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The Discs Structures of A4-Graph for the Held Group

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Abstract:

Let G be a finite group and X be a G-conjugacy of elements of order 3. The A4-graph of G is a simple graph with vertex set X and two vertices $x,y \in X$ are linked if $x \neq y$ and xy^{-1} is an involution element. This paper aims to investigate the A4-graph properties for the monster Held group He.

Keywords: Finite simple groups; A4-graph; Connectivity, Diameter.
Mathematics Subject Classification 2020: 20D05, 68R01, 05C40, 05C69

هياكل أقطار البيان–A4 لزمرة هيلد

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الخلاصة

لنفرض أن G زمرة منتهية، و X يمثل G–صف تكافئ لعناصر من الرتبة 3. البيان– A_4 للزمرة G هو بيان بسيط له X كمجموعة رؤوس، مع كون الرأسين $x, y \in X$ مرتبطة في حالة $x \neq y$, و xy^{-1} عنصر الالتفاف . *هذا البحث يهدف الى استقصى خواص* البيان– A_4 *لزمرة الوحش هيلد* He.

1.Introduction

A group action on a graph is one of the most effective ways for understanding group structure. Several recent research, including [1,2] and [3], have established the effectiveness of this strategy. For a finite group G, it contains a subset X as G-conjugacy of an element of order 3. Aubad [4] proposed the A4-graph, which is a simple graph identified as $\mathcal{A}_4$(G,X), with the X representing the vertex set and x, y ∈ X are adjacent if the conditions $x \neq y$ and $(xy^{-1})^2 = 1$ are fulfilled. The most significant consequence in [4] is an analysis of the formation of $\mathcal{A}_4$(G,X) and $G \cong {}^3D_4(2)$, as well as offering valuable A4-graph general characteristics. Also, it seems to be worth noting that the alternating group A4 may be produced by any two connected vertices in the A4-graph. The authors of [5] provided a thorough examination of the A4-graphs for the Mathieu group M_{20} and the tits group T. In addition, for specific leech lattice groups, the authors in [6] provided a special study on A4-graphs. Moreover, the entire lists of simple Mathieu groups are explored in [7], as well as a comprehensive review of the A4-graph properties.

In this article, G will be referred to as a Held group He with $X=t^G$ a G-conjugacy class, such that $t \in G$ has order three. The goal of this paper is to develop a variety of A4-graph

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characteristics for the group G. Describing the structure of the discs and figuring the diameters for an A4-graph are part of the inquiry.

Assume that $x \in$X, for $i \in \mathbb{Z}^+$, then $\Delta_i(x)$ identifies the i^{th} disc of x and yields the following set, $\Delta_i(x)=\{r\in$ X| d(x,r)=$i\}$, where d(,) is the standard distance metric for the A4-graph. The fact that G is transitive on A4-graph vertices and that G acing on X by conjugation merges G into the A4-graph automorphisms group is a key feature of the A4-graph.

This is the layout of the paper. We give general properties as well as methods for studying the A4-graph in Section 2. In Section 3, the A4-graph of the Held finite simple group He is computationally investigated. Finally, in Section 4, we reach conclusions and develop a view of the future.

2. A4-Graph General Properties

We begin with the definition of the A4-graph follow by illustrating example:

Definition 2.1.[4] Let G be finite group and X be a G-conjugacy class for elements of order 3. The A4-graph of G is a simple graph indicated by $\mathcal{A4}\,(G,X)$, with vertex set X and x,$y\in$X linked by one edge if x $\neq$y and $(xy^{-1})^2=1$.

Example 2.2. Let G$\cong A_6$, and let X=t^G such that t= (4,5,6). Then the A4-graph is connected with diameter 3. The computational calculations inside GAP[8] led to the following results: $\Delta_0(t)=\{t\}$,$\Delta_1(t)$={ (3,4,6),(1,4,6),(2,4,6),(3,5,4),(1,5,4),(2,5,4),(3,6,5), (1,6,5),(2,6,5)}, $\Delta_2(t)$={(3,4,5),(1,4,5),(2,4,5),(3,5,6),(1,5,6),(2,5,6),(3,6,4), (1,6,4),(2,6,4),(2,3,4),(1,4,3),(1,2,4),(2,3,5),(1,5,3),(1,2,5),(2,3,6),(1,6,3),(1,2,6),(2,4,3), (1,3,4),(1,4,2),(2,5,3),(1,3,5),(1,5,2),(2,6,3),(1,3,6),(1,6,2)},$\Delta_3(t)$={(4,6,5),(1,2,3),(1,3,2)}.

Moreover, the girth is 3 and the clique number is 4 of the $\mathcal{A4}\,(A_6,X)$. This can be seen in the next figure:

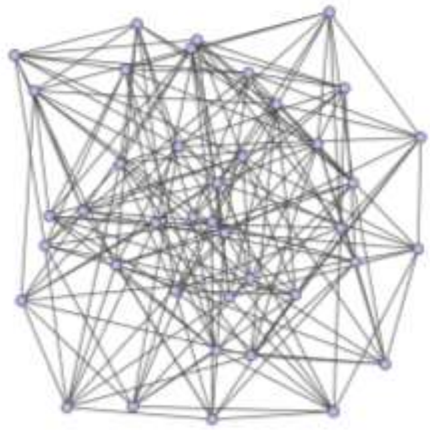

Figure 1-The A4-graph structure $\mathcal{A4}\,(A_6,X)$.

The next key observations on the A4 -graph for finite groups is mentioned in [4]. We will use G $\cong$ He and X=t^G in this study, with t being an element of order 3 in G. The first set of findings shows the A4-graph overall attributes, while the second set of results shows the discs structure of the graph.

Lemma 2.3.[4] Let G be a finite group and let X be a G-conjugacy class of elements of order 3. Then, A4-graph of G holds the following properties:

1- $\mathcal{A4}\,(G,X)$ is a simple undirected graph.

2- $\mathcal{A4}\,(G,X)$ is a regular graph.

3- The alternating group of degree 4 is created by two linked vertices.

Lemma 2.4.[4] Let G be a finite group and $X=t^G$ such that $t\in$G has order 3.Then we have the following results:

1- $\Delta_i(t)$ for the $\mathcal{A}4$ (G,X) breakdown into a union of $C_G(t)-$ orbits (such that $C_G(t)$ acts on X by conjugation).
2- If $x \in$ X such that $x \in \Delta_1(t)$, then tx has order 3.

Remark 2.5. From the foregoing, we may conclude that the fix $t \in$X will be picked at random.

The $C_G(t)-$ orbits of $\mathcal{A}4$ (G, X) revealed the following results:

Lemma 2.6.[7] Let T_i, T_j are different $C_G(t)-$orbits in the $\mathcal{A}4$ (G,X). Let $a \in T_i$ and $b \in T_j$ such that $a \in \Delta_1(b)$. Then $a^u \in \Delta_1(b^u)$ for $u \in C_G(t)$.

Read the following list to get started:

$X_C=\{w\in X|\ tw\in X\}$.

The set C is a particular G-conjugacy class. The set $X_C \neq Ø$ is made up of $C_G(t)$-orbits from the class X. One should first describe how the set X_C is split down into $C_G(I)$- orbits before you can determine if $\Delta_i(t)$ has vertices in X_C.

The size of the set X_C is specified by the following structured:

$$|X_C| = \frac{|G|}{|C_G(t)||C_G(w)|}\sum_{i=1}^{n}\frac{\chi_i(w)\chi_i(t)\overline{\chi_i(t)}}{\chi_i(1)}$$

where χ_i are the complex irreducible characters of G, for i=1,...,n, and w is a random element in the class C. As a result, $|X_C|$ may be readily computed using the computational algebraic system GAP, especially the function " ***Class Multiplication Coefficient***". Moreover, in this article, we deal with the He group computationally, and we build 2058-point permutation representations in GAP using the Online Atlas [9]. The GAP-recommended computational approach, as well as the Online Atlas, are the focal points of our plan. Finally, for the paper notation, we can use the same notations as in [7].

3. Discs Structures of the A4-Graphs for He

The A4-graph discs structure of the Held group He is discussed in this section. GAP refers to a collection of He-conjugacy constraints that partition order 3 elements between two classes, namely 3A and 3B. Thus, there are two A4-graphs, one for each He-conjugacy class. We utilize class names from the Online Atlas in the tables that follow. After we merge these classes, we reduce the letter component of the class name still more, and their letters are in alphabetical order to make things simpler. For Example, 21CD is shortcut to 21C∪ 21D in Tables 1 and 2. In the next, we provide information about these A4-graphs.

3.1 Discs Structure of $\mathcal{A}_4$ (He,3A)

Consider the class X=3A of the group G≅He, this class has size 533120. Also, for $t \in$3A, $C_G(t) \cong 3.A_7$ and the action of $C_G(t)$ on 3A produce 131-$C_G(t)$-orbits. The computational approach led to that $\mathcal{A}_4$ (He,3A) is connected of diameter 3. Moreover, the action of $C_G(t)$ on 3A breaks into the discs of the A4-graph as X_C sets for He-conjugacy class C. This can explain in the next table:

Table 1-The Discs Contraction of **$\mathcal{A}_4$ (He,3A)**

Class Name	|Orbit|	$\Delta_1(t)$	|Orbit|	$\Delta_2(t)$	|Orbit|	$\Delta_3(t)$
1A					1	1
2A			1	126		
2B			2	630		
3A	3	756			6	2910
3B					1	2520
4B			4	4410	4	4410
4C			2	3780		
5A			5	7686	1	2520

6A			7	25200	2	15120
6B					2	15120
7AB			2	90	1	315
7C					2	2160
7DE					1	1080
8A					8	60480
10A			10	30240	8	37800
12B					4	30240
14AB			2	3780	3	11340
14CD					2	15120
15A			7	47880	10	65520
17AB			2	15120	2	15120
21AB					2	2160
21CD					3	22680

In the above table, in some cases the distance between *t* and the vertices of $\mathcal{A}_4$ (He,3A) is determined by utilizing the G-conjugacy class C, resulting in $tx \in C$. Furthermore, these cases when the class C belongs to one of the G-conjugacy classes {1A,2AB,3B,4C,6B,7CDE,8A,12B,14CD,21ABCD}

3.2 Discs Structure of $\mathcal{A}_4$ (He,3B)

In this case, we let X=3B and then |X|=7996800. Let *t* be a fixed in 3B, then $C_G(t)$ isomorphic to 3.PSL(3,2). The computational calculation yields that $\mathcal{A}_4$ (He,3B) is connected with diameter 4. Furthermore, there are 16032 C_G(t)-orbits. These orbits break into the discs of the A4-graph as X_C for G-class C. The description of such sets is given in the next table:

Table 2-The Discs Contraction of $\mathcal{A}_4$ **(He,3B)**

Class name	|Orbit|	$\Delta_1(t)$	|Orbit|	$\Delta_2(t)$	|Orbit|	$\Delta_3(t)$	|Orbit|	$\Delta_4(t)$
1A					1	1		
2A			1	168	2	27		
2B					4	210		
3A					9	1400	2	336
3B	7	405	2	1008	59	19080	11	5096
4A			2	48	18	5424	4	2016
4B			2	504	50	23142	2	336
4C			6	2016	132	64764	4	2016
5A			4	1344	58	27888	14	7056
6A			10	3696	211	101976	10	5040
6B			4	378	586	292656	10	5040
7AB					18	8064	2	1008
7C			4	96	26	10128	4	2016
7DE			6	1512	137	68292	1	504
8A			14	6552	1066	536760	28	14112
10A					670	337680	18	9072
12A			12	4704	1324	660576	8	4032
12B			12	5376	1434	721056	38	19152
14AB					231	116424	4	2016
14CD			12	5544	1119	563724	8	4032
15A			10	3696	980	492576	54	25872
17AB			18	9072	903	455112	10	5040
21A			32	14784	708	354144	16	8064
21B			14	5712	738	369264	4	2016
21CD			16	7392	700	352464	15	7560
28AB			20	10080	568	286272	8	4032

In the above table, G-conjugacy class C, to which $tx\in$C specifically belongs in the situation that C $\in\{$1A,2B$\}$. This will classify the distance between t and the vertices of the A4-graph $\mathcal{A}_4$ (He,3B).

4. Main Theorem

We can establish the following theorem from the preceding results:

Theorem 4.1. Let G isomorphic to Held group He, then the A4-graph of G satisfies the following properties:

i. $\mathcal{A}_4$ (He,3A) is connected with diameter 3. In addition, if $t\in$3A we have $|\Delta_1(t)|$= 756 with 3-$C_G(t)$-orbits, $|\Delta_2(t)|$= 157932 with 50-$C_G(t)$-orbits and $|\Delta_3(t)|$= 374431 with 77-$C_G(t)$-orbits.

ii. $\mathcal{A}_4$ (He,3B) is connected with diameter 4. Moreover, if $t\in$3A we have $|\Delta_1(t)|$= 405 with 7-$C_G(t)$-orbits, $|\Delta_2(t)|$= 117282 with 273- $C_G(t)$-orbits, $|\Delta_3(t)|$= 7719456 with 15428- $C_G(t)$-orbits and $|\Delta_4(t)|$=159656 with 323- $C_G(t)$-orbits.

Proof. If we let $t\in G$ an element of order 3, this element can be randomly chosen by using Lemma 2.1**.** Then the first part of Lemma 2.2 allows us to split up the $C_G(t)$-orbits into certain $\Delta_i(t)$. Now, using Lemma 2.4, all we only need is to select a random element of $C_G(t)$-orbits in $\Delta_i(t)$, and then $\Delta_i(t)$ contains the whole orbits. Then by the employing this information computationally, we can obtain the Table 1 and Table 2**.** Finally, Theorem 4.1 immediately follows the previous table.

The number of subgroups from Held group He is formed by a random element of order 3 with additional other elements conjugate it in He. This may be observed in the following corollary.

Corollary 4.2. Let G be isomorphic to the Held group He. Then we have the following results:

i- Suppose that t be a fixed element in 3A. Then there are 756 elements $x\in$3A such that the subgroup generated by x and t is an isomorphic to A4.

ii- Suppose that t be a fixed element in 3B. Then there are 405 elements $x\in$3B such that the subgroup generated by x and t is an isomorphic to A4.

Proof. By using Lemma 2.1 part 3, we have two adjacent vertices in the A4-graph generated the alternating group A4. Then the results are followed from Theorem 5.1.

The girth and clique number of the A4-graph for the Held group He are calculated by the next result. The gap YAGS [10] may be used to compute the girth and clique number.

Theorem 4.3: Suppose that G isomorphic to the Held group. Then for the connected A4-graph, we have the following:

1- $\mathcal{A}_4$(He,3A) has a girth is 3 and the clique number is 16.

2- $\mathcal{A}_4$(He,3B) has a girth is 3 and the clique number is 64.

Proof: The YAGS gap packages can be used to accomplish computational tasks. Following that, the consequences are presented.

5. Conclusions

The construction of the A4-graph for each He-conjugacy class with representative order 3 for the Held group He is examined. The work includes an analysis the A4-graph discs structure along with calculating the diameter, namely the clique number and the girth of the graph. To accomplish this, computational methods to study the A4-graph are used.

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