathrm{M}}\}$ is *N*-primary submodule of M, then $r^n \in [J(\mathrm{M}):\mathrm{M}]$, where $n \in \mathbb{Z}^+$, hence$r \in \sqrt{[J(\mathrm{M}):\mathrm{M}]}$.

Conversely, let $rx = 0$, for some $r \in \mathrm{R}$ and $x \in \mathrm{M}$. Suppose that $x \notin \{0_{\mathrm{M}}\} + J(\mathrm{M})$, then $x \neq 0$. $\langle x \rangle$ is a nonzero submodule of an R-module M and hence by assumption $\mathrm{Ann}(\langle x \rangle) \subseteq \sqrt{[J(\mathrm{M}):\mathrm{M}]}$. But $r \in \mathrm{Ann}(\langle x \rangle)$, thus $r \in \sqrt{[J(\mathrm{M}):\mathrm{M}]}$, this implies that, there exists a positive integer s such that $r^s \in [J(\mathrm{M}):\mathrm{M}]$. Therefore $\{0_{\mathrm{M}}\}$ is *N*-primary submodule of M.

Recall that an R-module M is said to be multiplication if for each submodule N of M, there exists an ideal I of R such that $\mathrm{N} = \mathrm{IM}$, [4].

The following proposition gives a characterization for*N*S-primary submodule.

Proposition (2.5)

If M is a nonzero multiplication R-module, then $\{0_{\mathrm{M}}\}$is *N*-primary submodule of M, if and only, if it is *N*S-primary submodule.

Proof:

Let $f(m) = 0$, where $f \in \mathrm{End(M)}$, and $m \in \mathrm{M}$. Assume that $m \notin \{0_{\mathrm{M}}\} + J(\mathrm{M})$, then $m \neq 0$. We have to show that there exists a positive integer n such that $f^n(\mathrm{M}) \subseteq J(\mathrm{M})$, since $m \neq 0$, then $0 \neq \langle m \rangle = \mathrm{IM}$, for some ideal I of R. Now, if $f(\mathrm{M}) = 0$, then we are done,thus suppose that $f(\mathrm{M}) \neq 0$, hence there exists a nonzero ideal K of R such that $f(\mathrm{M}) = \mathrm{KM}$. Now, $0 = f(\langle m \rangle) = \mathrm{I}f(\mathrm{M}) = \mathrm{I}(\mathrm{K(M)}) = \mathrm{K(IM)}$, which implies that $\mathrm{K} \subseteq \mathrm{Ann(IM)}$. But by proposition(2.4), $\mathrm{Ann(IM)} \subseteq \sqrt{[J(\mathrm{M}):\mathrm{M}]}$, hence $\mathrm{K} \subseteq \sqrt{[J(\mathrm{M}):\mathrm{M}]}$, this implies that $f^n(\mathrm{M}) \subseteq J(\mathrm{M})$ for some positive integer *n*. Therefore $\{0_{\mathrm{M}}\}$ is *N*S-primary submodule of M. The converse side from (remark (2) in (2.2)).

Definition (2.6)

Let M be a nonzero R-module. If $\{0_{\mathrm{M}}\}$ is *N*S-primary submodule of M, then M is called *N*S-primary module.

Proposition (2.7)

Let M be a multiplication module, then N is *N*-primary submodule, if and only, if it is *N*-primary submoduleof M.

Proof:

Since M is a multiplication module, then $\frac{\mathrm{M}}{\mathrm{N}}$is also a multiplication module by [1, Corollary (3.22)]. From proposition (2.5) N is *N*-primary submodule of M, if and only, if it is *N*S-primary submodule of M.

Definition (2.8) [2]

Let M and M\` be R-modules, the module M\` is called M-projective, if for every homomorphism $f: \mathrm{M}` \to \frac{\mathrm{M}}{\mathrm{K}}$; K is a submodule of M can be lifted to a homomorphism$g: \mathrm{M}` \to \mathrm{M}$.

We are ready to prove the following proposition.

Proposition (2.9)

Let $f: \mathrm{M} \to \mathrm{M}`$ be an R-module epimorphism. If N is*N*S-primary submodule of M with$\ker f \subseteq \mathrm{N}$, then $f(\mathrm{N})$ is *N*S-primary submoduleof M\`, where M\` is M-projective module.

Proof:

First, we must prove $f(\mathrm{N})$ is a proper submodule of a module M\`. Suppose that $f(\mathrm{N}) = \mathrm{M}`$, then $f(\mathrm{N}) = f(\mathrm{M})$. Therefore $\mathrm{M} = \mathrm{N}$, which is a contradiction. Now, let $h(m`) \in f(\mathrm{N})$, where $h \in \mathrm{End(M`)}$,$m` \in \mathrm{M}`$. Suppose that, $m` \notin f(\mathrm{N}) + J(\mathrm{M}`)$. We have provethat $h^n(\mathrm{M}`) \subseteq f(\mathrm{N}) + J(\mathrm{M}`)$; n is a positive integer. f is an epimorphism and $m` \in \mathrm{M}`$ therefore there exists $m \in \mathrm{M}$ with$f(m) = m`$. Consider the following diagram:

$$\begin{array}{ccc} & \text{M`} & \\ k \swarrow & \downarrow h & \\ \text{M} \underset{f}{\rightarrow} & \text{M`} & \rightarrow o \end{array}$$

Since M` is M-projective, then there exists, a homomorphism $k$,such that $f \circ k = h$. But $h(m`) \in f(\text{N})$, this implies that $(f \circ k)(m`) \in f(\text{N})$, and hence$(f \circ k)(f(m)) \in f(\text{N})$.But $\ker f \subseteq \text{N}$, thus $(k \circ f)(m) \in \text{N}$.Since N is *N*S-primary submodule of M and $m \notin \text{N} + J(\text{M})$, then there exists, a positive integer *n* such that $(k \circ f)^n(\text{M}) \subseteq \text{N} + J(\text{M})$. Therefore $f((k \circ f)^n(\text{M})) \subseteq f(\text{N}) + J(\text{M`})$. By a simple calculation and since $f \circ k = h$, we conclude that $h^n(\text{M`}) \subseteq f(\text{N}) + J(\text{M`})$. This means that $f(\text{N})$ is *N*S-primary submodule of M`.

Corollary (2.10): If N is *N*S-primary submodule of M and K ,is a submodule of M with $\text{K} \subseteq \text{N}$, then$\frac{\text{N}}{\text{K}}$is *N*S-primary submodule of $\frac{\text{M}}{\text{K}}$, whenever $\frac{\text{M}}{\text{K}}$ is an M-projective module.

Definition (2.11)[3]

A submodule N of an R-module M, is said to be fully invariant if$f(\text{N}) \subseteq \text{N}$, for each $f \in \text{End}(\text{M})$.

We need the following two proposition to get a characterization of *N*S-primary submodules.

Proposition (2.12)

Let N be a proper fully invariant submodule of an R-module M. Suppose that $[\text{N}: f(\text{K})] \subseteq [\text{N} + J(\text{M}): f^n(\text{M})]$ for all submodule K of M with $\text{N} + J(\text{M}) \subsetneq \text{K}$, for all $f \in \text{End}(\text{M})$ and for a positive integer n , then N is *N*S-primary submoduleof M.

Proof:

Let $h(m) \in \text{N}$where $h \in \text{End}(\text{M})$, and $m \in \text{M}$, suppose that $m \notin \text{N} + J(\text{M})$, we must show that $h^n(\text{M}) \subseteq \text{N} + J(\text{M})$ where n is a positive integer.Now,$\text{N} \subsetneq \text{N} + \langle m \rangle$, hence by assumption$[\text{N}: h(\text{N} + \langle m \rangle)] \subseteq [\text{N} + J(\text{M}): h^n(\text{M})]$. But $1 \in [\text{N}: h(\text{N} + \langle m \rangle)]$, therefore$1 \in [\text{N} + J(\text{M}): h^n(\text{M})]$, which implies that $h^n(\text{M}) \subseteq \text{N} + J(\text{M})$. Therefore N is *N*S-primary submodule of M.

Proposition (2.13):

Let N be *N*S-primary submodule of an R-module Mthen, $[\text{N}: f(\text{K})] \subseteq \sqrt{[\text{N} + J(\text{M}): f^n(\text{M})}$, for all submodule K of M with $\text{N} + J(\text{M}) \subsetneq \text{K}$,for all $f \in \text{End}(\text{M})$ and *n*, is a positive integer.

Proof:

The submodule$\text{N} + J(\text{M})$ of an R-module M contained in K properly, thus there exists $x \in \text{K}$ and $x \notin \text{N} + J(\text{M})$. Assume that $r \in [\text{N}: f(\text{K})]$ this implies that $rf(x) \in \text{N}$. Now, define $h: \text{M} \rightarrow \text{M}$, by $h(m) = rf(m)$, for all $m \in \text{M}$. Clearly $h \in \text{End}(\text{M})$, also $h(x) = rf(x) \in \text{N}$. But N is *N*S-primary submodule of M and $x \notin \text{N} + J(\text{M})$, thus there exists a positive integer n such that $h^n(\text{M}) \subseteq \text{N} + J(\text{M})$, this implies that $r^n f^n(\text{M}) \subseteq \text{N} + J(\text{M})$, hence $r \in \sqrt{[\text{N} + J(\text{M}): f^n(\text{M})}$.Therefore$[\text{N}: f(\text{K})] \subseteq \sqrt{[\text{N} + J(\text{M}): f^n(\text{M})}$, for some $n \in \mathbb{Z}^+$.

Combining (2.12) with (2.13) we have at once the following Theorem.

Theorem (2.14)

If N is a proper fully invariant submodule of an R-module M, then N is *N*S-primary submodule of M, if and only, if $[\text{N}: f(\text{K})] \subseteq \sqrt{[\text{N} + J(\text{M}): f^n(\text{M})}$ for all ,submodule K of M with $\text{N} + J(\text{M}) \subsetneq \text{K}$, for all $f \in \text{End}(\text{M})$ and $n \in \mathbb{Z}^+$.

References

1. Ameri, R. **2007.** On the Prime Submodules of Multiplication Modules. *Int. J. of Mathematics and Mathematical Science*, **27**: 1715-1724.
2. Azumaya, G., Mbuntum F. and Varadarajan K. **1975.** On M-Projective and M-Injective Modules. *Pacific J. Math*. **95**: 9-16.
3. Dung, N.V., Huynh, D.V., Smith, P.F., and Wishbauer, R. **1994.** *Extending Modules*. Pitman Research Notes in Mathematics series (Longman, Harlow).
4. EL-Bast, Z.A. and Smith, P.F. **1988.** Multiplication Modules. *Comm. in Algebra*, **16**: 755-779.
5. Lu, C.P. **1989.** M-Radical of submodules in Modules. *Math. Japan*, **34**: 211-219.
6. Mohammed Baqer. H. **2016.** Nearly SemiprimeSubmodules. M.Sc. Thesis, Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.
7. Shireen Dakheel, O. **2010.** S-Prime Submodules and Some Related Concepts. Ph.D. Thesis, University of Baghdad.