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Graceful Antimagic Path and Cycle Related Graphs

Noor Khalil Shawkat*, Mohammed Ali Ahmed
[1]*Department of Mathematics, College of Education for Pure Sciences (Ibn AL Haitham), University of Baghdad, Baghdad, Iraq*

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Abstract
A graph G with p vertices and q edges is said to be antimagic if its edges can be labeled with $1,2, \dots, q$ such that the weights of vertices of G are pairwise distinct. The graceful labeling of a graph G with q edges is an assignment of integers from the set $\{0,1, \dots, q\}$ to the vertices of G, such that no two vertices receive the same label, where each edge is assigned the absolute value of the difference between the labels of its end vertices and the resulting edge labeling runs from 1 to q is inclusive. Moreover, if the induced edge labeling is simultaneously antimagic, that is, the sums of the labels of all edges incident to a given vertex are pairwise distinct for different vertices, we call the graceful labeling graceful antimagic. In this study, we will exhibit the existence of graceful antimagic labeling for two families of graphs the first derived from the path graph P_n, and the second from the cycle graph C_n. Both families were derived using the idea of a rooted product between the two graphs.

Keywords: Antimagic graph, graceful graph, graceful antimagic graph.
Mathematics Subject Classification: 05C78.

الرسومات الرشيقة المتباينة المرتبطة بالمسار والدائرة

نور خليل شوكت*, محمد علي أحمد
قسم الرياضيات, كلية التربية ابن الهيثم, جامعة بغداد, بغداد, العراق

الخلاصة
الرسم G والذي يحتوي على p من الرووس و q من الحواف, يدعى رسم متباين ان امكن وضع العلامات على الحواف باستخدام الاعداد الصحيحة $1,2, \dots, q$, بحيث ان اوزان جميع الرؤوس تكون مختلفة فيما بينها. التوسيم الرشيق للرسم G والذي يحتوي على q من الحواف, هوعملية وضع العلامات على الرؤوس باستخدام بعض الاعداد الصحيحة من المجموعة $\{0,1,2, \dots, q\}$,بحيث انه لايوجد رأسين يشتركان بنفس الرقم, من جانب اخر فان علامات الحواف تكون ناتجة من القيمة المطلقة لحاصل طرح علامة الرأسين لتلك الحافة, ويكون ناتج علامات جميع تلك الحواف محصورا بين الرقم (1) وعدد الحواف (q) بدون تكرار. بالاضافة الى ذلك ، اذا كانت اوزان جميع الرؤوس مختلفة بنفس الوقت, فان التوسيم في هذه الحالة يدعى توسيم رشيق متباين. في هذا الدراسة سوف نبحث ايجاد التوسيم الرشيق المتباين, لعائلتين من الرسوم, الأولى مشتقة من الرسم البياني

*Email: nour.khaleel1203a@ihcoedu.uobaghdad.edu.iq

للمسار P_n ، والثانية من الرسم البياني للدائرة C_n. هذه العائلتين من الرسوم كلاهما مشتقة باستخدام فكرة ضرب الجذور للرسم.

Introduction

Let $G=(V,E)$ be a finite, simple, and undirected graph without loops or multiple edges, where $V(G)$ and $E(G)$ represent vertex and edge sets of G, respectively. A graceful labeling of a graph G with q edges is an assignment of integers from the set $\{0,1,\dots,q\}$ to the vertices of G, such that no two vertices receive the same label, where each edge is assigned the absolute value of the difference between the labels of its end vertices, and the resulting edge labeling running from 1 to q is inclusive. The subject of graceful labeling was first introduced by Rosa in 1966 [1] , where it was named as a β-valuation, and this labeling was later renamed graceful labeling by Golomb [2].

The Ringel-Kotzig conjecture that all trees are graceful is still open. Hartsfield and Rangel [3] , introduced the concept of antimagic labeling in 1990. A graph G with p vertices and q edges is said to be antimagic if its edges can be labeled with $1,2,\dots,q$ such that the weights of the vertices of G are pairwise distinct. Hartsfield and Ringel proved that paths P_n, complete graphs $K_n, n\geq 3$, wheels and cycles graphs are antimagic. Moreover, they conjectured that every tree except P_2 is antimagic, and every connected graph except P_2 is antimagic. Both conjectures are still open. Vasuki B, Shobana L and Ahmed [4] proved that (a, d) antimagic for some special graphs. For more information on graceful and antimagic labeling please refer to [5] , [6] and [7].

In recent years we have observed the popularity of labeling that simultaneously satisfies several conditions, see [8]. For the graceful labeling, if the induced edge labeling simultaneously admits vertex antimagic labeling, then it is called graceful antimagic labeling. The concept of graceful antimagic labeling was first introduced by Ahmed et al. [9] , who proved that the path graph $P_n, n\geq 3$, star and double star graphs, all non-isomorphic trees up to eight vertices, cycles C_n for odd numbers, complete graphs and some related graphs are graceful antimagic. N Shawkat and M Ahmed [9] proved that existence of graceful antimagic labeling for split of the star graph $K_{1,n}$, $K_{2,n}$ graph, $P_2+\overline{K}_n$ graph, jellyfish graph $J_{n,n}$, Dragon graph $T_{3,n}$, kite graph $(T_{4,n})$ and the double comb graph DCO_n. A graceful antimagic labelling f is an injection from the vertex set of G into the set $\{0,1,\dots,|E(G)|\}$ such that the induced edge labeling f^*, defined as $f^*(uv)=|f(u)-f(v)|$ for every edge $uv\in E(G)$, has the following two properties: First $f^*(uv)\neq f^*(zw)$ for all pairs of distinct edges $uv, zw\in E(G)$, i.e, the labelling f is graceful, and second, for all pairs of distinct vertices $u,v\in V(G)$ is $wt_{f^*}(u)\neq wt_{f^*}(v)$, where $wt_{f^*}(u)=\sum_{uv\in E(G)} f^*(uv)$, i.e., $f^*$is an antimagic. [10].
In this study, we address the problem of finding graceful antimagic labeling for some families of graphs related to paths and cycles.

1. MAIN RESULTS

1.1 Path related graphs

In this subsection, we deal with some graphs derived from the path. The rooted product of graph G and rooted graph H, denoted by $G\circ H$, is defined as follows: take $|V(G)|$ copies of H, and for every vertex v_i of G, identify v_i with the root node of the i-th copy of H. The graceful antimagic of a graph $P_6\circ C_4$ are given in figure 1.

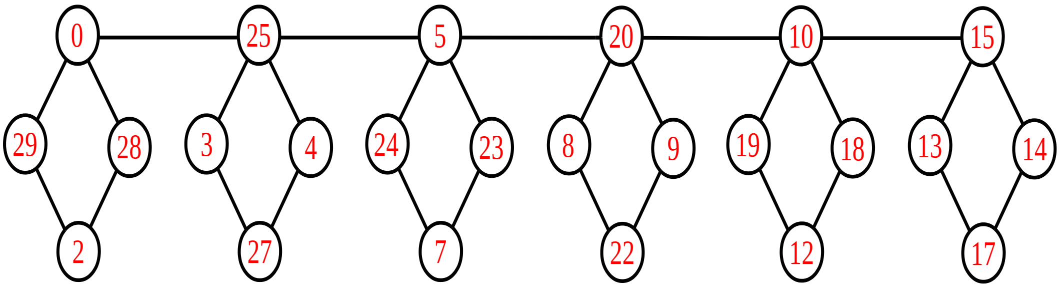

Figure 1: Graceful antimagic labeling of $P_6 \circ C_4$.

Theorem 1.1.1: The graph $P_n \circ C_4$ is graceful antimagic for $n \geq 4$, $n \not\equiv 0 \pmod 8$.
Proof: Let the vertex set and the edge set of the graph $P_n \circ C_4$ be
$V(P_n \circ C_4) = \{v_i : i = 1, 2, \cdots, 4n\}$,
$E(P_n \circ C_4) = \{v_i v_{i+1} : i = 1,2, \cdots, n-1\}$
$\cup \{v_i v_{n+i}, v_i v_{2n+i}, v_{n+i} v_{3n+i}, v_{2n+i} v_{3n+i} : i = 1,2, \cdots, n\}$.
For $n \geq 4$, $n \not\equiv 0 \pmod 8$, we define a vertex labeling $\beta_1 : V(P_n \circ C_4) \rightarrow \{0, 1, \cdots, 5n-1\}$ as follows

$$\beta_1(v_i) = \begin{cases} \frac{5i-5}{2} & \text{when } i \text{ is odd}, 1 \leq i \leq n \\ \frac{10n-5i}{2} & \text{when } i \text{ is even}, 2 \leq i \leq n \\ \frac{15n-5i+3}{2} & \begin{cases} when\ i = n+1, n+3, \cdots, 2n-1, & \text{for } n\ even \\ when\ i = n+1, n+3, \cdots, 2n, & \text{for } n\ odd \end{cases} \\ \frac{5i-5n-4}{2} & \begin{cases} when\ i = n+2, n+4, \cdots, 2n, & \text{for } n\ even \\ when\ i = n+2, n+4, \cdots, 2n-1, & \text{for } n\ odd \end{cases} \\ \frac{20n-5i+1}{2} & \begin{cases} when\ i = 2n+1, 2n+3, \cdots, 3n-1, & \text{for } n\ even \\ when\ i = 2n+1, 2n+3, \cdots, 3n, & \text{for } n\ odd \end{cases} \\ \frac{5i-10n-2}{2} & \begin{cases} when\ i = 2n+2, 2n+4, \cdots, 3n, & \text{for } n\ even \\ when\ i = 2n+2, 2n+4, \cdots, 3n-1, & \text{for } n\ odd \end{cases} \\ \frac{5i-15n-1}{2} & \begin{cases} when\ i = 3n+1, 3n+3, \cdots, 4n-1, & \text{for } n\ even \\ when\ i = 3n+1, 3n+3, \cdots, 4n, & \text{for } n\ odd \end{cases} \\ \frac{25n+4-5i}{2} & \begin{cases} when\ i = 3n+2, 3n+4, \cdots, 4n, & \text{for } n\ even \\ when\ i = 3n+2, 3n+4, \cdots, 4n-1, & \text{for } n\ odd\ . \end{cases} \end{cases}$$

Evidently all vertices have distinct labels. First, we prove that β_1 is a graceful labeling, thus we check the induced edge labels and show that they are distinct.
For $i = 1, 2, \cdots, n-1$
$\beta_1^*(v_i v_{i+1}) = |\beta_1(v_i) - \beta_1(v_{i+1})|$

$$= \begin{cases} \left|\frac{5i-5}{2} - \frac{10n-5(i+1)}{2}\right| & \text{when } i \text{ is odd} \\ \left|\frac{5(i+1)-5}{2} - \frac{10n-5i}{2}\right| & \text{when } i \text{ is even} \end{cases}$$

$= 5n - 5i.$
For $i = 1, 2, \cdots, n$
$\beta_1^*(v_i v_{n+i}) = |\beta_1(v_i) - \beta_1(v_{n+i})|$

$$= \begin{cases} \left|\frac{5i-5}{2} - \frac{15n-5(n+i)+3}{2}\right| & \text{when } i \text{ is odd} \\ \left|\frac{10n-5i}{2} - \frac{5(n+i)-5n-4}{2}\right| & \text{when } i \text{ is even} \end{cases}$$

$$= \begin{cases} 5n - 5i + 4 & \text{when } i \text{ is odd} \\ 5n - 5i + 2 & \text{when } i \text{ is even.} \end{cases}$$

$$\beta_1^*(v_i v_{2n+i}) = |\beta_1(v_i) - \beta_1(v_{2n+i})|$$
$$= \begin{cases} \left|\frac{5i-5}{2} - \frac{20n-5(2n+i)+1}{2}\right| & \text{when } i \text{ is odd} \\ \left|\frac{10n-5i}{2} - \frac{5(2n+i)-10n-2}{2}\right| & \text{when } i \text{ is even} \end{cases}$$
$$= \begin{cases} 5n-5i+3 & \text{when } i \text{ is odd} \\ 5n-5i+1 & \text{when } i \text{ is even.} \end{cases}$$
$$\beta_1^*(v_{n+i} v_{3n+i}) = |\beta_1(v_{n+i}) - \beta_1(v_{3n+i})|$$
$$= \begin{cases} \left|\frac{15n-5(n+i)+3}{2} - \frac{5(3n+i)-15n-1}{2}\right| & \text{when } i \text{ is odd} \\ \left|\frac{5(n+i)-5n-4}{2} - \frac{25n+4-5(3n+i)}{2}\right| & \text{when } i \text{ is even} \end{cases}$$
$$= \begin{cases} 5n-5i+2 & \text{when } i \text{ is odd} \\ 5n-5i+4 & \text{when } i \text{ is even.} \end{cases}$$
$$\beta_1^*(v_{2n+i} v_{3n+i}) = |\beta_1(v_{2n+i}) - \beta_1(v_{3n+i})|$$
$$= \begin{cases} \left|\frac{20n-5(2n+i)+1}{2} - \frac{5(3n+i)-15n-1}{2}\right| & \text{when } i \text{ is odd} \\ \left|\frac{5(2n+i)-10n-2}{2} - \frac{25n+4-5(3n+i)}{2}\right| & \text{when } i \text{ is even} \end{cases}$$
$$= \begin{cases} 5n-5i+1 & \text{when } i \text{ is odd} \\ 5n-5i+3 & \text{when } i \text{ is even.} \end{cases}$$

Combining the previous, we obtain that the edges receive the numbers $1, 2, \cdots, 5n-1$. Therefore, labeling β_1 is graceful.

Secondly, we will deal with the antimagic property of antimagic labeling.

$$wt_{\beta_1^*}(v_1) = |\beta_1(v_1) - \beta_1(v_2)| + |\beta_1(v_{n+1}) - \beta_1(v_1)| + |\beta_1(v_{2n+1}) - \beta_1(v_1)|$$
$$= \left|0 - \frac{10n-10}{2}\right| + \left|\frac{15n-5(n+1)+3}{2} - 0\right| + \left|\frac{20n-5(2n+1)+1}{2} - 0\right|$$
$$= 15n - 8.$$

For $i = 2, 3, \cdots, n-1$,

$$wt_{\beta_1^*}(v_i) = |\beta_1(v_i) - \beta_1(v_{i-1})| + |\beta_1(v_i) - \beta_1(v_{i+1})| + |\beta_1(v_{n+i}) - \beta_1(v_i)| + |\beta_1(v_{2n+i}) - \beta_1(v_i)|$$
$$= \begin{cases} (5n-5i+5) + (5n-5i) + (5n-5i+4) + (5n-5i+3) & \text{when } i \text{ is odd} \\ (5n-5i+5) + (5n-5i) + (5n-5i+2) + (5n-5i+1) & \text{when } i \text{ is even} \end{cases}$$
$$= \begin{cases} 20n-20i+12 & \text{when } i \text{ is odd} \\ 20n-20i+8 & \text{when } i \text{ is even.} \end{cases}$$
$$wt_{\beta_1^*}(v_n) = |\beta_1(v_n) - \beta_1(v_{n-1})| + |\beta_1(v_n) - \beta_1(v_{2n})| + |\beta_1(v_n) - \beta_1(v_{3n})|$$
$$= \begin{cases} \left|\frac{5n}{2} - \frac{5n-10}{2}\right| + \left|\frac{5n}{2} - \frac{5n-4}{2}\right| + \left|\frac{5n}{2} - \frac{5n-2}{2}\right| & \text{when } n \text{ is odd} \\ \left|\frac{5n-5}{2} - \frac{5n-5}{2}\right| + \left|\frac{5n-5}{2} - \frac{5n+3}{2}\right| + \left|\frac{5n-5}{2} - \frac{5n+1}{2}\right| & \text{when } n \text{ is even} \end{cases}$$
$$= \begin{cases} 8 & \text{when } n \text{ is even} \\ 12 & \text{when } n \text{ is odd.} \end{cases}$$

For $i = 1, 2, \cdots, n$,

$$wt_{\beta_1^*}(v_{n+i}) = |\beta_1(v_i) - \beta_1(v_{n+i})| + |\beta_1(.v_{n+i}) - \beta_1(v_{3n+i})|$$
$$= \begin{cases} (5n-5i+4) + (5n-5i+2) & \text{when } i \text{ is odd} \\ (5n-5i+2) + (5n-5i+4) & \text{when } i \text{ is even} \end{cases}$$
$$= 10n - 10i + 6.$$
$$wt_{\beta_1^*}(v_{2n+i}) = |\beta_1(v_i) - \beta_1(v_{2n+i})| + |\beta_1(v_{2n+i}) - \beta_1(v_{3n+i})|$$
$$= \begin{cases} (5n-5i+3) + (5n-5i+1) & \text{when } i \text{ is odd} \\ (5n-5i+1) + (5n-5i+3) & \text{when } i \text{ is even} \end{cases}$$
$$= 10n - 10i + 4.$$

$wt_{\beta_1^*}(v_{3n+i}) = |\beta_1(v_{n+i}) - \beta_1(v_{3n+i})| + |\beta_1(v_{2n+i}) - \beta_1(v_{3n+i})|$
$= \begin{cases} (5n-5i+2)+(5n-5i+1) & \text{when } i \text{ is odd} \\ (5n-5i+4)+(5n-5i+3) & \text{when } i \text{ is even} \end{cases}$
$= \begin{cases} 10n-10i+3 & \text{when } i \text{ is odd} \\ 10n-10i+7 & \text{when } i \text{ is even.} \end{cases}$
Thus,
$wt_{\beta_1^*}(v_{3n}) < wt_{\beta_1^*}(v_{2n}) < wt_{\beta_1^*}(v_{4n}) < wt_{\beta_1^*}(v_n) < wt_{\beta_1^*}(v_{4n-1}) < wt_{\beta_1^*}(v_{3n-1}) < wt_{\beta_1^*}(v_{2n-1}) < wt_{\beta_1^*}(v_{3n-2}) < wt_{\beta_1^*}(v_{2n-3}) < wt_{\beta_1^*}(v_{4n-3}) < wt_{\beta_1^*}(v_{n-1}) < \cdots < wt_{\beta_1^*}(v_1) < wt_{\beta_1^*}(v_2)$.

This implies that the vertex weights are distinct, and we can observe the following.
1. $wt_{\beta^*}(v_1) < wt_{\beta^*}(v_2)$ for every $n \geq 5$, while $wt_{\beta^*}(v_1) > wt_{\beta^*}(v_2)$ only when $n = 4$.
2. The antimagic labeling holds for every $n \geq 4$, $n \not\equiv 0 \pmod 8$, however when $n = 8$, we can easily show that $wt_{\beta^*}(v_1) = wt_{\beta^*}(v_3)$ and similarly when $n = 16$, then $wt_{\beta^*}(v_1) = wt_{\beta^*}(v_5)$ and so on.

Thus, the $P_n \circ C_4$ graph admits a graceful antimagic labeling, for every $n \geq 4$, $n \not\equiv 0 \pmod 8$.

Theorem 1.1.2: The graph $P_n \circ K_{1,4}$ is graceful antimagic for $n \geq 2$, $n \not\equiv 0 \pmod 6$.
Proof: Let the vertex and the edge set of the graph $P_n \circ K_{1,4}$ be
$V(P_n \circ K_{1,4}) = \{v_i : i = 1,2,\dots,5n\}$,
$E(P_n \circ K_{1,4}) = \{v_iv_{i+1} : i = 1,2,\dots,n-1\} \cup \{v_iv_{n+i}, v_{2n+i}, v_iv_{3n+i}, v_iv_{4n+i} : i = 1,2,\dots n\}$.
For $n \geq 2, n \not\equiv 0 \pmod 6$, we define the vertex labeling $\beta_2 : V(P_n \circ K_{1,4}) \to \{0,1,\dots,5n-1\}$ as follows:

$$\beta_2(v_i) = \begin{cases} \frac{5i-5}{2} & \text{when } i \text{ is odd}, 1 \leq i \leq n \\ \frac{10n-5i}{2} & \text{when } i \text{ is even}, 2 \leq i \leq n \\ \frac{15n-5i+3}{2} & \begin{cases} \text{when } i = n+1, n+3, \cdots, 2n-1, & \text{for } n \text{ even} \\ \text{when } i = n+1, n+3, \cdots, 2n, & \text{for } n \text{ odd} \end{cases} \\ \frac{5i-5n-8}{2} & \begin{cases} \text{when } i = n+2, n+4, \cdots, 2n, & \text{for } n \text{even} \\ \text{when } i = n+2, n+4, \cdots, 2n-1, & \text{for } n \text{ odd} \end{cases} \\ \frac{20n-5i+1}{2} & \begin{cases} \text{when } i = 2n+1, 2n+3, \cdots, 3n-1, & \text{for } n \text{ even} \\ \text{when } i = 2n+1, 2n+3, \cdots, 3n, & \text{for } n \text{ odd} \end{cases} \\ \frac{5i-10n-6}{2} & \begin{cases} \text{when } i = 2n+2, 2n+4, \cdots, 3n, & \text{for } n \text{ even} \\ \text{when } i = 2n+2, 2n+4, \cdots, 3n-1, & \text{for } n \text{ odd} \end{cases} \\ \frac{25n-5i-1}{2} & \begin{cases} \text{when } i = 3n+1, 3n+3, \cdots, 4n-1, & \text{for } n \text{ even} \\ \text{when } i = 3n+1, 3n+3, \cdots, 4n, & \text{for } n \text{ odd} \end{cases} \\ \frac{5i-15n-4}{2} & \begin{cases} \text{when } i = 3n+2, 3n+4, \cdots, 4n, & \text{for } n \text{even} \\ \text{when } i = 3n+2, 3n+4, \cdots, 4n-1, & \text{for } n \text{ odd} \end{cases} \\ \frac{30n-5i-3}{2} & \begin{cases} \text{when } i = 4n+1, 4n+3, \cdots, 5n-1, & \text{for } n \text{ even} \\ \text{when } i = 4n+1, 4n+3, \cdots, 5n, & \text{for } n \text{ odd} \end{cases} \\ \frac{5i-20n-2}{2} & \begin{cases} \text{when } i = 4n+2, 4n+4, \cdots, 5n, & \text{for } n \text{ even} \\ \text{when } i = 4n+2, 4n+4, \cdots, 5n-1, & \text{for } n \text{ odd.} \end{cases} \end{cases}$$

First, we will check the induced edge labeling.
For $i = 1, 2, \cdots, n-1$
$\beta_2^*(v_iv_{i+1}) = |\beta_2(v_i) - \beta_2(v_{i+1})|$
$= \begin{cases} \left|\frac{5i-5}{2} - \frac{10n-5(i+1)}{2}\right| & \text{when } i \text{ is odd} \\ \left|\frac{10n-5i}{2} - \frac{5(i+1)-510n-5i}{2}\right| & \text{when } i \text{ is even} \end{cases}$

$= 5n - 5i.$

For $i = 1, 2, \cdots, n$

$\beta_2^*(v_i v_{n+i}) = |\beta_2(v_i) - \beta_2(v_{n+i})|$

$$= \begin{cases} \left|\frac{5i-5}{2} - \frac{15n-5(n+i)+3}{2}\right| & \text{when } i \text{ is odd} \\ \left|\frac{10n-5i}{2} - \frac{5(n+i)-5n-8}{2}\right| & \text{when } i \text{ is even} \end{cases}$$

$= 5n - 5i + 4.$

$\beta_2^*(v_i v_{2n+i}) = |\beta_2(v_i) - \beta_2(v_{2n+i})|$

$$= \begin{cases} \left|\frac{5i-5}{2} - \frac{20n-5(2n+i)+1}{2}\right| & \text{when } i \text{ is odd} \\ \left|\frac{10n-5i}{2} - \frac{5(2n+i)-10n-6}{2}\right| & \text{when } i \text{ is even} \end{cases}$$

$= 5n - 5i + 3.$

$\beta_2^*(v_i v_{3n+i}) = |\beta_2(v_i) - \beta_2(v_{3n+i})|$

$$= \begin{cases} \left|\frac{5i-5}{2} - \frac{25n-5(3n+i)-1}{2}\right| & \text{when } i \text{ is odd} \\ \left|\frac{10n-5i}{2} - \frac{5(3n+i)-15n-4}{2}\right| & \text{when } i \text{ is even} \end{cases}$$

$= 5n - 5i + 2.$

$\beta_2^*(v_i v_{4n+i}) = |\beta_2(v_i) - \beta_2(v_{4n+i})|$

$$= \begin{cases} \left|\frac{5i-5}{2} - \frac{30n-5(4n+i)-3}{2}\right| & \text{when } i \text{ is odd} \\ \left|\frac{10n-5i}{2} - \frac{5(4n+i)-20n-2}{2}\right| & \text{when } i \text{ is even} \end{cases}$$

$= 5n - 5i + 1.$

This implies that the edges receive distinct labels from the set $1, 2, \dots 5n - 1$. Thus, the labeling β_2 is graceful.

Now we check the induced vertex weights

$$wt_{\beta_2^*}(v_1) = |\beta_2(v_1) - \beta_2(v_2)| + |\beta_2(v_1) - \beta_2(v_{n+1})| + |\beta_2(v_1) - \beta_2(v_{2n+1})| + |\beta_2(v_{3n+1})| + |\beta_2(v_1) - \beta_2(v_{4n+1})|$$

$= |0 - 5n - 5| + |0 - 5n - 1| + |0 - 5n - 2| + |0 - 5n - 3| + |0 - 5n - 4|$

$= 25n - 15.$

For $i = 2, 3, \cdots, n - 1$

$$wt_{\beta_2^*}(v_i) = |\beta_2(v_i) - \beta_2(v_{i-1})| + |\beta_2(v_i) - \beta_2(v_{i+1})| + |\beta_2(v_i) - \beta_2(v_{n+i})| + |\beta_2(v_i) - \beta_2(v_{2n+i})| + |\beta_2(v_i) - \beta_2(v_{3n+i})| + |\beta_2(v_i) - \beta_2(v_{4n+i})|$$

$$= \begin{cases} \left|\frac{5i-5}{2} - \frac{10n-5i+5}{2}\right| + \left|\frac{5i-5}{2} - \frac{10n-5i-5}{2}\right| + \left|\frac{5i-5}{2} - \frac{10n-5i+3}{2}\right| + \\ \left|\frac{5i-5}{2} - \frac{10n-5i+1}{2}\right| + \left|\frac{5i-5}{2} - \frac{10n-5i-1}{2}\right| + \left|\frac{5i-5}{2} - \frac{10n-5i-3}{2}\right| & \text{when } i \text{ is odd} \\ \left|\frac{10n-5i}{2} - \frac{5i-10}{2}\right| + \left|\frac{10n-5i}{2} - \frac{5i}{2}\right| + \left|\frac{10n-5i}{2} - \frac{5i-8}{2}\right| + \\ \left|\frac{10n-5i}{2} - \frac{5i-6}{2}\right| + \left|\frac{10n-5i}{2} - \frac{5i-4}{2}\right| + \left|5n - \frac{5i}{2} - \frac{5i-2}{2}\right| & \text{when } i \text{ is even} \end{cases}$$

$= 30n - 30i + 15.$

$$wt_{\beta_2^*}(v_n) = |\beta_2(v_n) - \beta_2(v_{n-1})| + |\beta_2(v_n) - \beta_2(v_{2n})| + |\beta_2(v_n) - \beta_2(v_{3n})| + |\beta_2(v_n) - \beta_2(v_{4n})| + |\beta_2(v_n) - \beta_2(v_{5n})|$$

$$= \begin{cases} \left|\frac{5n}{2} - \frac{5n-10}{2}\right| + \left|\frac{5n}{2} - \frac{5n-8}{2}\right| + \left|\frac{5n}{2} - \frac{5n-6}{2}\right| + \left|\frac{5n}{2} - \frac{5n-4}{2}\right| + \left|\frac{5n}{2} - \frac{5n-2}{2}\right| \\ \text{when } n \text{ is even} \\ \left|\frac{5n-5}{2} - \frac{5n+5}{2}\right| + \left|\frac{5n-5}{2} - \frac{5n+3}{2}\right| + \left|\frac{5n-5}{2} - \frac{5n+1}{2}\right| + \left|\frac{5n-5}{2} - \frac{5n-1}{2}\right| + \\ \left|\frac{5n-5}{2} - \frac{5n-3}{2}\right| \text{ when } n \text{ is odd} \end{cases}$$

$= 15$.

For $i = 1, 2, \cdots, n$

$wt_{\beta_2^*}(v_{n+i}) = |\beta_2(v_{n+i}) - \beta_2(v_i)|$

$$= \begin{cases} \left|\frac{10n-5i+3}{2} - \frac{5i-5}{2}\right| \text{ when } i \text{ is odd} \\ \left|\frac{5i-8}{2} - \frac{10n-5i}{2}\right| \text{ when } i \text{ is even} \end{cases}$$

$= 5n - 5i + 4.$

$wt_{\beta_2^*}(v_{2n+i}) = |\beta_2(v_{2n+i}) - \beta_2(v_i)|$

$$= \begin{cases} \left|\frac{10n-5i+1}{2} - \frac{5i-5}{2}\right| \text{ when } i \text{ is odd} \\ \left|\frac{5i-6}{2} - \frac{10n-5i}{2}\right| \text{ when } i \text{ is even} \end{cases}$$

$= 5n - 5i + 3.$

$wt_{\beta_2^*}(v_{3n+i}) = |\beta_2(v_{3n+i}) - \beta_2(v_i)|$

$$= \begin{cases} \left|\frac{10n-5i-1}{2} - \frac{5i-5}{2}\right| \text{ when } i \text{ is odd} \\ \left|\frac{5i-4}{2} - \frac{10n-5i}{2}\right| \text{ when } i \text{ is even} \end{cases}$$

$= 5n - 5i + 2.$

$wt_{\beta_2^*}(v_{4n+i}) = |\beta_2(v_{4n+i}) - \beta_2(v_i)|$

$$= \begin{cases} \left|\frac{10n-5i-3}{2} - \frac{5i-5}{2}\right| \text{ when } i \text{ is odd} \\ \left|\frac{5i-2}{2} - \frac{10n-5i}{2}\right| \text{ when } i \text{ is even} \end{cases}$$

$= 5n - 5i + 1.$

From the vertex weights we can observe the following:

1. The weights for the pendent vertices are arranged in decreasing order, where
$wt_{\beta_2^*}(v_{5n}) < wt_{\beta_2^*}(v_{4n}) < wt_{\beta_2^*}(v_{3n}) < wt_{\beta_2^*}(v_{2n}) < wt_{\beta_2^*}(v_{5n-1}) < wt_{\beta_2^*}(v_{4n-1}) <$
$wt_{\beta_2^*}(v_{3n-1}) < wt_{\beta_2^*}(v_{2n-1}) < \cdots < wt_{\beta_2^*}\left(v_{5n-(n-1)}\right) wt_{\beta_2^*}\left(v_{4n-(n-1)}\right)$
$< wt_{\beta_2^*}\left(v_{3n-(n-1)}\right) < wt_{\beta_2^*}\left(v_{2n-(n-1)}\right)$.

2. For the vertices of the path graph (center vertices), we have
$wt_{\beta_2^*}(v_n) < wt_{\beta_2^*}(v_{n-1}) < wt_{\beta_2^*}(v_{n-2}) < ... < wt_{\beta_2^*}(v_2)$, for every $n \geq 2$, $n \not\equiv 0$ (mod 6) and for the vertex v_1 it holds $wt_{\beta_2^*}(v_2) < wt_{\beta_2^*}(v_1)$ for $n \leq 5$, while for n $\geq 7, n \not\equiv 0$ (mod 6), $wt_{\beta_2^*}\left(v_{\lfloor n/6 \rfloor+1}\right) < wt_{\beta_2^*}(v_1) < wt_{\beta_2^*}\left(v_{\lfloor n/6 \rfloor+2}\right)$.
However, when $n = 6$ we can easily show that $wt_{\beta_2^*}(v_1) = wt_{\beta_2^*}(v_2)$ and similarly when $n = 12$, $wt_{\beta_5^*}(v_1) = wt_{\beta_5^*}(v_3)$ and so on.

3. When $n \geq 4$, $wt_{\beta_2^*}(v_{2n-3}) < wt_{\beta_2^*}(v_n) < wt_{\beta_2^*}(v_{5n-4})$.

This means that the vertex weights are pairwise distinct.
Thus, the graph $P_n \circ K_{1,4}$ is a graceful antimagic, for every $n \geq 2, n \not\equiv 0$ (mod 6).

Theorem 1.1.3: The graph $K_2 \circ nP_3$ is graceful antimagic for $n \geq 2,\ n \not\equiv 1$ (mod 3).
Proof: Let the vertex set and the edge set of the graph $K_2 \circ nP_3$ be
$V(K_2 \circ nP_3) = \{v_i : i = 1, 2, \ldots, 4n + 2\}$,

$E(K_2 \circ nP_3) = \{v_1v_2\} \cup \{v_1v_{n+i}, v_iv_{n+i}, v_2v_{3n+i}, v_{3n+i}v_{2n+i} : i = 3, 4, \dots n+2\}$.
For $n \geq 2, n \not\equiv 1 \pmod 3$, we define the vertex labeling $\beta_3: V(K_2 \circ nP_3) \to \{0, 1, 2, \dots, 4n + 1\}$ in the following way

$$\beta_3(v_i) = \begin{cases} 6n - 3in - i + 2 & \text{for } i = 1, 2, \\ 2n - i + 3 & \text{for } i = 3, 4, \dots, n+2, \\ i - n - 2 & \text{for } i = n+3, n+4, \dots, v_{2n+2}, \\ i - 2 & \text{for } i = 2n+3, 2n+4, \dots, 3n+2, \\ 7n - i + 4 & \text{for } i = 3n+3, 3n+4, \dots, 4n+2. \end{cases}$$

First, we will check the induced edge labeling.

$\beta_3^*(v_1v_2) = |\beta_3(v_1) - \beta_3(v_2)|$
$= |(6n - 3n - 1 + 2) - (6n - 3.2.n - 2 + \quad 2)| = 3n + 1.$

For $i = 3, 4, \dots n+2$

$\beta_3^*(v_1v_{n+i}) = |\beta_3(v_1) - \beta_3(v_{n+i})|$
$= \left|(6n - 3n - 1 + 2) - \left((n+i) - n - 2\right)\right|$
$= 3n - i + 3,$

$\beta_3^*(v_{n+i}v_i) = |\beta_3(v_{n+i}) - \beta_3(v_i)|$
$= \left|\left((n+i) - n - 2\right) - (2n - i + 3)\right|$
$= 2n - 2i + 5,$

$\beta_3^*(v_2v_{3n+i}) = |\beta_3(v_2) - \beta_3(v_{3n+i})|$
$= |(6n - 3(2)n - 2 + 2) - (7n - 3n - i + 4)| = 4n - i + 4,$

$\beta_3^*(v_{3n+i}v_{2n+i}) = |\beta_3(v_{3n+i}) - \beta_3(v_{2n+i})|$
$= |(7n - (3n+i) + 4) - (2n + i - 2)|$
$= 2n - 2i + 6.$

This implies that the edges are labeled with numbers $1, 2, \dots, 4n + 1$. Therefore, β_3 is a graceful labeling.

Now we prove that the induced edge labeling is antimagic.

$wt_{\beta_3^*}(v_1) = \left|\beta_3^*(v_1) - \beta_3^*(v_2)\right| + \sum_{i=3}^{n+2} |\beta_3(v_1) - \beta_3(v_{n+i})|$
$= \left|(3n+1)\right| + \sum_{i=3}^{n+2} \left|(3n+1) - (i-2)\right|$
$= 3n + 1 + \sum_{i=3}^{n+2} (3n - i + 3)$
$= \dfrac{5n^2 + 7n + 2}{2},$

$wt_{\beta_3^*}(v_2) = \left|\beta_3^*(v_2) - \beta_3^*(v_1)\right| + \sum_{i=3}^{n+2} |\beta_3(v_2) - \beta_3(v_{3n+i})|$
$= \left|0 - (3n+1)\right| \sum_{i=3}^{n+2} \left|0 - (4n - i + 4)\right|$
$= 3n + 1 + \sum_{i=3}^{n+2} (4n - i + 4)$
$= \frac{7n^2 + 9n + 2}{2}$

For $i = 3, 4, \dots n+2$

$wt_{\beta_3^*}(v_i) = |\beta_3(v_i) - \beta_3(v_{n+i})|$
$= |(2n - i + 3) - (i - 2)|$
$= 2n - 2i + 5,$

$wt_{\beta_3^*}(v_{n+i}) = |\beta_3(v_{n+i}) - \beta_3(v_i)| + |\beta_3(v_{n+i}) - \beta_3(v_1)|$
$= |(i-2) - (2n - i + 3)| + |(i-2) - (3n+1)|$
$= 5n - 3i + 8,$

$wt_{\beta_3^*}(v_{2n+i}) = |\beta_3(v_{2n+i}) - \beta_3(v_{3n+i})|$
$= |(2n + i - 2) - (4n - i + 4)|$
$= 2n - 2i + 6,$

$wt_{\beta_3^*}(v_{3n+i}) = |\beta_3(v_{3n+i}) - \beta_3(v_{2n+i})| + |\beta_3(v_{3n+i}) - \beta_3(v_2)|$

$= |(4n - i + 4) - (2n + i - 2)| + |(4n - i + 4) - (0)|$
$= 6n - 3i + 10.$
From the vertex weights we can observe the following.
1. $wt_{\beta_3^*}(v_i) < wt_{\beta_3^*}(v_{2n+i})$ for $i = 3, 4, \dots, n+2$, and for $i = 3, 4, \dots, n+2$ the weights of vertices v_i are odd numbers while the weights of vertices v_{2n+i} are even numbers.
2. $wt_{\beta_3^*}(v_{2n+i}) < wt_{\beta_3^*}(v_{n+i})$ for $i = 3, 4, \dots, n+2$,
3. $wt_{\beta_3^*}(v_{n+i}) < wt_{\beta_3^*}(v_{3n+i})$ for $i = 3, 4, \dots, n+2$,
4. $wt_{\beta_3^*}(v_{3n+3}) < wt_{\beta_3^*}(v_1) < wt_{\beta_3^*}(v_2)$.

This implies that the vertex weights are pairwise distinct when $n \not\equiv 1 \pmod 3$.
When $n = 4$ we can easily show that $wt_{\beta_3^*}(v_{n+3}) = wt_{\beta_3^*}(v_{4n+1})$, $wt_{\beta_3^*}(v_{2n}) = wt_{\beta_3^*}(v_{4n+2})$ and when $n = 7$, $wt_{\beta_3^*}(v_{n+3}) = wt_{\beta_3^*}(v_{3n+6})$, $wt_{\beta_3^*}(v_{n+4}) = wt_{\beta_3^*}(v_{4n})$ and so on. Thus, the graph $K_2 \circ nP_3$ is graceful antimagic, for $n \geq 2, n \not\equiv 1 \pmod 3$.

Theorem 1.1.4: The graph $P_2 \circ K_{n,2}$ is graceful antimagic for every odd $n, n \geq 3$.
Proof: Let the vertex set and the edge set of the graph $P_2 \circ K_{n,2}$ be
$V(P_2 \circ K_{n,2}) = \{v_i : i = 1,2, \dots, 2n+4\}$,
$E(P_2 \circ K_{n,2}) = \{v_i v_{2n+1}, v_i v_{2n+2}, v_{n+i} v_{2n+3}, v_{n+i} v_{2n+4} : i = 1,2, \dots, n\} \cup \{v_{2n+2} v_{2n+4}\}$.
For odd n, $n \geq 3$, we define the vertex labeling $\beta_4 : V(P_2 \circ K_{n,2}) \rightarrow \{0,1, \dots, 4n+1\}$ such that

$$\beta_4(v_i) = \begin{cases} 2i - 2 & \text{for } i = 1, 2, \dots n, \\ 2i - 1 & \text{for } i = n+1, n+2, \dots, 2n, \\ 2n + i - 1 & \text{for } i = 2n+1, 2n+2, \\ i - 4 & \text{for } i = 2n+3, 2n+4. \end{cases}$$

We start with checking the induced edge labeling.
For $i = 1, 2, \cdots, n$
$\beta_4^*(v_i v_{2n+1}) = |\beta_4(v_i) - \beta_4(v_{2n+1})|$
$= |(2i - 2) - (2n + 2n + 1 - 1)|$
$= 4n - 2i + 2,$
$\beta_4^*(v_i v_{2n+2}) = |\beta_4(v_i) - \beta_4(v_{2n+2})|$
$= |2i - 2 - (2n + 2n + 2 - 1)|$
$= 4n - 2i + 3,$
$\beta_4^*(v_{n+i} v_{2n+3}) = |\beta_4(v_{n+i}) - \beta_4(v_{2n+3})|$
$= |2(n+i) - 1 - (2n + 3 - 4)| = 2i,$
$\beta_4^*(v_{n+i} v_{2n+4}) = |\beta_4(v_{n+i}) - \beta_4(v_{2n+4})|$
$= |2(n+i) - 1 - (2n + 4 - 4)| = 2i - 1,$
$\beta_4^*(v_{2n+4} v_{2n+2}) = |\beta_4(v_{2n+4}) - \beta_4(v_{2n+2})|$
$= |2n - (4n+1)| = 2n + 1.$

Thus, the edges receive the numbers $1,2, \dots, 4n+1$. Therefore, β_4 is a graceful labeling.
Second, we check the antimagicness.
For $i = 1, 2, \cdots, n$
$wt_{\beta_4^*}(v_i) = |\beta_4(v_i) - \beta_4(v_{2n+2})| + |\beta_4(v_i) - \beta_4(v_{2n+1})|$
$= (4n - 2i + 3) + (4n - 2i + 2)$
$= 8n - 4i + 5,$
$wt_{\beta_4^*}(v_{n+i}) = |\beta_4(v_{n+i}) - \beta_4(v_{2n+4})| + |\beta_4(v_{n+i}) - \beta_4(v_{2n+3})|$
$= (2i - 1) + (2i) = 4i - 1,$
$wt_{\beta_4^*}(v_{2n+1}) = \sum_{i=1}^{n} |\beta_4(v_{2n+1}) - \beta_4(v_i)|$
$wt_{\beta_4^*}(v_{2n+1}) = \sum_{i=1}^{n} |\beta_4(v_{2n+1}) - \beta_4(v_i)|$

$= \sum_{i=1}^{n}(4n-2i+2) = 3n^2+n,$
$wt_{\beta_4^*}(v_{2n+2}) = \left|\beta_4^*(v_{2n+2}) - \beta_4^*(v_{2n+4})\right| + \sum_{i=1}^{n}|\beta_4(v_{2n+2}) - \beta_4(v_i)|$
$= 2n+1+\sum_{i=1}^{n}(4n-2i+3)$
$= (2n+1)+(3n^2+2n)$
$= 3n^2+4n+1,$
$wt_{\beta_4^*}(v_{2n+3}) = \sum_{i=1}^{n}|\beta_4(v_{2n+3}) - \beta_4(v_{n+i})|$
$= \sum_{i=1}^{n}(2i) = n^2+n.$
Finally,
$wt_{\beta_4^*}(v_{2n+4}) = \left|\beta_4^*(v_{2n+4}) - \beta_4^*(v_{2n+2})\right| + \sum_{i=1}^{n}|\beta_4(v_{2n+4}) - \beta_4(v_{n+i})|$
$= 2n+1+\sum_{i=1}^{n}(2i-1)$
$= n^2+2n+1.$

This implies that the weights of the vertices are arranged in decreasing order, where $wt_{\beta_4^*}(v_{n+1}) < wt_{\beta_4^*}(v_{n+2}) < \cdots < wt_{\beta_4^*}(v_{2n}) < wt_{\beta_4^*}(v_n) < wt_{\beta_4^*}(v_{n-1}) < \cdots < wt_{\beta_4^*}(v_1) < wt_{\beta_4^*}(v_{2n+3}) < wt_{\beta_4^*}(v_{2n+4}) < wt_{\beta_4^*}(v_{2n+1}) < wt_{\beta_4^*}(v_{2n+2})$.
Thus, the weights are all distinct and we can observe the following particular note $wt_{\beta_4^*}(v_1) < wt_{\beta_4^*}(v_{2n+3})$ for every odd $n \geq 9$, while $wt_{\beta_4^*}(v_{2n+3}) < wt_{\beta_4^*}(v_1)$ when $n \leq 9$.
Hence, the graph $P_2 \circ K_{2,n}$ is a graceful antimagic, for every odd $n, n \geq 3$.

1.2 Cycle related graphs

In this subsection, we study some graphs derived from the cycle graph. We start with graph C_n^3 which is obtained from a cycle on n vertices with one chord of length three.

Theorem 1.2.1: The graph C_n^3 is graceful antimagic for $n \equiv 1 (\mathrm{mod}\ 4)$ and for $n \equiv 2 (\mathrm{mod}\ 4)$, where $n \geq 9$..
Proof: Let the vertex set and the edge set of the graph C_n^3 be
$V(C_n^3) = \{v_i : i = 1,2,\ldots,n\}$,
$E(C_n^3) = \{v_i v_{i+1} : i = 1,2,\ldots n-1\} \cup \{v_n v_1\} \cup \{v_n v_3\}$.
For $n = 9, 10$ and 13 the desired labeling is given in figure 2.
For $n \equiv 1, 2\ (\mathrm{mod}\ 4)$, $n \geq 14$, we define the vertex labeling $\beta_5 : V(C_n^3) \to \{0, 1, \ldots, n+1\}$ as follows

$$\beta_5(v_i) = \begin{cases} \frac{i-1}{2} & \text{when } i \text{ is odd}, 1 \leq i \leq n \\ n+2-\frac{i}{2} & \text{when } i \text{ is even}, 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + 1 \\ n-\frac{i}{2} & \text{when } i \text{ is even}, \left\lceil \frac{n}{2} \right\rceil + 3 \leq i \leq n. \end{cases}$$

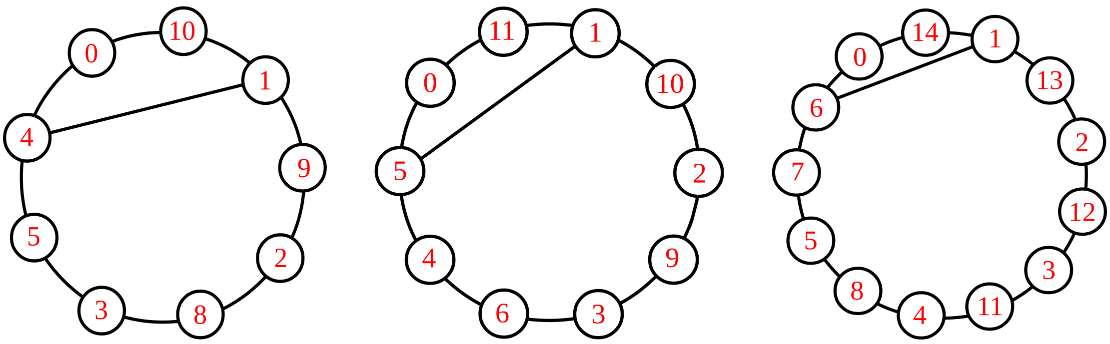

Figure 2: Graceful antimagic labeling of C_9^3, C_{10}^3 and C_{13}^3.

First, we prove that this labeling is graceful.
For $i = 1, 2, \cdots, n-1$
$\beta_5^*(v_i v_{i+1}) = |\beta_5(v_i) - \beta_5(v_{i+1})|$

$$= \begin{cases} \left|\frac{i-1}{2} - \left(n+2-\frac{i+1}{2}\right)\right| & when\ i \text{ is odd}, 1 \le i \le \left\lceil\frac{n}{2}\right\rceil \\ \left|\left(n+2-\frac{i}{2}\right) - \frac{(i+1-1)}{2}\right| & \text{when } i \text{ is even}, 2 \le i \le \left\lceil\frac{n}{2}\right\rceil + 1 \\ \left|\frac{i-1}{2} - [n - \frac{(i+1)}{2}]\right| & \text{when } i \text{ is odd}, \left\lceil\frac{n}{2}\right\rceil + 2 \le i \le n-1 \\ \left|(n - \frac{i}{2}) - \frac{i+1-1}{2}\right| & \text{when } i \text{ is even}, \left\lceil\frac{n}{2}\right\rceil + 3 \le i \le n-1 \end{cases}$$

$$= \begin{cases} n - i + 2 & \text{when } 1 \le i \le \left\lceil\frac{n}{2}\right\rceil + 1 \\ n - i & \text{when } \left\lceil\frac{n}{2}\right\rceil + 2 \le i \le n-1 \end{cases}$$

$\beta_5^*(v_n v_1) = |\beta_5(v_n) - \beta_5(v_1)|$

$$= \begin{cases} \frac{n}{2} & \text{when } n \text{ is even} \\ \frac{n-1}{2} & \text{when } n \text{ is odd} \end{cases}$$

$\beta_5^*(v_n v_3) = |\beta_5(v_n) - \beta_5(v_3)|$

$$= \begin{cases} \frac{n-2}{2} & \text{when } n \text{ is even} \\ \frac{n-3}{2} & \text{when } n \text{ is odd.} \end{cases}$$

This implies that the edge labels are $1, 2, \dots, n+1$. Hence, the labeling β_5 is graceful.
Now we check the weights of vertices under the induced edge labeling β_5^*.
$wt_{\beta_5^*}(v_1) = |\beta_5(v_1) - \beta_5(v_n)| + |\beta_5(v_1) - \beta_5(v_2)|$

$$= \begin{cases} \left|0 - \frac{n}{2}\right| + |0 - (n+1)| & \text{when } n \text{ is even} \\ \left|0 - \frac{n-1}{2}\right| + |0 - (n+1)| & \text{when } n \text{ is odd,} \end{cases}$$

$$= \begin{cases} \frac{3n+2}{2} & \text{when } n \text{ is even} \\ \frac{3n+1}{2} & \text{when } n \text{ is odd} \end{cases}$$

$wt_{\beta_5^*}(v_2) = |\beta_5(v_2) - \beta_5(v_1)| + |\beta_5(v_2) - \beta_5(v_3)|$
$= |(n+1) - 0| + |(n+1) - (1)|$
$= 2n + 1,$
$wt_{\beta_5^*}(v_3) = |\beta_5(v_3) - \beta_5(v_2)| + |\beta_5(v_3) - \beta_5(v_n)| + |\beta_5(v_3) - \beta_5(v_4)|$

$$= \begin{cases} |1 - (n+1)| + \left|1 - \frac{n}{2}\right| + |1 - n| & \text{when } n \text{ is even} \\ |1 - (n+1)| + \left|1 - \frac{(n-1)}{2}\right| + |1 - n| & \text{when } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \frac{5n-4}{2} & \text{when } n \text{ is even} \\ \frac{5n-5}{2} & \text{when } n \text{ is odd.} \end{cases}$$

For $i = 4, 5, \dots n-1$
$wt_{\beta_5^*}(v_i) = |\beta_5(v_i) - \beta_5(v_{i-1})| + |\beta_5(v_i) - \beta_5(v_{i+1})|$

$$=\begin{cases} \left|(n+2-\frac{i}{2}-\frac{i-2}{2}\right|+\left|\left(n+2-\frac{i}{2}\right)-\frac{i}{2}\right| & \text{when } i \text{ is even}, 4\le i\le \left\lceil\frac{n}{2}\right\rceil+1\\ \left|\frac{i-1}{2}-\frac{2n-i+5}{2}\right|+\left|\frac{i-1}{2}-\frac{2n-i+3}{2}\right| & \text{when } i \text{ is odd}, 5\le i\le \left\lceil\frac{n}{2}\right\rceil\\ \left|\frac{n+2}{4}-\frac{3n+6}{4}\right|+\left|\frac{n+2}{4}-\frac{3n-6}{4}\right| & \text{when } i=\frac{n}{2}+2 \text{ and } n \text{ is even}\\ \left|\frac{n+3}{4}-\frac{3n+5}{4}\right|+\left|\frac{n+3}{4}-\frac{3n-7}{4}\right| & \text{when } i=\frac{n+1}{2}+2 \text{ and } n \text{ is odd}\\ \left|\left(n-\frac{i}{2}\right)-\frac{i-2}{2}\right|+\left|\left(n-\frac{i}{2}\right)-\frac{i}{2}\right| & \text{when } i \text{ is even}, \left\lceil\frac{n}{2}\right\rceil+3\le i\le n-1\\ \left|\frac{i-1}{2}-\frac{2n-i+1}{2}\right|+\left|\frac{i-1}{2}-\frac{2n-i-1}{2}\right| & \text{when } i \text{ is odd } \left\lceil\frac{n}{2}\right\rceil+4\le i\le n-1 \end{cases}$$

$$=\begin{cases} 2n-2i+5 & \text{when } 4\le i\le \left\lceil\frac{n}{2}\right\rceil+1\\ n-1 & \text{when } i=\frac{n}{2}+2 \text{ and } n \text{ is even}\\ n-2 & \text{when } i=\frac{n+1}{2}+2 \text{ and } n \text{ is odd}\\ 2n-2i+1 & \text{when } \left\lceil\frac{n}{2}\right\rceil+3\le i\le n-1. \end{cases}$$

Finally,

$$wt_{\beta_5^*}(v_n)=|\beta_5(v_n)-\beta_5(v_{n-1})|+|\beta_5(v_n)-\beta_5(v_1)|+|\beta_5(v_n)-\beta_9(v_3)|$$

$$=\begin{cases} \left|\frac{n}{2}-\frac{n-2}{2}\right|+\left|\frac{n}{2}-0\right|+\left|\frac{n}{2}-1\right| & \text{when } n \text{ is even}\\ \left|\frac{n-1}{2}-\frac{n+1}{2}\right|+\left|\frac{n-1}{2}-0\right|+\left|\frac{n-1}{2}-1\right| & \text{when } n \text{ is odd.} \end{cases}$$

$$=\begin{cases} n & \text{when } n \text{ is even}\\ n-1 & \text{when } n \text{ is odd.} \end{cases}$$

Hence, from the weights of vertices, we can observe the following, for every odd $n\ge 17, n\equiv 1(\text{mod } 4)$ and for every even $n\ge 14, n\equiv 2(\text{mod } 4)$.

1. For $4\le i\le n-2$, $wt_{\beta_5^*}(v_{i+1})<wt_{\beta_5^*}(v_i)$, moreover the weight of the vertex v_1 is even, while the weights of vertices v_i, for $4\le i\le n-2$, are all odd.
2. $wt_{\beta_5^*}(v_n)<wt_{\beta_5^*}(v_1)<wt_{\beta_5^*}(v_2)<wt_{\beta_5^*}(v_3)$.

Now, we need to check that the weights of the vertices v_i, for $4\le i\le n-2$ and the remaining four vertices, v_1, v_2, v_3 and v_n are pairwise distinct.

1. For even n we have

$$wt_{\beta_5^*}\left(v_{\frac{n+4}{2}}\right)<wt_{\beta_5^*}(v_n)<wt_{\beta_5^*}\left(v_{\frac{n+2}{2}}\right)<wt_{\beta_5^*}\left(v_{\frac{n-2}{2}}\right)<wt_{\beta_5^*}(v_1)<wt_{\beta_5^*}(v_2)<wt_{\beta_5^*}(v_3).$$

2. For odd n we have

$$wt_{\beta_5^*}\left(v_{\frac{n+5}{2}}\right)<wt_{\beta_5^*}(v_n)<wt_{\beta_5^*}\left(v_{\frac{n+3}{2}}\right)<wt_{\beta_5^*}\left(v_{\frac{n-3}{2}}\right)<wt_{\beta_5^*}(v_1)<wt_{\beta_5^*}(v_2)<wt_{\beta_5^*}(v_3).$$

Hence, the graph C_n^3 is a graceful antimagic labeling, for every odd $n, n\ge 9, n\equiv 1(\text{mod } 4)$ and for every even $n, n\ge 10, n\equiv 2(\text{mod } 4)$.

Let G and H be two graphs, then the graph $G+H$ is obtained by joining every vertex of the graph G with all the vertices of H.

Theorem 1.2.2: The graph $\overline{K}_n+C_3$ is graceful antimagic for every $n\ge 3$.

Proof: Let the vertex set and the edge set of the graph $\overline{K}_n+C_3$ be

$V(\overline{K}_n+C_3)=\{v_i: i=1,2,\dots,n+3\}$,

$E(\overline{K}_n+C_3)=\{v_iv_{n+1}, v_iv_{n+2}, v_iv_{n+3}: i=1,2,\dots,n\}\cup\{v_{n+1}v_{n+2}, v_{n+2}v_{n+3}, v_{n+3}v_{n+1}\}$.

For $n \geq 3$ we define the vertex labeling
$\beta_6: V(\overline{K}_n + C_3) \longrightarrow \{0,1,2, \dots, 3n+3\}$ as follows

$$\beta_6(v_i) = \begin{cases} i \quad \text{when } i = 1, 2, \dots, n \\ ni + i - n^2 - 2n - 1 \quad \text{when } i = n+1, n+3 \\ 3n+3 \quad \text{when } i = n+2. \end{cases}$$

First, we investigate the properties of the induced edge labeling $\beta_6^*(v_i)$.
For $i = 1, 2, \cdots, n$
$\beta_6^*(v_i v_{n+1}) = |\beta_6(v_i) - \beta_6(v_{n+1})|$
$= \left| i - (n(n+1) + (n+1) - n^2 - 2n - 1) \right|$
$= i$
$\beta_6^*(v_i v_{n+2}) = |\beta_6(v_i) - \beta_6(v_{n+2})|$
$= \left| i - (3n+3) \right|$
$= 3n + 3 - i,$
$\beta_6^*(v_i v_{n+3}) = |\beta_6(v_i) - \beta_6(v_{n+3})|$
$= \left| i - (n(n+3) + (n+3) - n^2 - 2n - 1) \right|$
$= 2n + 2 - i$
$\beta_6^*(v_{n+1} v_{n+2}) = |\beta_6(v_{n+1}) - \beta_6(v_{n+2})|$
$= \left| (n(n+1) + (n+1) - n^2 - 2n - 1) - (3n+3) \right|$
$= 3n + 3$
$\beta_6^*(v_{n+2} v_{n+3}) = |\beta_6(v_{n+2}) - \beta_6(v_{n+3})|$
$= \left| (3n+3) - (n(n+3) + (n+3) - n^2 - 2n - 1) \right|$
$= n + 1$
$\beta_6^*(v_{n+3} v_{n+1}) = |\beta_6(v_{n+3}) - \beta_6(v_{n+1})|$
$= \left| (n(n+3) + (n+3) - n^2 - 2n - 1) - (n(n+1) + (n+1) - n^2 - 2n - 1) \right|$
$= 2n + 2.$
As the edge labels are $1, 2, \dots, 3n+3$ we get that β_6 is a graceful labeling.
Now we check the corresponding vertex weights.
For $i = 1, 2, \cdots, n$
$wt_{\beta_6^*}(v_i) = |\beta_6(v_i) - \beta_6(v_{n+1})| + |\beta_6(v_i) - \beta_6(v_{n+2})| + |\beta_6(v_i) - \beta_6(v_{n+3})|$
$= \left| i - 0 \right| + \left| i - (3n+3) \right| + \left| i - (2n+2) \right|$
$= 5n + 5 - i$
$wt_{\beta_6^*}(v_{n+1}) = |\beta_6(v_{n+1}) - \beta_6(v_{n+2})| + |\beta_6(v_{n+1}) - \beta_6(v_{n+3})| + \sum_{i=1}^{n} |\beta_6(v_{n+1}) - \beta_6(v_i)|$
$= \left| 0 - (3n+3) \right| + \left| 0 - (2n+2) \right| + \sum_{i=1}^{n} \left| 0 - i \right|$
$= \frac{n^2 + 11n + 10}{2}$
$wt_{\beta_6^*}(v_{n+2}) = |\beta_6(v_{n+2}) - \beta_6(v_{n+1})| + |\beta_6(v_{n+2}) - \beta_6(v_{n+3})| + \sum_{i=1}^{n} |\beta_6(v_{n+2}) - \beta_6(v_i)|$
$= \left| (3n+3) - 0 \right| + \left| (3n+3) - (2n+2) \right| + \sum_{i=1}^{n} |(3n+3) - i|$
$= (3n+3) + (n+1) + \frac{5n(n+1)}{2}$
$= \frac{5n^2 + 13n + 8}{2}$
$wt_{\beta_6^*}(v_{n+3}) = |\beta_6(v_{n+3}) - \beta_6(v_{n+1})| + |\beta_6(v_{n+3}) - \beta_6(v_{n+2})| + \sum_{i=1}^{n} |\beta_6(v_{n+3}) - \beta_6(v_i)|$
$= \left| (2n+2) - 0 \right| + \left| (2n+2) - (3n+3) \right| + \sum_{i=1}^{n} \left| (2n+2) - i \right|$
$= (2n+2) + (n+1) + \frac{3n(n+1)}{2}$
$= \frac{3n^2 + 9n + 6}{2}.$

It is clear that the weights of the vertices v_i, for $1 \le i \le n$ form an arithmetic progression with the common difference 1, which is given by $4n+5, 4n+6, \cdots, 5n+4$ on the other hand $5n+4 < wt_{\beta_{21}^*}(v_{n+1}) < wt_{\beta_{21}^*}(v_{n+3}) < wt_{\beta_{21}^*}(v_{n+2})$. Thus, the graph $\overline{K}_n + C_3$ is graceful antimagic for every $n \ge 3$. □

In the next theorem we deal with graph KC_n^3 which is obtained from the graph C_n^3 by joining the vertices of graph $\overline{K}_n$ with one vertex of degree three of C_n^3.

Theorem 1.2.3: The graph KC_n^3 is graceful antimagic for every $n \ge 11, n \equiv 3 (\text{mod } 4)$.
Proof: Let the vertex set and the edge set of the graph KC_n^3 be
$V(KC_n^3) = \{v_i : i = 1,2,\ldots,2n\}$,
$E(KC_n^3) = \{v_i v_{i+1} : i = 1,2,\ldots,n-1\} \cup \{v_n v_1\} \cup \{v_n v_3\} \cup \{v_n v_{n+i} : i = 1,2,\ldots,n\}$.
For $n \ge 11, n \equiv 3 (\text{mod } 4)$, we define the vertex labeling $\beta_7 : V(KC_n^3) \to \{0,1,\ldots,2n+1\}$ as follows

$$\beta_7(v_i) = \begin{cases} \frac{i-1}{2} & \text{when } i = 1,3,\ldots,n \\ \frac{4n+4-i}{2} & \text{when } i = 2,4,\ldots,n-1 \\ \frac{5n+5-2i}{2} & \text{when } i = n+1, n+2, \ldots, \frac{3n+5}{2} \\ \frac{5n+1-2i}{2} & \text{when } i = \frac{3n+7}{2}, \frac{3n+9}{2}, \ldots, 2n. \end{cases}$$

First, we show that β_7 is a graceful labeling.
For $i = 1, 2, \cdots, n-1$

$$\beta_7^*(v_i v_{i+1}) = |\beta_7(v_i) - \beta_7(v_{i+1})|$$
$$= \begin{cases} \left|\frac{i-1}{2} - \left(2n+2-\frac{i+1}{2}\right)\right| & \text{when } i = 1,3,\ldots,n-2 \\ \left|\left(2n+2-\frac{i}{2}\right) - \frac{(i+1)-1}{2}\right| & \text{when } i = 2,4,\ldots,n-1 \end{cases}$$
$$= 2n+2-i$$

$$\beta_7^*(v_n v_1) = |\beta_7(v_n) - \beta_7(v_1)|$$
$$= \left|\frac{n-1}{2} - \frac{1-1}{i}\right|$$
$$= \frac{n-1}{2}$$

$$\beta_7^*(v_n v_3) = |\beta_7(v_n) - \beta_7(v_3)|$$
$$= \left|\frac{n-1}{2} - \frac{3-1}{2}\right|$$
$$= \frac{n-3}{2}.$$

For $i = 1, 2, \cdots, n$

$$\beta_7^*(v_n v_{n+i}) = |\beta_7(v_n) - \beta_7(v_{n+i})|$$
$$= \begin{cases} \left|\frac{n-1}{2} - \frac{5n-2(n+i)+5}{2}\right| & \text{when } i = 1,2,\ldots,\frac{n+5}{2} \\ \left|\frac{n-1}{2} - \frac{5n-2(n+i)+1}{2}\right| & \text{when } i = \frac{n+7}{2}, \frac{n+9}{2}, \ldots, n \end{cases}$$
$$= \begin{cases} n+3-i & \text{when } i = 1,2,\ldots,\frac{n+5}{2} \\ n+1-i & \text{when } i = \frac{n+7}{2}, \frac{n+9}{2}, \ldots, n. \end{cases}$$

As the edges receive the numbers $1,2,\ldots,2n+1$ the labeling β_7 is graceful.
Second, for the antimagic labeling we will get:
$wt_{\beta_7^*}(v_1) = |\beta_7(v_1) - \beta_7(v_n)| + |\beta_7(v_1) - \beta_7(v_2)|$

$= \left|0-\frac{n-1}{2}\right| + |0-(2n+1)|$
$= \frac{5n+1}{2}$
$wt_{\beta_7^*}(v_2) = |\beta_7(v_2) - \beta_7(v_1)| + |\beta_7(v_2) - \beta_7(v_3)|$
$= |2n+1-0| + |(2n+1)-1|$
$= 4n+1,$
$wt_{\beta_7^*}(v_3) = |\beta_7(v_3) - \beta_7(v_2)| + |\beta_7(v_3) - \beta_7(v_4)| + |\beta_7(v_3) - \beta_7(v_n)|$
$= |1-(2n+1)| + |1-2n| + \left|1-\frac{n-1}{2}\right|$
$= \frac{9n-5}{2}.$
For $i = 4,5,\cdots,n-1$
$wt_{\beta_7^*}(v_i) = \left|\beta_7^*(v_i) - \beta_7^*(v_{i+1})\right| + \left|\beta_7^*(v_i) - \beta_7^*(v_{i-1})\right|$

$$= \begin{cases} \left|\left(2n+2-\frac{i}{2}\right)-\frac{i}{2}\right| + \left|\left(2n+2-\frac{i}{2}\right)-\frac{i-2}{2}\right| & \text{when } i \text{ is even} \\ \left|\frac{i-1}{2}-\frac{4n-i+3}{2}\right| + \left|\frac{i-1}{2}-\frac{4n-i+5}{2}\right| & \text{when } i \text{ is odd} \end{cases}$$
$= 4n+5-2i.$
For $i = 1,2,\cdots,n$
$wt_{\beta_7^*}(v_{n+i}) = |\beta_7^*(v_{n+i}) - \beta_7^*(v_n)|$
$$= \begin{cases} \left|\frac{3n-2i+5}{2}-\frac{n-1}{2}\right| & \text{when } i = 1,2,\dots,\frac{n+5}{2} \\ \left|\frac{3n-2i+1}{2}-\frac{n-1}{2}\right| & \text{when } i = \frac{n+7}{2},\frac{n+9}{2},\dots,n. \end{cases}$$
$$= \begin{cases} n+3-i & \text{when } i = 1,2,\dots,\frac{n+5}{2} \\ n+1-i & \text{when } i = \frac{n+7}{2},\frac{n+9}{2},\dots,n. \end{cases}$$
Finally,
$$wt_{\beta_7^*}(v_n) = \left|\beta_7^*(v_n) - \beta_7^*(v_1)\right| + \left|\beta_7^*(v_n) - \beta_7^*(v_{n-1})\right| + \sum_{i=1}^{\frac{n+5}{2}} \left|\beta_7^*(v_n) - \beta_7^*(v_{n+i})\right| + \sum_{i=\frac{n+7}{2}}^{n} \left|\beta_7^*(v_n) - \beta_7^*(v_{n+i})\right| + \left|\beta_7^*(v_n) - \beta_7^*(v_3)\right|$$
$$= \left|\frac{n-1}{2}-0\right| + \left|\frac{n-1}{2}-\frac{3n+5}{2}\right| + \sum_{i=1}^{\frac{n+5}{2}} \left|\frac{n-1}{2}-\frac{3n-2i+5}{2}\right| + \sum_{i=\frac{n+7}{2}}^{n} \left|\frac{n-1}{2}-\frac{3n-2i+1}{2}\right| + \left|\frac{n-1}{2}-1\right|$$
$= \frac{n^2+7n+12}{2}.$
Hence, from weights of vertices, we can observe the following for every $n \geq 11, n \equiv 3 (\text{mod } 4)$.
For $+1 \leq i \leq 2n-1,\ wt_{\beta_7^*}(v_{i+1}) < wt_{\beta_7^*}(v_i)$.
$wt_{\beta_7^*}(v_{n+1}) < wt_{\beta_7^*}(v_{n-1})$.
1. For $4 \leq i \leq n-2,\ wt_{\beta_7^*}(v_{i+1}) < wt_{\beta_7^*}(v_i)$, moreover the weight of the vertex v_1 is even while the weights of vertices v_i, for $4 \leq i \leq n-2$, are all odd.
2. $wt_{\beta_7^*}(v_1) < wt_{\beta_7^*}(v_4) < wt_{\beta_7^*}(v_2) < wt_{\beta_7^*}(v_3) < wt_{\beta_7^*}(v_n)$.

Thus, the graph KC_n^3 admits a graceful antimagic labeling, for $n \geq 11, n \equiv 3 (\text{mod } 4)$.

2. Conclusions

In this paper, we addressed a new concept of graceful labeling that induces edge labeling as an antimagic, which was recently presented by [1]. We proved that some families of graphs admit this labeling, and our future work involves finding another family of graphs that admits graceful antimagic labeling.

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