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Detour Polynomial of Theta Graph

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Abstract

Let Γ be any connected graph with vertices set $V(\Gamma)$ and edges set $E(\Gamma)$. For any two distinct vertices y and z, the detour distance between y and z which is denoted by $D(y,z)$ is a longest path between y and z in a graph Γ. The detour polynomial of a connected graph Γ is denoted by $D(\Gamma;x)$; and is defined by $\sum_{y,z\in V(\Gamma)} x^{D(y,z)}$. In this paper, the detour polynomial of the theta graph and the uniform theta graph will be computed.

Keywords: Detour distance, Detour polynomial, Theta graph.

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ألخلاصة

نفرض أن Γ هو بيان متصل بمجموعة رؤوس $V(\Gamma)$ ومجموعة حافات $E(\Gamma)$. لأي رأسين مختلفين y و z ، تعرف مسافة ألألتفاف بين الرأسين y و z والتي يرمز لها بـ $D(y,z)$ على أنها الطول لأطول درب $y-z$ بين الرأسين y و z في البيان Γ . متعددة حدود ألألتفاف للبيان المتصل Γ و يرمز لها بـ $D(\Gamma;x)$، و تعرف $D(\Gamma;x) = \sum_{y,z\in V(\Gamma)} x^{D(y,z)}$، في هذا البحث تم أيجاد متعددة حدود ألألتفاف للبيان ثيتا و البيان ثيتا المتماثل.

1. Introduction

In a connected simple graph Γ, the detour distance [1] was defined as a length of a maximum $y-z$ path between two distinct vertices y and z in a vertex set $V(\Gamma)$ of a graph Γ. The Detour Polynomial of a graph Γ is defined by $D(\Gamma;x) = \sum_{\{y,z\}\subseteq V(\Gamma)} x^{D(y,z)}$. The Detour index is defined as the sum of the detour distances between unordered pairs of vertices of the graph Γ. If $D(y,z)$ equals to the standard distance between every pair of vertices y and z of a graph Γ, the graph Γ is called a detour graph [2].

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The use of the detour index was analyzed and its application was compared by Trinajsti'c et al. [3]. Along with that, they compared Wiener's index in structure-boiling point modeling. Whereas the detour index was probed by Rücker [4] as a descriptor for boiling points of acyclic and cyclic alkanes. The detour index was first defined by Amic and Trinajstic in [3] and then the definition of detour polynomial was introduced by Ali and Gashaw in 2012 [2].

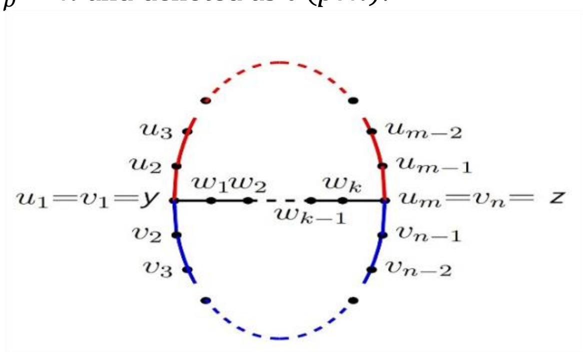
Hence, an algorithm was proposed by Lukovits and Razinger [5] for the detection of the longest path between any two vertices of a graph. It was then used for derivation of analytical formulas for the detour index of fused bicyclic structures. Collaboratively, computer methods were proposed by Trinajsti' c et al. [6] and Rücker and Rücker [4]for computing the detour distances and hence for computing the detour index. There are several recent studies to find detour indices with chemical applications, see [7]and [8]. In 2010 [9], the authors found the detour Hosoya polynomials of some compound graphs, the detour polynomial of Ladder graph and Corona graph was found by [2] and [10].

In this paper, the detour polynomial was found for the theta graph, and in order to obtain this polynomial it was necessary to divide its graph in two cases and to give a general formula for each of the cases.

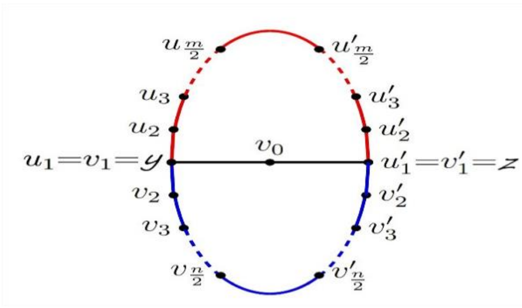
Definition 1.1,[12] Theta graph is a simple graph which encompasses two vertices y and z they are interlinked by three internally disjoint paths L_1, L_2 and L_3 of length $m-1, n-1$ and $k+1$ with common end vertices y and z as m, n and $k+2$ as the number of vertices on respective paths, denoted by $\theta(k, m, n)$ as depicted in Figure 1.

For $k=1$, we denote the graph by $\theta(m, n)$ as depicted in Figures 1(b) and 1(c).

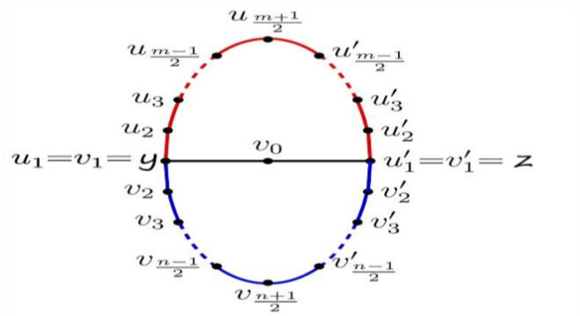
Definition 1.2,[11] Generalized theta graph is a graph includes of two vertices y and z are interlinked by a number of internally disjoint paths $L_1, L_2, \ldots, L_p$ with common end vertices each of length ≥ 1 denoted as $\theta(L_1, L_2, \ldots, L_p: n_1, n_2, \ldots, n_p)$ with $n_1, n_2, \ldots, n_p$ the number of vertices on respective paths, as depicted in Figure 2. Theta graph is called a uniform theta graph, if $n_1 = n_2 = \cdots = n_p = n$ and denoted as $\theta(p: n)$.



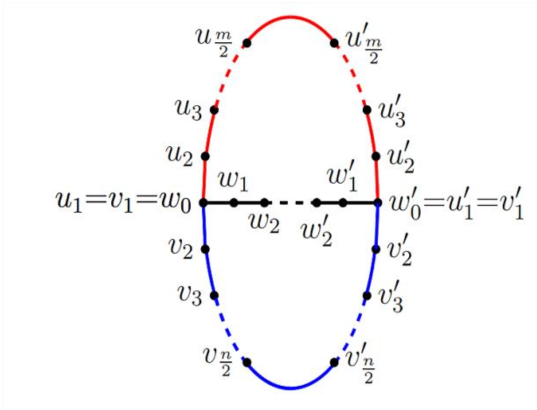
(a) $\theta(k, m, n)$



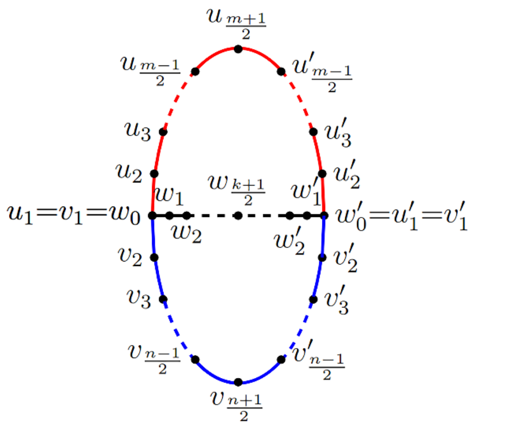
(b) $\theta(m, n)$ m and n are even



(c) $\theta(m, n)$ m and n are odd



(d) $\theta(k,m,n)$ *k*, *m* and *n* are even



(e) $\theta(k,m,n)$ *k*, *m* and *n* are odd

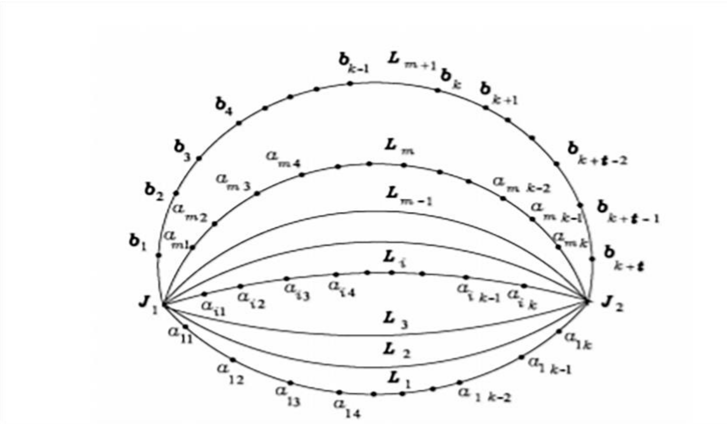


Figure 2: Generalized theta graph

In [12], Herish and Ivan obtained the restricted detour polynomial of the theta Graphs. In this article, the detour polynomial of the theta graph and the uniform theta graph will be computed.

2. Detour Polynomial of the Theta Graph $\theta(m,n)$

In this section, we determine the detour polynomial of the theta graph $\theta(m,n)$, where $m \leq n$, and m and n both are even, both are odd or $m = n$.

Theorem 2.1 If m and n are both even (or both odd) and $m, n \geq 6$, then

$$D(\theta(m,n);x) = 2\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \sum_{j=i+1}^{\lfloor \frac{m+1}{2} \rfloor} x^{m+n+i-(j+2)} + 2\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=i+1}^{\lfloor \frac{n+1}{2} \rfloor} x^{m+n+i-(j+2)}$$
$$+ \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} x^{n-3+(i+j)} + \sum_{i=1}^{\frac{n-m}{2}} \sum_{j=1}^{\frac{n-m}{2}+1-i} x^{n+1-(i+j)}$$
$$+ \sum_{i=1}^{\frac{n-m}{2}} \sum_{j=\frac{n-m}{2}+2-i}^{\lfloor \frac{n}{2} \rfloor} x^{m+i+j-3} + \sum_{i=\frac{n-m}{2}+1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} x^{m+i+j-3}$$
$$+2\sum_{i=2}^{\lfloor \frac{m}{2} \rfloor} \sum_{j=2}^{\lfloor \frac{n}{2} \rfloor} x^{m+n-(i+j)} + 2\sum_{i=2}^{\lfloor \frac{m+1}{2} \rfloor} \sum_{j=2}^{\lfloor \frac{n+1}{2} \rfloor} x^{m+n+1-(i+j)} - x^{n-1} - x^{\frac{n+m}{2}}.$$

Proof. We start the proof when m and n are both even. There are two main cases that can be distinguished for u and v as in the following discussion:

1. If the two vertices are on the same path, then we have the following four subcases

(a) If $u = u_i$ and $u = u_j$, then for $i = 1, \ldots, \frac{m}{2} - 1, j = i + 1, \ldots, \frac{m}{2}$, then we have the path $P_1: u = u_i, u_{i-1}, \ldots, u_1 = v_1, v_2, \ldots, v'_1 = u'_1, u'_2, \ldots, u_j = v$ is a longest $u - v$ path, and

$D(u,v) = D(u_i,u_j) = m + n + i - j - 2.$
By the same way, if $u = u_i'$ and $v = u_j'$ for $i = 1, \dots, \frac{m}{2} - 1, j = i + 1, \dots, \frac{m}{2}$, then
the polynomial ,in this case, is given by
$F_1(x) = 2\sum_{i=1}^{\frac{m}{2}-1} \sum_{j=i+1}^{\frac{m}{2}} x^{m+n+i-j-2}.$
(b) If $u = u_i$ and $v = u_j' i = 1, \dots, \frac{m}{2}, j = 1, \dots, \frac{m}{2}$, then we have the path $P_2\colon u = u_i, u_{i-1}, \dots, u_1 = v_1, v_2, \dots, v_2', v_1' = u_i', \dots, u_j' = v$ is a longest path, and
$D(u,v) = d(u_i,u_j') = n - 3 + (i + j)$, and we get the polynomial
$F_2(x) = \sum_{i=1}^{\frac{m}{2}} \sum_{j=1}^{\frac{m}{2}} x^{n-3+i+j}$
(c) If $u = v_i$ and $v = v_j$, for $i = 1, \dots, \frac{n}{2} - 1, j = i + 1, \dots, \frac{n}{2}$, then we have the path $P_3\colon u = v_i, v_{i-1}, \dots, v_1 = u_1, u_2, \dots, u_1' = v_1', v_2', \dots, v_j = v$ is a longest path, and
$D(u,v) = D(v_i,v_j) = m + n + i - j - 2.$
By the same way, if $u = v_i'$ and $v = v_j'$ for $= 1, \dots, \frac{m}{2} - 1, j = i + 1, \dots, \frac{m}{2}$, then
the corresponding polynomial in this case is given by
$F_3(x) = 2\sum_{i=1}^{\frac{n}{2}-1} \sum_{j=i+1}^{\frac{n}{2}} x^{m+n+i-(j+2)}.$
(d) If $u = v_i$ and $v = v_j'$ and since $m \le n$, then we have the following two subcases
i. For $i = 1, \dots, \frac{n-m}{2}$ and $j = 1, \dots, \frac{n-m}{2} + 1 - i$, then the path $P_4\colon u = v_i, v_{i+1}, \dots, v_{\frac{n}{2}}, v_{\frac{n}{2}}', \dots, v_j' = v$ is a longest $u - v$ path and $D(v_i,v_j') = n + 1 - (i + j)$ and for $i = 1, \dots, \frac{n-m}{2}$ and $j = \left(\frac{n-m}{2}\right) + 2 - i, \dots, \frac{n}{2}$, then the path $P_5\colon u = v_i, v_{i-1}, \dots, u_i, \dots, u_1' = v_1', \dots, v_j' = v$ is a longest $u - v$ path and $D(v_i,v_j') = m + i + j - 3.$
ii. For $i = \frac{n-m}{2} + 1, \dots, \frac{n}{2}, j = 1, \dots, \frac{n}{2}$, then the path $P_6\colon u = v_i, v_{i-1}, \dots, u_i, \dots, u_1' = v_1', \dots, v_j' = v$ is a longest $u - v$ path and $D(u,v) = D(v_i,v_j') = m + i + j - 3.$
Notice that the edge $v_1 v_1'$ is counted twice.
Hence, case (d) leads to the following polynomial

$$F_4(x) = \sum_{i=1}^{\frac{n-m}{2}} \sum_{j=1}^{\frac{n-m}{2}+1-i} x^{n+1-(i+j)} + \sum_{i=1}^{\frac{n-m}{2}} \sum_{j=\frac{n-m}{2}+2-i}^{\frac{n}{2}} x^{m+i+j-3} + \sum_{i=\frac{n-m}{2}+1}^{\frac{n}{2}} \sum_{j=1}^{\frac{n}{2}} x^{m+i+j-3} - x^{n-1}.$$

2. If the vertices u and v are on different paths, then we consider the following cases:
(a) If $(u = u_i$ and $v = v_j)$ or $(u = u_i'$ and $v = v_j')$, then for $i = 2, \dots, \frac{m}{2}, j = 2, \dots, \frac{n}{2}$ the path $P_7\colon u = u_i, u_{i+1}, \dots, u_i', \dots, u_1' = v_1', v_2', \dots, v_j = v$ is a longest $u - v$ path and
$D(u,v) = D(u_i,v_j) = D(u_i',u_j') = m + n - i - j$. This leads to the following polynomial
$F_5(x) = \sum_{i=2}^{\frac{m}{2}} \sum_{j=2}^{\frac{n}{2}} x^{m+n-(i+j)}.$
(b) If $(u = u_i$ and $v = v_j')$ or $(u = u_i'$ and $v = v_j)$, then for $i = 2, \dots, \frac{m}{2}, j = 2, \dots, \frac{n}{2}$ the path $P_8\colon u = u_i, u_{i+1}, \dots, u_i', \dots, u_1' = v_1', v_1, v_2, \dots, v_i, \dots, v_j' = v$ is a longest $u - v$ path and
$D(u,v) = D(u_i,v_j') = D(u_i',v_j) = m + n + 1 - (i + j)$. This leads to the following polynomial
$F_6(x) = 2\sum_{i=2}^{\frac{m}{2}} \sum_{j=2}^{\frac{n}{2}} x^{m+n+1-(i+j)}.$
Now, if m and n both are odd positive integers (see Figure 1c), then the proof of all cases are similar to the technique of proof when m and n are even and we get the following polynomial

$$D(\theta(m,n);x) = 2\sum_{i=1}^{\frac{m-1}{2}} \sum_{j=i+1}^{\frac{m+1}{2}} x^{m+n+i-(j+2)} + \sum_{i=1}^{\frac{m-1}{2}} \sum_{j=1}^{\frac{m-1}{2}} x^{n-3+(i+j)}$$
$$+2\sum_{i=1}^{\frac{n-1}{2}} \sum_{j=i+1}^{\frac{n+1}{2}} x^{m+n+i-(j+2)} + \sum_{i=2}^{\frac{m-1}{2}} \sum_{j=2}^{\frac{n-1}{2}} x^{m+n-(i+j)}$$
$$+2\sum_{i=2}^{\frac{m+1}{2}} \sum_{j=2}^{\frac{n+1}{2}} x^{m+n+1-(i+j)} + \sum_{i=1}^{\frac{n-m}{2}} \sum_{j=1}^{\frac{n-m}{2}+1-i} x^{n+1-(i+j)}$$
$$+\sum_{i=1}^{\frac{n-m}{2}} \sum_{j=\frac{n-m}{2}+2-i}^{\frac{n-1}{2}} x^{m+i+j-3} + \sum_{i=\frac{n-m}{2}+1}^{\frac{n-1}{2}} \sum_{j=1}^{\frac{n-1}{2}} x^{m+i+j-3} - x^{n-1}$$

Combining and simplifying the polynomials of m and n both even and both odd in previous cases we obtain the detour polynomial of $\theta(m,n)$ as given in the statement of the theorem.

Now, putting $m = n$ from Theorem 2.1, we get the following interesting result.

Corollary 2.2. The detour polynomial of $\theta(n,n)$, for $n \geq 6$ is given by

$$D(\theta(n,n);x) = 4\sum_{i=1}^{\left\lfloor\frac{n+1}{2}-1\right\rfloor} \sum_{j=i+1}^{\left\lfloor\frac{n+1}{2}\right\rfloor} x^{2n+i-(j+2)} + 2\sum_{i=1}^{\left\lfloor\frac{n}{2}\right\rfloor} \sum_{j=1}^{\left\lfloor\frac{n}{2}\right\rfloor} x^{n-3+i+j}$$
$$+2\sum_{i=2}^{\left\lfloor\frac{n}{2}\right\rfloor} \sum_{j=2}^{\left\lfloor\frac{n}{2}\right\rfloor} x^{2n-(i+j)} + 2\sum_{i=2}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \sum_{j=2}^{\left\lfloor\frac{n+1}{2}\right\rfloor} x^{2n-(i+j-1)}.$$

Proof. From Theorem 2.1, suppose that $n = m$, we will get the formula $D(\theta(n,n);x)$ after making some algebraic simplifications.

The next two corollaries are direct consequences of Theorem 2.1.

Corollary 2.3. If $n > m \geq 6$, m is an even integer and n is an odd integer, then the detour polynomial of $\theta(m,n)$ is given by

$$D(\theta(m,n);x) = 2\sum_{i=1}^{\frac{m}{2}-1} \sum_{j=i+1}^{\frac{m}{2}} x^{m+n+i-(j+2)} + 2\sum_{i=1}^{\frac{n-1}{2}} \sum_{j=i+1}^{\frac{n+1}{2}} x^{m+n+i-(j+2)}$$
$$+\sum_{i=1}^{\frac{m}{2}} \sum_{j=1}^{\frac{m}{2}} x^{n-3+i+j} + \sum_{i=1}^{\left\lceil\frac{n-m}{2}\right\rceil} \sum_{j=1}^{\left\lceil\frac{n-m}{2}\right\rceil+1-i} x^{n+1-(i+j)}$$
$$+\sum_{i=1}^{\left\lceil\frac{n-m}{2}\right\rceil} \sum_{j=\left\lceil\frac{n-m}{2}\right\rceil+2-i}^{\frac{n-1}{2}} x^{m-3+i+j} + \sum_{i=\left\lceil\frac{n-m}{2}\right\rceil+1}^{\frac{n-1}{2}} \sum_{j=1}^{\frac{n-1}{2}} x^{m-3+i+j}$$
$$+2\sum_{i=2}^{\frac{m}{2}} \sum_{j=2}^{\frac{n-1}{2}} x^{m+n-(i+j)} + 2\sum_{i=2}^{\frac{m}{2}} \sum_{j=2}^{\frac{n+1}{2}} x^{m+n+1-(i+j)} - x^{n-1}.$$

Proof. We get the previous formula in the same way as proof Theorem 2.1.

Corollary 2.4. If $n \geq m \geq 6$, m is an odd integer and n is an even integer, then the detour polynomial of $\theta(m,n)$ is given by

$$D(\theta(m,n);x) = 2\sum_{i=1}^{\frac{m-1}{2}} \sum_{j=i+1}^{\frac{m+1}{2}} x^{m+n+i-(j+2)} + 2\sum_{i=1}^{\frac{n}{2}-1} \sum_{j=i+1}^{\frac{n}{2}} x^{m+n+i-(j+2)}$$
$$+\sum_{i=1}^{\frac{m-1}{2}} \sum_{j=1}^{\frac{m-1}{2}} x^{n-3+i+j} + \sum_{i=1}^{\left\lceil\frac{n-m}{2}\right\rceil} \sum_{j=1}^{\left\lceil\frac{n-m}{2}\right\rceil+1-i} x^{n+1-(i+j)}$$
$$+\sum_{i=\left\lceil\frac{n-m}{2}\right\rceil+1}^{\frac{n}{2}} \sum_{j=1}^{\frac{n}{2}} x^{m-3+i+j} + \sum_{i=1}^{\left\lceil\frac{n-m}{2}\right\rceil} \sum_{j=\left\lceil\frac{n-m}{2}\right\rceil+2-i}^{\frac{n}{2}} x^{m-3+i+j}$$
$$+2\sum_{i=2}^{\frac{m-1}{2}} \sum_{j=2}^{\frac{n}{2}} x^{m+n-(i+j)} + \sum_{i=2}^{\frac{m+1}{2}} \sum_{j=2}^{\frac{n}{2}} x^{m+n+1-(i+j)} - x^{n-1}.$$

Proof. We get the above formula in the same way as proof Theorem 2.1.

The next theorem computes the detour polynomial of the theta graph $\theta(m,n,k)$ depicted in Figure 1(a).

Theorem 2.5. If $n \geq m \geq k > 6$ and m,n and k all are even positive integers (or all are odd positive integers), then

$$D(\theta(k,m,n);x) = 2\sum_{i=1}^{\left\lfloor\frac{m}{2}\right\rfloor-1}\sum_{j=i+1}^{\left\lfloor\frac{m+1}{2}\right\rfloor} x^{m+n+i-(j+2)} + 2\sum_{i=1}^{\left\lfloor\frac{n}{2}\right\rfloor-1}\sum_{j=i+1}^{\left\lfloor\frac{n+1}{2}\right\rfloor} x^{m+n+i-(j+2)}$$

$$+\sum_{i=1}^{\left\lfloor\frac{m}{2}\right\rfloor}\sum_{j=1}^{\left\lfloor\frac{m}{2}\right\rfloor} x^{n-3+(i+j)} \sum_{i=1}^{\frac{n-m}{2}}\sum_{j=1}^{\frac{n-m}{2}+1-i} x^{n+1-(i+j)}$$

$$+\sum_{i=1}^{\frac{n-m}{2}}\sum_{j=\frac{n-m}{2}+2-i}^{\left\lfloor\frac{n}{2}\right\rfloor} x^{m+i+j-3} + \sum_{i=\frac{n-m}{2}+1}^{\left\lfloor\frac{n+1}{2}\right\rfloor}\sum_{j=1}^{\left\lfloor\frac{n}{2}\right\rfloor} x^{m+i+j-3}$$

$$+2\sum_{i=2}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{j=2}^{\left\lfloor\frac{n}{2}\right\rfloor} x^{m+n-(i+j)} + 2\sum_{i=0}^{\left\lfloor\frac{k}{2}\right\rfloor-1}\sum_{j=i+1}^{\left\lfloor\frac{k+1}{2}\right\rfloor} x^{n+k+i-j}$$

$$+\sum_{i=0}^{\left\lfloor\frac{k}{2}\right\rfloor}\sum_{j=0}^{\left\lfloor\frac{k}{2}\right\rfloor} x^{n+i+j-1} + 2\sum_{i=2}^{\left\lfloor\frac{m}{2}\right\rfloor}\sum_{j=2}^{\left\lfloor\frac{n-(k+2)}{2}\right\rfloor} x^{m+n-(i+j)}$$

$$+2\sum_{i=2}^{\left\lfloor\frac{m}{2}\right\rfloor}\sum_{j=\left\lfloor\frac{n-(k+2)}{2}\right\rfloor+1}^{\left\lfloor\frac{n}{2}\right\rfloor} x^{m+k+j-i} + \sum_{i=2}^{\left\lfloor\frac{m+1}{2}\right\rfloor}\sum_{j=2}^{\left\lfloor\frac{n-(k+2)}{2}\right\rfloor} x^{m+n-(i+j)}$$

$$+\sum_{i=2}^{\left\lfloor\frac{m+1}{2}\right\rfloor}\sum_{j=\left\lfloor\frac{n-(k+2)}{2}\right\rfloor+1}^{\left\lfloor\frac{n+1}{2}\right\rfloor} x^{m+k+j-i} + \sum_{i=2}^{\left\lfloor\frac{m}{2}\right\rfloor}\sum_{j=2}^{\left\lfloor\frac{n+1}{2}\right\rfloor} x^{m+n+k+1-(i+j)}$$

$$+\sum_{i=2}^{\left\lfloor\frac{m+1}{2}\right\rfloor}\sum_{j=2}^{\left\lfloor\frac{n}{2}\right\rfloor} x^{m+n+k+1-(i+j)} + \sum_{i=2}^{\left\lfloor\frac{m-(k+2)}{2}\right\rfloor+2}\sum_{j=1}^{\left\lfloor\frac{k}{2}\right\rfloor} x^{m+n+j-(i+1)}$$

$$+\sum_{i=\left\lfloor\frac{m-(k+2)}{2}\right\rfloor+3}^{\left\lfloor\frac{m}{2}\right\rfloor}\sum_{j=1}^{i-\left\lfloor\frac{m-(k+2)}{2}\right\rfloor+1} x^{n+k+j-(i+1)}$$

$$+\sum_{i=\left\lfloor\frac{m-(k+2)}{2}\right\rfloor+3}^{\left\lfloor\frac{m+1}{2}\right\rfloor}\sum_{j=i-\left\lfloor\frac{m-(k+2)}{2}\right\rfloor}^{\left\lfloor\frac{k}{2}\right\rfloor} x^{n+m+j-(i+1)}$$

$$+\sum_{i=2}^{\left\lfloor\frac{m-(k+2)}{2}\right\rfloor+2}\sum_{j=1}^{\left\lfloor\frac{k+1}{2}\right\rfloor} x^{m+n+i-(j+1)}$$

$$+\sum_{i=\left\lfloor\frac{m-(k+2)}{2}\right\rfloor+3}^{\left\lfloor\frac{m+1}{2}\right\rfloor}\sum_{j=1}^{i-\left\lfloor\frac{m-(k+2)}{2}\right\rfloor+1} x^{n+k+i-(j+1)}$$

$$+\sum_{i=\left\lfloor\frac{m-(k+2)}{2}\right\rfloor+3}^{\left\lfloor\frac{m+1}{2}\right\rfloor}\sum_{j=i-\left\lfloor\frac{m-(k+2)}{2}\right\rfloor}^{\left\lfloor\frac{k+1}{2}\right\rfloor} x^{n+m+j-(i+1)}$$

$$+2\sum_{i=2}^{\left\lfloor\frac{m}{2}\right\rfloor}\sum_{j=1}^{\left\lfloor\frac{k+1}{2}\right\rfloor} x^{m+n+k-(i+j)} + \sum_{i=2}^{\left\lfloor\frac{m+1}{2}\right\rfloor}\sum_{j=1}^{\left\lfloor\frac{k}{2}\right\rfloor} x^{m+n+k-(i+j)}$$

$$+\sum_{i=2}^{\left\lfloor\frac{n-(k+2)}{2}\right\rfloor+2}\sum_{j=1}^{\left\lfloor\frac{k}{2}\right\rfloor} x^{m+n+j-(i+1)}$$

$$+\sum_{i=\left\lfloor\frac{n-(k+2)}{2}\right\rfloor+3}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{j=1}^{i-\left\lfloor\frac{n-(k+2)}{2}\right\rfloor+1} x^{m+k+i-(j+1)}$$

$$+\sum_{i=\left\lfloor\frac{n-(k+2)}{2}\right\rfloor+3}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{j=i-\left\lfloor\frac{n-(k+2)}{2}\right\rfloor}^{\left\lfloor\frac{k}{2}\right\rfloor} x^{n+m+j-(i+1)}$$

$$+\sum_{i=2}^{\left\lfloor\frac{n-(k+2)}{2}\right\rfloor+2}\sum_{j=1}^{\left\lfloor\frac{k+1}{2}\right\rfloor} x^{m+n+j-(i+1)}$$

$$+\sum_{i=\left\lfloor\frac{n-(k+2)}{2}\right\rfloor+3}^{\left\lfloor\frac{n+1}{2}\right\rfloor}\sum_{j=1}^{i-\left\lfloor\frac{n-(k+2)}{2}\right\rfloor+1} x^{m+k+i-(j+1)}$$

$$+\sum_{i=\left\lfloor\frac{n-(k+2)}{2}\right\rfloor+3}^{\left\lfloor\frac{n+1}{2}\right\rfloor}\sum_{j=i-\left\lfloor\frac{n-(k+2)}{2}\right\rfloor}^{\left\lfloor\frac{k+1}{2}\right\rfloor} x^{n+m+j-(i+1)}$$

$$+\sum_{i=2}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{j=1}^{\left\lfloor\frac{k}{2}\right\rfloor} x^{m+n+k-(i+j)} + \sum_{i=2}^{\left\lfloor\frac{n+1}{2}\right\rfloor}\sum_{j=1}^{\left\lfloor\frac{k+1}{2}\right\rfloor} x^{m+n+k-(i+j)} - 2x^{(n-1)}.$$

Proof. We start the proof for m, n and k all are even positive integer.
Let u and v, then there are two main cases which can be distinguished for u and v.
Case 1. If u and v are on the same path, then we have six subcases which can be highlighted

(a) If $u = u_i$ and $v = u_j$ (or $u = u'_i$ and $= u'_j$), the same as Case 1 (a) of Theorem12..
(b) If $u = u_i$ and $v = u'_j$, the same as Case 1 (b) of Theorem2.1.
(c) If $u = v_i$ and $v = v_j$ (or $u = v'_i$ and $= v'_j$), the same as Case 1 (c) of Theorem2.1.
(d) If $u = v_i$ and $v = v'_j$, the same as Case 1 (d) of Theorem 2.1.
(e) If $u = w_i$ and $v = w_j$ (or $u = w'_i$ and $= w'_j$), then for $i = 0,1, \dots, \frac{k}{2} - 1, j = i + 1, \dots, \frac{k}{2}$, the path $P_1: u = w_i, w_{i-1}, \dots, w_0 = v_1, v_2, v'_i, \dots, v'_1 = w'_0, \dots, w_j = v$. is a longest $u - v$ path and then $D(u, v) = D(w_i, w_j) = n + k + i - j$.
In the same way, if $u = w'_i$ and $v = w'_j$.
Then, from this case obtain the polynomial

$$F_1(x) = 2 \sum_{i=0}^{\frac{k}{2}-1} \sum_{j=i+1}^{\frac{k}{2}} x^{n+k+i-j}.$$

(f) If $u = w_i$ and $v = w'_j$, for $i = 0, \dots, \frac{k}{2}, j = 0, \dots, \frac{k}{2}$, then the path $P_2: u = w_i, w_{i-1}, \dots, w_0 = v_1, v_2, \dots, v'_i, \dots, w'_j = v$, P_2 is a longest $u - v$ path, and then
$D(u, v) = D(w_i, w'_j) = n + i + j - 1$.
Notice that the detour distance between w_0 and w'_o is counted in case (b) with length $n - 1$.
Then, the corresponding polynomial in this case is

$$F_2(x) = \sum_{i=0}^{\frac{k}{2}} \sum_{j=0}^{\frac{k}{2}} x^{n+i+j-1} - x^{n-1}.$$

Case 2. If the two vertices u and v are on different paths, then we have six subcases as follows
(a) If $u = u_i$ and $v = v_j$ (or $u = u'_i$ and $= v'_j$), then for $i = 2, \dots, \frac{m}{2}, j = 2, \dots, \frac{n}{2}$, we have two subcases,
(i) For $i = 2, \dots, \frac{m}{2}, j = 2, \dots, \frac{n-k-2}{2}$; then the path $P_3: u = u_i, u_{i+1}, \dots, u'_1 = v'_1, \dots, v'_i, \dots, v_j = v$, is the longest $u - v$ path and then $D(u, v) = D(u_i, v_j) = m + n - (i + j)$.
(ii) For $i = 2, \dots, \frac{m}{2}, j = \frac{n-k-2}{2}, \dots, \frac{n}{2}$; then the path $P_4: u = u_i, u^2_{i+1}, \dots, u'_1 = w'_1, \dots, w_i, \dots, v_j = v$ is the longest $u - v$ path and then $D(u, v) = D(u_i, v_j) = m + k + j - i$.
In the same way, all cases are repeated if $u = u'_i$ and $v = v'_j$, for $i = 2, \dots, \frac{m}{2}, j = 2, \dots, \frac{n}{2}$.
From this case we get a polynomial given by

$$F_3(x) = 2 \sum_{i=2}^{\frac{m}{2}} \sum_{j=2}^{\frac{n-k-2}{2}} x^{m+n-(i+j)} + 2 \sum_{i=2}^{\frac{m}{2}} \sum_{j=\frac{n-k-2}{2}+1}^{\frac{n}{2}} x^{m+k+j-i}.$$

(b) If $u = u_i$ and $v = v'_j$ (or $u = u'_i$ and $= v_j$), then for $i = 2, \dots, \frac{m}{2}, j = 2, \dots, \frac{n}{2}$; then the path $P_5: u = u_i, u_{i+1}, \dots, u'_1 = w'_0, \dots, w_i, u_1 = v_1, v_2, \dots, v'_j = v$, is a longest $u - v$ path and then $D(u, v) = D(u_i, v'_j) = m + n + k + 1 - (i + j)$.
In the same way, all cases are repeated if $u = u'_i$ and $v = v_j$, for $i = 2, \dots, \frac{m}{2}, j = 2, \dots, \frac{n}{2}$. From this case we get the corresponding polynomial given by

$$F_4(x) = 2 \sum_{i=2}^{\frac{m}{2}} \sum_{j=2}^{\frac{n}{2}} x^{m+n+k+1-(i+j)}.$$

(c) If $u = u_i$ and $v = w_j$ (or $u = u'_i$ and $v = w'_j$), then for $i = 2, \dots, \frac{m}{2}, j = 1, \dots, \frac{k}{2}$, we have three subcases
(i) For $i = 2, \dots, \frac{m-k-2}{2} + 2$ and $j = 1, \dots, \frac{k}{2}$, the path $P_6: u = u_i, u_{i+1}, \dots, u'_i, \dots, u'_1 = v'_1, v'_2, \dots, w_0, \dots, w_j = v$ is a longest $u - v$ path and then
$D(u, v) = D(u_i, w_j) = n + m + j - (i + 1)$.
(ii) For $i = \frac{m-k-2}{2} + 3, \dots, \frac{m}{2}$ and $j = 1, \dots, \frac{k}{2}$; if $j = 1, \dots, i - \left(\frac{m-k-2}{2} + 1\right)$, then the path $P_7: u = u_i, u_{i-1}, \dots, w_0 = v_1, v_2, \dots, v'_i, \dots, v'_1 = w'_0, \dots, w_j = v$ is a longest $u - v$ path and then

$D(u,v) = D(u_i, w_j) = n + k + i - (j+1)$, and if $j = i - \left(\frac{m-k-2}{2} + 2\right), \dots, \frac{k}{2}$, then the path $P_8: u = u_i, u_{i+1}, \dots, u'_1 = w'_0, \dots, v'_i, \dots, v_i, v_1 = w_0, \dots, w_j = v$ is a longest $u - v$ path and then $D(u,v) = D(u_i, w_j) = n + k + i - (j+1)$.
All other cases are repeated if $u = u'_i$ and $v = w'_j$.
From this case we get the corresponding polynomial given by

$$F_5(x) = 2\sum_{i=2}^{\frac{m-(k+2)}{2}} \sum_{j=1}^{\frac{k}{2}} x^{m+n+j-(i+1)} + 2\sum_{i=\frac{m-(k+2)}{2}+3}^{\frac{m}{2}} \sum_{j=1}^{i-\frac{m-(k+2)}{2}+1} x^{n+k+i-(j+1)}$$
$$+2\sum_{i=\frac{m-(k+2)}{2}+3}^{\frac{m}{2}} \sum_{j=i-\frac{m-(k+2)}{2}} x^{m+n+j-(i+1)}.$$

(d) If $u = u_i$ and $v = w'_j$ (or $u = u'_i$ and $= w_j$), and for $i = 2, \dots, \frac{m}{2}, j = 1, \dots, \frac{k}{2}$, the path $p_8: u = u_i, u_{i+1}, \dots, u'_2, u'_1 = w'_0, v'_2, v_i, \dots, v_1 = w_0, w_1, \dots, w'_j = v$. is a longest $u - v$ path and then $D(u,v) = D(u_i, w'_j) = m + n + k - (i+j)$.
The case is similar if $u = u'_i$ and $v = w_j$ for $i = 2, \dots, \frac{m}{2}, j = 1, \dots, \frac{k}{2}$.
The corresponding polynomial in this case is given by

$$F_6(x) = 2\sum_{i=2}^{\frac{m}{2}} \sum_{j=2}^{\frac{k}{2}} x^{m+n+k-(i+j)}.$$

(e) If $u = v_i$ and $v = w_j$ (or $u = v'_i$ and $= w'_j$) then, for $i = 2, \dots, \frac{n}{2}, j = 1, \dots, \frac{k}{2}$, we have two subcases
(1) If $= 2, \dots, \frac{n-(k+2)}{2} + 2$ and $j = 1, \dots, \frac{k}{2}$, then the path $P_9: u = v_i, v_{i+1}, \dots, v'_i, w'_0 = u'_1, u'_2, \dots, u'_i, u_i, \dots, u_1 = w_0, w_1, w_j = v$ is a longest $u - v$ path and then $D(u,v) = D(v_i, w_j) = m + n + j - (i+1)$.
(2) If $i = \frac{n-(k+2)}{2} + 3, \dots, \frac{n}{2}$ and $j = 1, \dots, i - \left(\frac{n-(k+2)}{2} + 1\right)$, then the path $P_{10}: u = v_i, v_{i-1}, \dots, w_0 = u_1, u_2, \dots, u'_i, \dots, v'_1 = w'_0, w'_1, \dots, w_j = v$ is a longest $u - v$ path and then $D(u,v) = D(v_i, w_j) = m + k = i - (j+1)$. If $i = \frac{n-(k+2)}{2} + 3, \dots, \frac{n}{2}$ and $j = i - \frac{n-(k+2)}{2}$, then the path $P_{11}: u = v_i, v_{i+1}, \dots, v'_i, w'_0 = u'_1, u'_2, \dots, u'_i, u_i, \dots, u_1 = w_0, w_1, w_j = v$ is a longest $u - v$ path and then $D(u,v) = D(v_i, w_j) = m + k = i - (j+1)$.
Similarly all cases are repeated if $u = v'_i$ and $v = w'_j$.
So, this case leads to the following polynomial

$$F_7(x) = 2\sum_{i=2}^{\frac{n-(k+2)}{2}} \sum_{j=2}^{\frac{k}{2}} x^{m+n+j-(i+1)} + 2\sum_{i=\frac{n-(k+2)}{2}+3}^{\frac{n}{2}} \sum_{j=1}^{i-\frac{n-(k+2)}{2}+1} x^{m+k+i-(j+1)}$$
$$+2\sum_{i=\frac{n-(k+2)}{2}+3}^{\frac{n}{2}} \sum_{j=i-\frac{n-(k+2)}{2}}^{\frac{k}{2}} x^{m+n+j-(i+1)}.$$

(f) If $u = v_i$ and $v = w'_j$ (or $u = v'_i$ and $= w_j$), for $i = 2, \dots, \frac{n}{2}, j = 1, \dots, \frac{k}{2}$, then the path $p_{12}: u = v_i, v_{i+1}, \dots, v'_i, \dots, v'_1 = w'_0, \dots, u'_i, \dots, u'_i, \dots, w'_j = v$. is a longest $u - v$ path, and then $D(u,v) = D(v_i, w'_j) = m + n + k - (i+j)$.
If $u = v'_i$ and $v = w_j$ then the case is similar for $i = 2, \dots, \frac{m}{2}$ and $j = 1, \dots, \frac{k}{2}$.
The corresponding polynomial in this case is given by

$$F_8(x) = \sum_{i=2}^{\frac{n}{2}} \sum_{j=1}^{\frac{k}{2}} x^{m+n+k-(i+j)}.$$

If n, m and k all are odd positive integers as shown in Figure 1(e), then the technique proof of all cases are similar as the technique proof where n, m and k are even positive integers.

Now, the result is a consequence of combining and simplifying the polynomials of n, m and k which are obtained, odd or even.

The next result computes the detour polynomial of the theta graph in which the three paths have the same order.

Corollary 2.5. If $n \geq 6$, we have

$$D(\theta(n,n,n);x) = 6\sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor-1} \sum_{j=i+1}^{\left\lfloor\frac{n+1}{2}\right\rfloor} x^{2n+i-(j+2)} + 3\sum_{i=1}^{\left\lfloor\frac{n}{2}\right\rfloor} \sum_{j=i+1}^{\left\lfloor\frac{n}{2}\right\rfloor} x^{n+i+j-3}$$
$$+3\sum_{i=2}^{\left\lfloor\frac{n}{2}\right\rfloor} \sum_{j=2}^{i-1} x^{2(n-1)+i-j} + 3\sum_{i=1}^{\left\lfloor\frac{n}{2}\right\rfloor} \sum_{j=1}^{\left\lfloor\frac{n}{2}\right\rfloor} x^{2(n-1)+j-i}$$
$$+3\sum_{i=2}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \sum_{j=2}^{i-1} x^{2(n-1)+i-j} + 3\sum_{i=2}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \sum_{j=i}^{\left\lfloor\frac{n+1}{2}\right\rfloor} x^{2(n-1)+j-i}$$
$$+3\sum_{i=2}^{\left\lfloor\frac{n}{2}\right\rfloor} \sum_{j=2}^{\left\lfloor\frac{n+1}{2}\right\rfloor} x^{2n-(i+j)} + 3\sum_{i=2}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \sum_{j=2}^{\left\lfloor\frac{n}{2}\right\rfloor} x^{3n-(i+j)} - 2x^{n-1}.$$

Proof. Put $k = m = n$ in Theorem 2.4 and simplifying, then we get the result.

The Detour polynomial of a uniform theta graph $\theta(p{:}\,n)$ is given in the next result. From the definition of a uniform theta graph, we have p disjoint paths each of order n, and by Corollary 2.5, we get the result.

Corollary 2.6. If $n \geq 6$, we have

$$D(\theta(p{:}\,n);x) = p\sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor-1} \sum_{j=i+1}^{\left\lfloor\frac{n+1}{2}\right\rfloor} x^{2n+i-(j+2)} + \frac{p(p-1)}{2}\sum_{i=1}^{\left\lfloor\frac{n}{2}\right\rfloor} \sum_{j=1}^{\left\lfloor\frac{n}{2}\right\rfloor} x^{n+i+j-3}$$
$$+\frac{p(p-1)}{2}\sum_{i=2}^{\left\lfloor\frac{n}{2}\right\rfloor} \sum_{j=2}^{i-1} x^{2(n-1)+i-j} + \frac{p(p-1)}{2}\sum_{i=1}^{\left\lfloor\frac{n}{2}\right\rfloor} \sum_{j=1}^{\left\lfloor\frac{n}{2}\right\rfloor} x^{2(n-1)+j-i}$$
$$+\frac{p(p-1)}{2}\sum_{i=2}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \sum_{j=2}^{i-1} x^{2(n-1)+i-j} + \frac{p(p-1)}{2}\sum_{i=2}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \sum_{j=i}^{\left\lfloor\frac{n+1}{2}\right\rfloor} x^{2(n-1)+j-i}$$
$$+\frac{p(p-1)}{2}\sum_{i=2}^{\left\lfloor\frac{n}{2}\right\rfloor} \sum_{j=2}^{\left\lfloor\frac{n+1}{2}\right\rfloor} x^{2n-(i+j)} + \frac{p(p-1)}{2}\sum_{i=2}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \sum_{j=2}^{\left\lfloor\frac{n+1}{2}\right\rfloor} x^{3n-(i+j)} - (p-1)x^{n-1}.$$

Proof: For every two vertices that are on the same path or are on different paths, all cases of Theorem1 are repeated and using Corollary 2.5 we get the result.

Conclusion:

It is difficult to find a detour polynomial for graphs that have multiple separate paths internally between any two distinct vertices, and for the purpose of obtaining a general formula, we split all the proof that can be given to the theta graph, and multiple results are obtained for special cases depending on the general formulas.

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