DOI: 10.24996/ijs.2024.65.2.24

ISSN: 0067-2904

Generalization of Local Modules

Majid Mohammed Abed
Department of Mathematics, College of Education for Pure Sciences, University Of Anbar, Baghdad, Iraq

Received: 4/12/2022 Accepted: 11/3/2023 Published: 29/2/2024

Abstract

In this paper we introduce a generalization of local module. The notion of 2-Local module can be considered a new concept (generalization of local modules). Let T be a ring with identity. 2-maximal submodule can define as, any submodule B_1 of B is called 2-maximal if (B/B_1) is a 2-regular module. 2-Local module means every submodule A of a T-module B where $\frac{B}{A}$ is a unique 2-maximal submodule. Almost any unique maximal submodule of a T-module B gives 2-local module. Also, we can present a generalization of local module by, if B is a fully prime module over the ring T and every submodule B_1 of a T-module B is a unique almost max-submodule, this means B is a 2-local. Every semi-maximal submodule B_1 over chained module B is maximal (2-maximal submodule) and hence B is 2-local module. Some properties and examples of 2-Local module are given. Finally some implications of 3-local module have been presented.

Keywords: 2-Local module; 2-pure submodule; Regular module; Semi-simple module; Cyclic submodule.

تعميم المقاسات المحلية

ماجد محمد عبد
قسم الرياضيات ; كلية التربية للعلوم الصرفة ; جامعة الانبار ; بغداد ; العراق

الخلاصة

في هذا البحث قدمنا تعميم للمقاس المحلي. فكرة 2−local يمكن اعتبارها مفهومًا جديدًا (تعميم للمقاسات المحلية). −2local تعني كل مقاس جزئي Aمن B علي الحلقة T حيث A / B هي مقاس جزئي وحيد −2 (maximal. على الاغلب أي مقاس جزئي وحيد اعظمي من المقاسB يعطي مقاس 2−local. أيضًا، يمكننا تقديم تعميم للمقاس المحلي من خلال، إذا كانت B هي مقاس اولي كامل علي الحلقة T وكل مقاس جزئي B_1من المقاس B هي على الاغلب مقاس جزئي اعظمي وحيد، وهذا يعني أن B عبارة عن وحدة ثنائية محلية. كل مقاس جزئي شبه اعظمي B_1 فوق المقاس السلسلة B هي اعظمي (2−maximal) ومن ثم −2). B (localبعض خصائص وأمثلة الي (2−Local module) معطاة. أخيراً؛ تم تقديم بعض النتائج للمقاس المعمم −3)(local مع نتائج جديدة حول هذا التعميم.

*Email: majid_math@uoanbar.edu.iq

1. Introduction

Through this paper T represents commutative ring with 1 and B a unitary T-module. We need to present a clear definition of maximal submodule (shortly max-submodule). Any submodule B_1 of T-module B is called maximal if there exists a non-zero

submodule B_2 of B such that $B_1 \subseteq B_2 \subseteq B$, so B_2=B. Also, semi-max-submodule studied by [1], a submodule B_1 of T-module is called semi-max-submodule if (B/B_1) is semi-simple module over the ring T. A T-module M is called regular if given any element m ∈M there exists f ∈ Hom_T (M, T) with (mf)m = m. Another generalization of max-submodule namely weak max-submodule is present by Shwkaea 2012, any submodule B_1 of B is called weak maximal if (B/B_1) is regular T-module [2].

T-module B is called local if it has a one max-submodule. Any module B is called simple if it has only two submodules are {0} and itself B [3]. Any submodule B_1 is called simple T-module if (B/B_1) is simple T-module ([2], [16]). Note that Any submodule B_1 of B is called pure submodule of B if $JB \cap B_1 = JB_1$ and every T-module B is called regular if B is pure and B_1 is a submodule of B is called 2-pure if $J^2B \cap B_1 = J^2B_1$ where J is an ideal of T [4]. A class of endoprimitive rings is defined which contains all strongly prime rings and all weakly primitive rings. Endoprimitive rings and more information in [5].

Any T-module B is called 2-regular if there exists submodule B_1 of B is 2-pure. Also, multiplication module and cyclic modules in [7]. About regular property [8], any T-module B is called 2-Local module if B has only one 2-max- submodule, where 2-maximal submodule can define by: any submodule B_1 of B is called 2-maximal if (B/B_1) is a 2-regular module. Closed-CS-module in [9]. In [10], any module B over integral domain is said to be divisible if tB=B, t ∈ T. So $B_1 \leq B$ any submodule B_1 of B is called almost max-submodule if and only if B_1 is large maximal submodule (shortly La-max-submodule) if and only if B_1 is a maximal submodule**.** More details about algebraic structures which related to this article in ([11], [12], [13]). Semi $\boldsymbol{T}$-Small Submodules in [14]. A module B is multiplication if, A=JB where J is an ideal in T and A submodule of B [15]. From ([16], [17]), faithful multiplication module and Injective module.

In this paper, we introduce and study a new definition namely 2-maximal submodule and we make a generalization of local module. Some new results of the topic have been obtained. Also, we present a comprehensive study of 2-local module. Some results are analogous to the properties of pure submodules.

2. Local Modules

In this section, we study the concept of 2-local module as a generalization of local module. Also, we present more details about maximal and pure submodule which will use later in this section.

Definition 2.1. [1] Let B be a T-module. Then any submodule B_1 of B is called maximal if there exists K submodule of B such that B_1 is a proper subset of K⊆B, then K=B or B_1=K.
Recall that all the submodules of Z_{12} are (2), (3), (4), (6) and (0), so only (2) and (3) are max-submodules of B. Note that any T-module B is called 2-regular if every submodule B_1 of B is 2-pure. Therefore, 2-max-submodule can define it by depending on 2- regular.

Definition 2.2. [18] Any submodule B_1 of a T-module B is called unique 2-maximal if and only if the factor module (B/B_1) is 2-regular T-module.

Remarks and Examples 2.3.
1. Any unique max-submodule B_1 of a T-module B is a unique 2-maximal.
2. A unique 2-max-submoduleis not maximal.
3. B_1={0, 6, 12, 18, …} is a unique 2-max-submodule of B={…, -2, -1, 0, 1, 2, 3, …},
4. B_1 is not a unique max-submodule of B (because $(B/B_1) \cong B_2$ and B_2 = {0, 1, 2, 5} is not simple module).
5. Any module B maybe contain more than one max-submodule.
6. (0) is a max-submodule of any module B (by definition of maximal submodule).
7. Let (B/B_1) be a quotient module (Field). If B_1 is a submodule of B then B_1 is a maximal submodule of B.

Lemma 2.4. [2] Let B be a T-module. Then every unique weak max-submodule A of B is a unique 2-maximal.

Lemma 2.5. [4] Let B be a T-module. If B_1 is a finitely generated submodule of B, then (B/B_1) is 2-regular module over the ring T.
In the next theorem, we prove that there is an important relationship between cyclic submodule and generalization of local module.

Theorem 2.6. Let B be a T-module. If every submodule of B is cyclic and 2-pure, then B is 2-local module.

Proof: Since B_1 is cyclic submodule, so B_1 is a f. generated submodule of B. Also, B is a 2-pure. So, from definition of 2-regular we get, B is 2-regular [18]. Hence (B/B_1) is also 2-regular. Therefore, the submodule B_1 is a unique 2-maximal in B. Thus, B is 2-Local module.

Proposition 2.7. Let B be a T-module and B_1, B_2 are 2-local modules. Then $B_1 \cap B_2$ is 2-local module.

Proof: Suppose that B_1 and B_2 are 2-local modules. So B_1 and B_2 have a unique 2-maximal submodule. Then $\frac{B}{B1}$, $\frac{B}{B2}$ are 2-regular T-modules. But $\frac{B}{B1} \oplus \frac{B}{B2}$ is 2-rsgular T-module. Then $\frac{B}{B1 \cap B2} \cong \frac{B}{B1} \oplus \frac{B}{B2}$ [19]. Hence; $\frac{B}{B_1 \cap B_2}$ is 2-regular. So$B_1 \cap B_2$ is a unique 2-max-submodule and finally, B is 2-local module.

Corollary 2.8. Let B be an T-module. If B is 2-regular T-module, then B is 2-local module.
Proof: Suppose that B is 2-regular T-module. If $A \leq B$, then $\frac{B}{A}$ is 2-regular T-module. Then A is a unique 2-maximal submodule. Thus B is 2-local module.

Proposition 2.9. Let B be an T-module over (P.I.R) T. If s belong to B and t belong to T, there exists 1 in T such that $t^2 s = t^2 1 t^2 s$, then B is 2-local module.

Proof: We need to prove that B has a unique 2-max-submodule. Assume that $B_1 \leq B$ and A is an ideal of T. Let $s \in t^2 B \cap B_1$. So $s \in t^2 B$ and $s \in B_1$. Hence $\exists\, b \in B \ni s = t^2 b$. But $1 \in T \ni t^2 b = s = t^2 1 t^2 b$. Then $s \in t^2 B_1$. We have T is a (P.I.R). Then $J^2 B \cap B_1 = J^2 B_1$. So B_1 is 2-pure submodule of B. Therefore B is a 2-regular T-module. Hence B_1 is a unique 2-max-submodule of B. Thus B is 2-local module.

Definition 2.10. [19] Let B be an T-module. Any submodule B_1 of B is called Large-maximal (La-max-submodule) if $\exists\, B_2 \leq B,\ B_1 \leq B_2 \leq B$, so $B_2 \leq_{ess} B$.

Remark 2.11. Any essential submodule B_1 of B is a La-max-submodule and hence B_1 is a maximal submodule, because from [6], if $B_1 \leq B \ni B_1 \leq_{ess} W$ and let $B_1 < B_2 < B$, so $B_1 \leq_{ess} B_2 \leq_{ess} B$. So $B_2 \leq_{ess} B$. Hence B_1 is a La-max-submodule of B.

Example 2.12. In $Z_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ we see $\{\bar{0}, \bar{2}\}$ is a maximal and so is La-max-submodule, because $\{\bar{0}, \bar{2}\} \leq Z_4 \leq Z_4$ and $Z_4 \leq_{ess} Z_4$. The converse is not true in general, for example In Z as Z-module, 4Z is L-maximal submodule since $4Z < 2Z \leq Z$ and $2Z \leq_{ess} Z$ but 4Z is not maximal submodule since $2Z \neq Z$ [19]. But La-max-submodule is maximal in case B is semisimple module.

Now from [20], we study some properties of hollow and local modules. Also other modules have some relations with 2-local modules which is namely chained and semi-simple modules. Recall that any T-module B is called chained if $\forall\, B_1 \leq B, B_2 \leq B$, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Recall that, any $B_1 \leq B$ is called almost maximal if $B_2 \leq_{ess} B$; $B_1 \leq B_2 \leq B$, so $B_2 = B$. A nonzero module B over the ring T is called prime if Ann(B)=Ann(B_1), where $0 \neq B_1 \leq B$.

Theorem 2.13. Let B be a chained T-module. If $B_1 \leq B$ where B_1 is almost unique maximal, then B is a 2-local module.
Proof: Clear. Because any almost unique maximal submodule B_1 of chained module B is a unique maximal. So B_1 is a unique 2-maximal. Thus B is a 2-local module.

Corollary 2.14. Every semi-maximal submodule B_1 over chained module B is maximal and hence B is 2-local module.
Proof: Clear from Theorem 2.13.
Recall that the module B is said to be a fully prime module (resp. fully semiprime module) if every proper fully invariant submodule A of B is a prime (resp. semiprime) submodule.

Proposition 2.15. Let B be fully prime module over the ring T. If $B_1 \leq B$ is unique almost max-submodule, then B is a 2-local module.
Proof: Suppose that B_1 is almost max-submodule, and let $B_1 \leq B_2 \leq B$ such that $0 \neq B_2 \leq B$. Since B is a fully prime module, so B_2 is semi-essential in B not essential (If B is a fully prime module, then $0 \neq B_2 \leq B$ is a semi-essential) with B_1 almost maximal, imply $B_2 = B$ (B_1 is semi-max-submodule). So B_1 is a unique maximal and hence is unique 2-maximal submodule. Thus B is a 2-local module.

Definition 2.16. For every T-module B is called 3-regular if $B_1 \leq B$ is 3-pure where a submodule B_1 of B is called 3-pure if $J^3B \cap B_1 = J^3B_1$ such that J is an ideal of T. Also, any module is called 3-maximal if and only if B/B_1 is 3-regular T-module.

Definition 2.17. [7] Any module B over the ring R is simple if B has only two submodules are {0} and itself B.
Remark 2.18. [7] Every simple submodule is cyclic. So, every simple submodule is 3-maximal submodule.

Proposition 2.19. Let R be a simple ring. Then every homomorphic image of multiplication module is a 3-max-submodule.

Proof: Assume that B is a multiplication module, d: B→B^* is T-homomorphism and K=d(B). Suppose that h belongs to K. So, k= d(x), x ∈ B. By assumption**,** there exists an ideal J of T such that Tx=JB. Hence Tk=T(d(x)) = d(Tx) =d(JB) =JK. Therefore, K is a multiplication module. But T is simple ring. Then K is simple module and hence is 3-max-submodule.

Corollary 2.20. Let B be a divisible module over integral domain T. If JB=B_1 where J is an ideal of T for any submodule B_1 of B, then B is a simple module (B_1 is simple submodule) and hence B_1 is 3-maximal.
Proof: Suppose that JB=B_1 for any submodule B_1 of B where J is an ideal of T. So, B is a multiplication module. But B is divisible module. Hence B_1=B. Therefore, B is a simple module (B_1 is simple submodule) and hence is 3-maximal.

Corollary 2.21. Let B be a faithful module with A=JB over the ring T for any submodule A of B. If B is a semi-simple module over semi-simple ring S=End_R(B), then 3-maximal submodule**.**
Proof: Assume that B is a semi-simple over S. But A=JB (multiplication module) over T. From Theorem 2.43 in [17], B is a faithful module. Since S=End_T(B) is semi-simple ring, then S is a semi-simple ring. From B is a faithful multiplication module and from Theorem 2.44 in [17], we get B is a f.generated multiplication module over T (B_1 is f. generated submodule). Hence B_1 is 3-max-submodule

Now, according to the above, we present some important results about:
1. Let B be an T-module. If $A \leq B$ such that A is cyclic and $AJ^3 = BJ^3 \cap A$. Then B is 3-local module.
2. Let B be an T-module. If B_1 and B_2 are 3-local module, then the intersection of B_1 and B_2 is 3-local module.
3. Let B be an T-module. If B is 3-regular T-module, then B is 3-local module.
4. Let B be an T-module. If $B_1 \leq B$ such that B_1 is almost unique maximal and 3-regular, then B is 3-local module.
5. Every semi-max-submodule B_1 over chained module B is 3-maximal and hence B is 3-local module.

3. Conclusion

The local module is very important notion in module theory. In this paper, we have defined local module and we achieve an important generalization of this module. Several results have been presented in this article. From Theorem 2.6, if B is a T-module with B_1 is cyclic submodule of B and $B_1J^2 = BJ^2 \cap B_1$, this imply B is 2-local module. We studied the relationship between the 2-maximal submodule and 2-local module. Also, if B is a T-module and $B_1 \leq B$ where B_1 is almost unique maximal, then B is a 2-local module. From Proposition 2.18, we studied another generalization of local module namely 3-local module because, if T is a simple ring, then every homomorphic image of multiplication module is a 3-max-submodule and hence is a 3-local. More implications in this paper are obtained.

Acknowledgements: I would like to present my thanks to the reviewers for their advice and for correcting errors in the paper and taking all their comments seriously.

References
[1] H. Y. Khalaf, "Semi-maximal Submodule", Ph. D Thesis, University of Baghdad, 2007.
[2] M. R. Shwkaea, "Weak Maximal Ideals and Weak Maximal Submodules", M. Sc. Thesis, University of Tikrit, 2012.
[3] F. Kach, "*Modules and Rings*", *Acad, Press*., *London*, 1982.

[4] A. H. Ghaleb, "Generalization of Regular Modules and Pure Submodules", ph. D, thesis, University of Baghdad, 2015.
[5] G. Desal and W. Nicholson, "Endoprimitive Rings", *J. Algebra*, vol.70, pp.545-560, 1981.
[6] Z. A. El-Bast, and P. P. Smith, "Multiplication Module", *Comm. In Algebra*, vol.16, pp.755-770,
[7] 1988.
[8] A. F. Talak and M. M. Abed, "P-(S. P) Submodules and C1 (Extending) Modules," *In Journal of Physics: Conference Series*., vol. 1804, no. 1, p. 012083, 2021.
[9] R. M. AL–Shaiban and N. S. AL-Mothafar, "Properties of J Regular modules", *Iraqi Journal of Science*, vol. 60, no.3, pp. 647-652, 2019.
[10] M. M. Abed, "A new view of closed-CS-module", *Italian Journal of Pure and Applied Mathematics*, no. 43, pp. 65-72, 2020.
[11] F. N. Hammad and M. M. Abed, "A New Results of Injective Module with Divisible Property," *In Journal of Physics: Conference Series*, vol. 1818, no. 1, pp. 012168, 2021.
[12] M. M. Abed and F. G. Al-Sharqi, "Classical Artinian Module and Related Topics," *In Journal of Physics: Conference Series*, vol. 1003, no. 1, p. 012065. IOP, 2018.
[13] M. M. Abed, F. G. Al-Sharqi, and S. H. Zail, "A Certain Conditions on Some Rings Give P.P. Ring", *In Journal of Physics: Conference Series*., vol. 1818, no., 2021.
[14] M. M. Abed, "On Indeterminacy (Neutrosophic) of Hollow Modules", *Iraqi Journal of Science*, vol. 63, no. 6, pp. 2650-2655, 2022.
[15] A. A. Elewi, "Semi $\boldsymbol{T}$-Small Submodules", *Iraqi Journal of Science*, Vol. 62, No. 6, pp. 1956- 1960, 2021.
[16] M. M. Abed, N. Hassan and F. Al-Sharqi, "On Neutrosophic Multiplication Module", *Neutrosophic Sets and Systems*, vol. 49, 2022.
[17] A. F. Talak and M. M. Abed, "Injective Module and C_1-Module", *AIP Conference Proceedings* 2386, 060004, 2022.
[18] K. Saeho, "On Faithful Multiplication Modules", A Thesis Approved in Partial Fulfilment of the Requirements for The Degree of Master of Science in Mathematics, Thaksin University, 2011.
[19] H. K. Mohammad and A. S. Mohammed M. D. Sallman, "2-Maximal Submodules and Related Concepts", *Journal of University of Anbar for pure science*, vol.11, no.3, 2017.
[20] A. A. Abduljaleel and S. M. Yaseen, "Large-Maximal submodules", *Journal of Physics: Conference Series* 1963, 012011, 2021.
[21] M. A. Salih, D. A. Jarwan, M. M. Abed, "n-Refined Indeterminacy of Some Modules", *International Journal of Neutrosophic Science*, vol. 20, no. 02, pp. 20-26, 2023.