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Dynamics of a Delayed Prey-Predator System With Crowley-Martin Functional Response

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Abstract:
ln this paper, an ecological predator-prey model with time delay is proposed and studied analytically and numerically. The stability and bifurcation analysis of the proposed model with a Crowley-Martin response function are covered in this article. In the beginning, equilibrium points of the proposed model are identified. Secondly, the local asymptotic stability of the equilibria and Hopf-bifurcation are discussed by the characteristic equations of the system. Finally, by creating a suitable Lyapunov function and LaSalle's invariance principle, the equilibria's global asymptotic stability is examined.

Keywords: Prey-predator model, Crowley-Martin, Stability analysis, Hopf-bifurcation

ديناميكيات نظام الفريسة-المفترس المتأخر مع استجابة كراولي- مارتن الدالية

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الخلاصة

في هذا البحث تم اقتراح نموذج بيئي للمفترس – الفريسة مع تأخر زمني حيث تمت دراسته تحليليًا وعدديا. تناول هذا البحث استقرار وتحليل التشعب للنموذج باستخدام دالة استجابة كرولى– مارتين (Crowley–Martin). في البداية ، حددنا نقاط التوازن. ثانياً ، تمت مناقشة الاستقرار المقارب الموضعي (local asymptotic stability) لنقاط التوازن وتشعب هوبف باستخدام المعادلات المميزة للنظام وأخيرًا، تم اختبار توازن الاستقرارالمقارب الكلي (equilibria's global asymptotic stability) باستخدام دالة (Lyapunov) ملائمة و مبدأ (LaSalle) للثبات (LaSalle's invariance principle).

1 Introduction:

The dynamic interaction between predators and their prey has long been and will remain one of central concepts among both mathematical and ecological science. The term competitive can be defined as one of the most ancient and fascinating concepts in community ecology which contends no more than n species can exist simultaneously on n infrastructure [1,2,3,4]. Gauss [3] validated it using tests on Paramecium cultures. It was thought to be true in the lab until Ayala's [5] experimental proof that two species of Drosophila could cohabit on a single prey Irakli Loladze [6] proposed that the prey consisted of a single species of alga

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and the predators were two different zooplankton species to explain their coexistent. A prey, a specialized, and a predator species make up a three-species food web, according to Gakkhar and Najl's observation [7].

Local stability is an important concept in many areas of study, including physics, engineering, mathematics, and biology. For example, in control theory, local stability is used to analyze the behavior of control systems near an equilibrium point, and to determine whether small deviations will result in a return to the same point, or go to another point which makes the system unstable. In ecology, local stability is used to describe the stability of populations of species in a specific region, and how changes in the environment or interactions with other species can affect that stability. The behavior of the prey population before the predator's arrival could then be compared to the behavior of the population after the predator arrives that allows researchers to better understand the impact of the predator on the ecosystem. [4]

In population biology, the impact of the past on a system's stability is a significant issue. Crowley-Martin response function is used because to the declining availability of prey in nature and ecology. Population dynamic models now include time delay to represent maturing, capturing, and other processes. Since a temporal delay could result in the equilibrium being unstable, delay differential equations generally have substantially more intricate dynamics than regular differential equations. The delay models have been the subject of extensive study see [8-16].

The first way of applying a mathematical model to investigate the biological system was put forth by Bertalanffy [17]. He emphasized that this strategy combines coordination, order, and purposed to create three fundamental concepts for examining the biological process, namely the trophic-level evaluation, system standpoint, and dynamic observation. Improving knowledge of population interconnections and their dependency on both internal and external variables is the primary goal of the population dynamics models [18].

The existence of the time delays is a significant factor in system instability, making time-delay research is another of the most difficult problems to solve. Time delays frequently appear in a variety of engineering projects, including biological, socioeconomic, and chemical processes. Differential-difference equations, a subset of functional differential equations, are used to model time delays [19]. Some researchers have explored some models with time delayed terms in depth in a variety of biological models [20-33].

The stability of a dynamic ecology system with a predator is a complex and important area of study in the field of ecology. In this type of system, the predator-prey interactions play a crucial role in determining the stability of the ecosystem as a whole. A key factor in the stability of a predator-prey system is the rate of increase of the prey population. If the prey population grows too rapidly, the predator population may not be able to keep up, this leads to a decline in the predator population and an increase in the prey population. On the other hand, if the prey population grows too slowly, the predator population may decline, leading to a decrease in the overall stability of the ecosystem. Another important factor in the stability of predator-prey systems is the functional response of the predator which describes the relationship between the rate of prey consumption and the density of the predator population. Different functional responses such as the type I, type II, and type III functional responses can have a significant impact on the stability of the predator-prey system. The stability of predator-prey systems can also be influenced by other factors such as environmental variability, resource competition, and disease. These factors can interact with predator-prey interactions to create complex and nonlinear dynamics that make it difficult to predict the stability of the system in the long term. Overall, the stability of a dynamic ecology system with a predator is a complex and multi-faceted issue, and a comprehensive understanding of the system requires a combination of mathematical modeling, empirical observations, and theoretical analysis. In the context of continuous dynamical systems, a prey-predator model may refer to a mathematical model that used to describe the behavior of a system before the arrival of a predator. The goal of such model would be to understand the behavior of the system's prey population before the arrival of the predator and how that behavior changes in response to the predator's presence. A time delay in a system refers to a lag in the response of the system to a change in its inputs. It occurs when the output of the system does not immediately respond to changes in the input, but it responds after a certain amount of time has passed. Time delays can occur in many types of systems, including physical systems, biological systems, and engineered systems, such as control systems and communication systems. Time delays can have a significant impact on the stability and performance of a system. For example, in control systems, a time delay can cause instability and decrease performance. As a result, the study of the time delay systems is an important area of research in control theory and engineering. Techniques such as delay compensation and the use of predictive control can be used to mitigate the effects of time delay in systems. [4], [8], [32], [34].

The Crowley-Martin functional response is a mathematical model that describes the relationship between the rate of consumption of a resource and the density of the consumers exploiting that resource. The Crowley-Martin functional response is a type of type III functional response which is characterized by a declining rate of consumption as the density of consumers increases. In this model, the rate of consumption is described as a logistic function with a saturation term that limits the rate of consumption as the density of consumers increases. The Crowley-Martin functional response is commonly used in the field of ecological and environmental studies to describe the interactions between species and their environment. It is also used in the study of predator-prey interactions. It also provides a theoretical framework for understanding how the rate of resource exploitation by predators can influence the dynamics of prey populations.[35],[36].

In this paper, we will consider the following prey-predator system with the Crowley-Martin functional response and time delay:

$$\begin{aligned}
\frac{dx}{dt} &= \left(r\left(1-\frac{x(t-\tau)}{K}\right)-\frac{a_1\beta y(t)}{(1+m_1x(t))(1+m_2y(t))}-\frac{a_2z(t)}{(1+m_3x(t))(1+m_4z(t))}\right)x(t),\\
\frac{dy}{dt} &= \left(\frac{e_1a_1\beta x(t-\tau)}{(1+m_1x(t-\tau))(1+m_2y(t-\tau))}-c_1\right)y(t),\\
\frac{dz}{dt} &= \left(\frac{e_2a_2x(t-\tau)}{(1+m_3x(t-\tau))(1+m_4z(t-\tau))}-c_2\right)z(t).
\end{aligned} \tag{1}$$

Where $x(t)$ is the density of prey population and $y(t)$, $z(t)$ are the density of predators population that feed on $x(t)$. Also, r is the intrinsic growth rate of the prey, K is the carrying capacity, $a_1, a_2, m_1, m_2, m_3, m_4$ represent the capture rate, handling the $x(t)$ and the magnitude of interference among $y(t)$, $z(t)$. β is the relative superiority of $y(t)$. e_1 and e_2 are the conversion rates of prey $x(t)$. The natural death rates for species are c_1 , c_2. All parameters are positive.

2. Analysis of the equilibria:

In this section, we study the existence of equilibrium points of the system (1), we can see that the system (1) has five possible equilibria which are listed as follows:

(i) Vanishing equilibrium point $E_0 = (0,0,0)$ and predators-free equilibrium point $E_1 = (K, 0,0)$.
(ii) The one predator equilibrium point $E_2 = (x^*, y^*, 0)$ exists in the Int R_+^2 of xy-plane if there is a positive solution to the following equations:

$$r\, x^* + \frac{a_1\, \beta\, y^*}{G^*} = r\, K \quad , \tag{2a}$$

$$\frac{e_1\, a_1\, \beta\, x^*}{G^*} = c_1. \tag{2b}$$

Where $G^* = (1 + m_1 x^*)(1 + m_2\, y^*)$.

From equation (2b), we get: $y^* = \frac{(e_1\, a_1\, \beta - c_1\, m_1)\, x^* - c_1}{c_1\, m_2\, (1 + m_1\, x^*)}$ (3a)

y^* is positive provided $0 < \frac{c_1}{e_1\, a_1\, \beta - c_1\, m_1} < x^*$. (3b)

By substituting y^* in equation (2a), we can get the following polynomial equation:

$$A_1\, x^{*3} + A_2\, x^{*2} + A_3\, x^* + A_4 = 0 \tag{4}$$

Where $A_1 = e_1\, m_1\, m_2\, r$, $A_2 = e_1\, m_2\, r\, (1 - m_1 k)$,
$A_3 = K\,(e_1\, a_1\, \beta - e_1\, m_2\, r - c_1\, m_1)$, $A_4 = -c_1\, k$

By Descarte's Rule of sign [37], [38] equation (4) has a unique positive root provided that
$A_2 > 0$ or $A_3 < 0$. (5)

(iii) The equilibrium point $E_3 = (\bar{x}, 0, \bar{z})$ exists in the Int R_+^2 of xz-plane, where;

$\bar{z} = \frac{(e_2\, a_2 - c_2\, m_3)\, \bar{x} - c_2}{c_2\, m_4\, (1 + m_3\, \bar{x})}$ is positive provided $0 < \frac{c_2}{e_2\, a_2 - c_2\, m_3} < \bar{x}$. (6)

And $\bar{x}$ is a positive root of polynomial equation:

$$B_1\, \bar{x}^3 + B_2\, \bar{x}^2 + B_3\, \bar{x} + B_4 = 0. \tag{7}$$

Where; $B_1 = e_2\, m_3\, m_4\, r$, $B_2 = e_2\, m_4\, r\, (1 - m_3\, K)$,
$B_3 = K\,(e_2\, a_2 - c_2\, m_3 - e_2 m_4 r)$, $B_4 = -c_2 K$

Which exists if $B_2 > 0$ or $B_3 < 0$. (8)

(iv) The positive equilibrium point $E_4 = (\hat{x}, \hat{y}, \hat{z})$ exists in the interior of R_+^3 if there is a positive solution to the non-linear system:

$$r\left(1 - \frac{\hat{x}}{K}\right) = \frac{a_1\, \beta\, \hat{y}}{\hat{G}} + \frac{a_2\, \hat{z}}{\hat{H}} \quad , \tag{9a}$$

$$\frac{e_1\, a_1\, \beta\, \hat{x}}{\hat{G}} = c_1, \tag{9b}$$

$$\frac{e_2\, a_2\, \hat{x}}{\hat{H}} = c_2\, . \tag{9c}$$

Where $\hat{G} = (1 + m_1\, \hat{x})\,(1 + m_2\, \hat{y})$,and $\hat{H} = (1 + m_3\, \hat{x})\,(1 + m_4\, \hat{z})$.

From equations (9b) and (9c), we get:

$\hat{y} = \frac{(e_1\, a_1\, \beta - c_1\, m_1)\, \hat{x} - c_1}{c_1\, m_2\, (1 + m_1\, \hat{x})}$ and $\hat{z} = \frac{(e_2\, a_2 - c_2\, m_3)\, \hat{x} - c_2}{c_2\, m_4\, (1 + m_3\, \hat{x})}$, respectively.

By substituting $\hat{y}$ and $\hat{z}$ in equation (9a), we get a polynomial equation: $(\hat{x}) = 0$.

Where $q(x) = C_1 x^4 + C_2 x^3 + C_3 x^2 + C_4 x + C_5$, (10)

$C_1 = r\, e_1\, e_2\, m_1\, m_2\, m_3\, m_4$, $C_2 = r\, e_1\, e_2\, m_2\, m_4\, (m_1 + m_3 - K\, m_1\, m_3)$
$C_3 = r\, e_1\, e_2\, m_2\, m_4\, (1 - K m_1 - K m_3) + K\, e_2\, m_3\, m_4\, (e_1\, a_1\, \beta - c_1\, m_1)$
$+ K\, e_1\, m_1\, m_2\, (e_2\, a_2 - c_2\, m_3)$,
$C_4 = K\, e_2\, m_4\, (e_1\, a_1\, \beta - c_1\, m_1 - c_1\, m_3) + K\, e_1\, m_2\, (e_2\, a_2 - c_2\, m_3 - c_2\, m_1)$
$- r\, K\, e_1\, e_2\, m_2\, m_4$, $C_5 = -K\,(c_1\, e_2\, m_4 + c_2\, e_1 m_2)$

Now according to the Descarte's Rule of sign, the above equation has a unique positive root such that one set of the following sets of conditions:

$$\begin{cases} C_2 > 0\, , C_4 < 0 \\ C_2 > 0\, , C_3 > 0 \\ C_3 < 0\, , C_4 < 0 \end{cases} \tag{11}$$

So it exists uniquely in the Int R_+^3 if the following conditions hold:

$$0 < \frac{c_2}{e_2\, a_2 - c_2\, m_3} < \hat{x} \quad \text{and} \quad 0 < \frac{c_1}{e_1\, a_1\, \beta - c_1\, m_1} < \hat{x} \tag{12}$$

3. Local stability and Hopf bifurcation:

In this section, we investigate the local stability at each of feasible equilibrium point of the system (1) by using the corresponding characteristic equations.

Theorem 3.1: For system (1) we have the following:

(a) The equilibrium point $E_0 = (0,0,0)$ is a saddle point.

(b) The predators-free equilibrium point $E_1 = (K, 0,0)$ is locally asymptotically stable if

$$K < \min \left\{ \frac{c_1}{e_1\, a_1\, \beta - m_1\, c_1}, \frac{c_2}{e_2\, a_2 - m_3\, c_2} \right\}, \tag{13.1}$$

$$r\, \tau < \frac{\pi}{2}. \tag{13.2}$$

Proof: (a): The Jacobian matrix of the equilibrium point $E_0 = (0,0,0)$ is:

$$J_{(t)}(E_0) = \begin{bmatrix} r & 0 & 0 \\ 0 & -c_1 & 0 \\ 0 & 0 & -c_2 \end{bmatrix} \tag{14}$$

The eigenvalues of (14) are $\lambda_1 = r > 0$, $\lambda_2 = -c_1 < 0$, and $\lambda_3 = -c_2 < 0$.

Thus, the equilibrium point $E_0 = (0,0,0)$ is a saddle point with locally stable manifold in the yz-plane and with locally unstable manifold in the x-direction.

(b) The Jacobian matrix of the equilibrium point $E_1 = (K, 0,0)$ is:

$$J_{(t)}(E_1) = \begin{bmatrix} -re^{-\lambda\tau} & \frac{-a_1\beta k}{1+m_1 k} & \frac{-a_2 k}{1+m_3 k} \\ 0 & \frac{e_1 a_1 \beta k}{1+m_1 k} - c_1 & 0 \\ 0 & 0 & \frac{e_2 a_2 k}{1+m_2 k} - c_2 \end{bmatrix}$$

The characteristic equation of predators-free equilibrium point $E_1 = (K, 0,0)$ is:

$$\left(\lambda + r\, e^{-\lambda\tau}\right)\left(\lambda - \frac{e_1\, a_1\, \beta\, K}{1 + m_1\, K} + c_1\right)\left(\lambda - \frac{e_2\, a_2\, K}{1 + m_3\, K} + c_2\right) = 0 \tag{15}$$

To find the value of λ that satisfies $\lambda + r\, e^{-\lambda\tau} = 0$

let $\lambda = \omega_i$, so we have:

$$r - \omega \sin(\omega\tau) + i\omega\cos(\omega\tau) = M(\omega) + i\, V(\omega) = 0 \tag{16}$$

$M(\omega) = r - \omega \sin(\omega\tau)$ and $V(\omega) = \omega \cos(\omega\tau)$.

From equation (16), we conclude that: $\omega_0 = 0$, $\omega_{n+1} = \frac{\pi}{2} + n\pi$ for $n \in I$.

Then the root of $V(\omega)$ are all real numbers.

By using equation (3.10) in[1, page 244], from the relation $\dot{V}M(\omega)$, we have

$$\frac{dV(\omega_n)}{d\omega_n} M(\omega_n) = \left(\cos(\omega_n\tau) - \omega_n\tau\sin(\omega_n\tau)\right)\left(r - \omega_n\sin(\omega_n\tau)\right) \tag{17}$$

$$= \begin{cases} r > 0 & if \quad n = 0 \\ (-\sin(n\pi) - \omega_n\, \tau\cos(n\pi))(r - \omega_n \cos(n\pi)) & if \quad n \neq 0 \end{cases} \tag{18}$$

$$\frac{dV(\omega_n)}{d\omega_n} = \frac{\pi^2}{4\tau}(2n+1)^2 - r(-1)^n(2n+1)\frac{\pi}{2} \tag{19}$$

This inequality $r\,(2n+1) < \frac{\pi}{2\tau}(2n+1)^2$ holds if $r\tau < \frac{\pi}{2}$.

That is equation (19) is positive if $r\tau < \frac{\pi}{2}$.

λ_1 has a negative real part if $r\tau < \frac{\pi}{2}$, also from equation (15), λ_2, λ_3 has negative real part if $K < \min \left\{ \frac{c_1}{e_1\, a_1\, \beta - m_1\, c_1}, \frac{c_2}{e_2\, a_2 - m_3\, c_2} \right\}$.

To study the local stability of the equilibrium point $E_2 = (x^*, y^*, 0)$, we have to evaluate the Jacobian matrix at the equilibrium point $E_2 = (x^*, y^*, 0)$ which is given by:

$$J_{(t)}(E_2)=\begin{bmatrix} \frac{c_1 m_1 y^*}{e_1(1+m_1x^*)}-\frac{-rx^*}{K}e^{-\lambda\tau} & \frac{c_1}{e_1(1+m_2y^*)} & \frac{-a_2x^*}{1+m_3x^*} \\ \frac{c_1y^*}{(1+m_1x^*)x^*}e^{-\lambda\tau} & \frac{-m_2c_1}{(1+m_2y^*)}e^{-\lambda\tau} & 0 \\ 0 & 0 & \frac{e_2a_2x^*}{(1+m_3x^*)}-c_2 \end{bmatrix}$$

Then, we have the characteristic equation of $J_{(t)}(E_2)$:

$$\left(\lambda-\frac{e_2\,a_2\,x^*}{(\,1+m_3x^*\,)}+c_2\right)\left[\left(\lambda-\frac{c_1\,m_1\,y^*}{e_1\,(\,1+m_1x^*\,)}+\frac{rx^*}{K}e^{-\lambda\tau}\right)\left(\lambda+\frac{m_2\,c_1\,y^*}{(\,1+m_2y^*\,)}e^{-\lambda\tau}\right)+\left(\frac{{c_1}^2\,y^*}{e_1\,x^*\,(\,1+m_1x^*\,)(\,1+m_2y^*\,)}e^{-\lambda\tau}\right)\right]=0$$

Further simplification yields the following:

$\lambda_1=\frac{e_2\,a_2\,x^*}{(\,1+m_3x^*\,)}-c_2$ and $\lambda^2+P\lambda+(\,Q+R\lambda\,)\,e^{-\lambda\tau}+S\,e^{-2\lambda\tau}=0$ (20)

Where $P=\frac{-c_1\,m_1\,y^*}{e_1(\,1+m_1x^*\,)}$, $Q=\left(\frac{1}{x^*}-m_1m_2y^*\right)\frac{c_1^3\,y^*}{e_1^2\,a_1\,\beta\,x^*}$, $R=\frac{rx^*}{K}+\frac{C_1\,m_2\,y^*}{(\,1+m_2y^*\,)}$, and $S=\frac{c_1\,m_2\,r\,x^*\,y^*}{K\,(\,1+m_2y^*\,)}$

When $\tau=0$, equation (20) reduces to: $\lambda^2+(\,P+R\,)\lambda+Q+S=0$ (21)

According to Routh-Hurwitz criterion, equation (21) has all negative roots if $P+R>0$ and $Q+S>0$.

Hence, for $\tau=0$, the prey-predators one equilibrium point $E_2=(x^*,y^*,0)$ is locally asymptotically stable whenever the following conditions are satisfied:

$$\frac{e_2\,a_2\,x^*}{(\,1+m_3x^*\,)}<c_2 \quad (22a)$$

$$\frac{m_1}{e_1\,(\,1+m_1\,x^*\,)}<\frac{m_2}{1+m_2\,y^*} \quad (22b)$$

$$\frac{c_1^2\,m_1\,y^*}{e_1^2a_1\,\beta\,x^*}<\frac{r\,x^*}{K\,(\,1+m_2\,y^*\,)} \quad (22c)$$

Now we can discuss the system in presence of delay around $E_2=(x^*,y^*,0)$. Let $\lambda(\tau)=M(\tau)\pm i\omega(\tau)$; $(\omega>0)$ but the change of stability around $E_2=(x^*,y^*,0)$ will occur for $M(\tau)=0$. Hence it is found that the position of stability for $\lambda(\tau)=\pm i\omega(\tau)$ if $i\omega$ $(\omega>0)$ is a solution of equilibrium point equation (21) separating real and imaginary parts:

$$Q\cos(\,\omega\tau\,)+R\omega\sin(\,\omega\tau\,)+S\cos(\,2\omega\tau\,)=\omega^2 \quad (23a)$$

$$Q\sin(\,\omega\tau\,)-R\omega\cos(\,\omega\tau\,)+S\sin(2\omega\tau)=P\omega \quad (23b)$$

Solving (23a) and (23b) we get:

$$\sin(\omega\tau)=\frac{R\,\omega^3+\omega\,(\,PQ-RS\,)}{\omega^4+P^2\,\omega^2-S^2} \quad (24a)$$

$$\cos\,(\omega\tau)=\frac{Q\,S+\omega^2\,(\,Q-PR\,)}{\omega^4+P^2\,\omega^2-S^2}\backslash \quad (24b)$$

squaring and adding (24a) and (24a) we get the polynomial equation:

$$\omega^8+P_1\,\omega^6+P_2\,\omega^4+P_3\,\omega^2+P_4=0. \quad (25)$$

Where $P_1=2P^2-R^2$, $p_2=P^4+2R^2\,S-Q^2-P^2\,R^2-2S^2$,
$P_3=4PQRS-2Q^2\,S-2P^2\,S^2-P^2\,Q^2-R^2\,S^2$, and $P_4=S^2\,(\,S^2-Q^2\,)$

Let $f(\omega)=\omega^8+P_1\omega^6+P_2\omega^4+P_3\omega^2+P_4$ and $h=\omega^2$ then equation (25) can be written as:

$$h^4+P_1\,h^3+P_2\,h^2+P_3\,h+P_4=0 \quad (26)$$

$$\text{If } \frac{m_2\, r\, x^*}{K\,(1+m_2 y^*)} < \left(\frac{1}{x^*} - m_1 m_2\, y^*\right)\frac{c_1^2}{e_1^2\, a_1\, \beta\, x^*}, \tag{27}$$

then equation (26) has at least one positive root, without loss of generality we assume that equation (26) with condition (27) has three positive roots h_1 , h_2 , h_3 then: $\omega_k = \sqrt{h_k}$ (k=1,2,3). From equation (24b), we get the corresponding value of time delay $\tau_k^j > 0$ such that equation (21) has a pair of purely imaginary roots $\pm\omega_k$

$$\tau_k^j = \frac{1}{\omega_k}\cos^{-1}\left(\frac{Q\,S + \omega_k^2\,(Q - PR)}{\omega_k^4 + P^2\,\omega_k^2 - S^2}\right) + \frac{2\,j\,\pi}{\omega_k} \quad , \; j = 0,1,2,\dots \tag{28}$$

Let τ_0^* be the smallest positive value of τ_k^0 for $= 1,2,3$, $\omega_0^* = \omega_{k_0}$.

Now differentiating both sides of equation (21) we get:

$$(2\lambda + P)\frac{d\lambda}{d\tau} - (Q + R\lambda)\,e^{-\lambda\tau}\left(\lambda + \tau\frac{d\lambda}{d\tau}\right) + R\,\frac{d\lambda}{d\tau}\,e^{-\lambda\tau} - 2S\left(\lambda + \tau\frac{d\lambda}{d\tau}\right)e^{-2\lambda\tau} = 0$$

Then, we obtain:

$$\frac{d\lambda}{d\tau} = \frac{\lambda\left(Q + R\lambda + 2Se^{-\lambda\tau}\right)e^{-\lambda\tau}}{P + 2\lambda - \tau(Q + R\lambda)\,e^{-\lambda\tau} + Re^{-\lambda\tau} - 2S\tau\, e^{-2\lambda\tau}}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(P + 2\lambda)e^{\lambda\tau}}{\lambda\,(Q + R\lambda + 2Se^{-\lambda\tau})} + \frac{R - \tau(Q + R\lambda) - 2S\tau e^{-\lambda\tau}}{\lambda(Q + R\lambda + 2Se^{-\lambda\tau})} \tag{29}$$

For $\lambda = \pm\omega_0^* i$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{\omega_0^*\big(P\cos(\omega_0^*\tau_0) - 2\omega_0^*\sin(\omega_0^*\tau_0)\big)(2S\sin(\omega_0^*\tau_0) - R\omega_0^*) + \big(P\sin(\omega_0^*\tau_0) + 2\omega_0^*\cos(\omega_0^*\tau_0)\big)\big(Q + 2S\cos(\omega_0^*\tau_0)\big)\omega_0^* + iL_1}{\omega_0^{*2}\big(2S - \sin(\omega_0^*\tau_0) - R\omega_0^*\big)^2 + \omega_0^{*2}\big(Q + 2S\cos(\omega_0^*\tau_0)\big)^2}$$
$$+ \frac{\big(R - Q\tau_0 - 2S\tau_0\cos(\omega_0^*\tau_0)\big)(2S\sin(\omega_0^*\tau_0) - R\omega_0^*)\omega_0^* + \omega_0^*(2S\tau_0\sin(\omega_0^*\tau_0) - R\omega_0^*\tau_0)\big(Q + 2S\cos(\omega_0^*\tau_0)\big) + iL_2}{\omega_0^{*2}\big(2S - \sin(\omega_0^*\tau_0) - R\omega_0^*\big)^2 + \omega_0^{*2}\big(Q + 2S\cos(\omega_0^*\tau_0)\big)^2}$$

Where;

$$L_1 = -\omega_0^*\big(Q + 2S\cos(\omega_0^*\tau_0)\big)\big(P\cos(\omega_0^*\tau_0) - 2\omega_0^*\sin(\omega_0^*\tau_0)\big)$$
$$+\omega_0^*(2S\sin(\omega_0^*\tau_0) - R\omega_0^*)\big(P\sin(\omega_0^*\tau_0) + 2\omega_0^*\cos(\omega_0^*\tau_0)\big)$$
$$L_2 = -\omega_0^*\big(Q + 2S\cos(\omega_0^*\tau_0)\big)\big(R - Q\tau_0 - 2S\tau_0\cos(\omega_0^*\tau_0)\big)$$
$$+\omega_0^*(2S\sin(\omega_0^*\tau_0) - R\omega_0^*)(2S\tau_0\sin(\omega_0^*\tau_0) - (\omega_0^*\tau_0)$$

$$\left(\frac{d\big(\mathrm{Re}(\lambda(\tau_0))\big)}{d\tau}\right)^{-1}$$
$$= \frac{1}{N_0}\big[\omega_0^*(2PS\sin(\omega_0^*\tau_0)\cos(\omega_0^*\tau_0) - PR\omega_0^*\cos(\omega_0^*\tau_0) - 4S\omega_0^*\sin^2(\omega_0^*\tau_0)$$
$$+ 2R\omega_0^{*2}\sin(\omega_0^*\tau_0) + PQ\sin(\omega_0^*\tau_0) + 2PS\sin(\omega_0^*\tau_0)\cos(\omega_0^*\tau_0)$$
$$+ 2Q\omega_0^*\cos(\omega_0^*\tau_0) + 4S\omega_0^*\cos^2(\omega_0^*\tau_0) + 2RS\sin(\omega_0^*\tau_0) - R^2\omega_0^*$$
$$- 2QS\tau_0\sin(\omega_0^*\tau_0) + RQ\omega_0^*\tau_0 - 4S^2\tau_0\sin(\omega_0^*\tau_0)\cos(\omega_0^*\tau_0)$$
$$+ 2RS\omega_0^*\tau_0\cos(\omega_0^*\tau_0) + 2QS\tau_0\sin(\omega_0^*\tau_0) + 4S^2\tau_0\sin(\omega_0^*\tau_0)\cos(\omega_0^*\tau_0)$$
$$- RQ\omega_0^*\tau_0 - 2RS\omega_0^*\tau_0\cos(\omega_0^*\tau_0))\big]$$

Where; $N_0 = \omega_0^{*2}(2S - \sin(\omega_0^*\tau_0) - R\omega_0^*)^2 + \omega_0^{*2}\big(Q + 2S\cos(\omega_0^*\tau_0)\big)^2 > 0$

$$\left(\frac{d\left(\mathrm{Re}\left(\lambda(\tau_0)\right)\right)}{d\tau}\right)^{-1}$$
$$= \frac{\omega_0^*}{N_0}[SP\sin(2\omega_0^*\tau_0) - PR\omega_0^*\cos(\omega_0^*\tau_0) - 4S\omega_0^*\sin^2(\omega_0^*\tau_0)$$
$$+ 4S\omega_0^*\cos(\omega_0^*\tau_0) + 2R\omega_0^{*2}\sin(\omega_0^*\tau_0) + PQ\sin(\omega_0^*\tau_0) + SP\sin(2\omega_0^*\tau_0)$$
$$+ 2\omega_0^*\cos(\omega_0^*\tau_0) + 2SR\sin(\omega_0^*\tau_0) - 2QS\tau_0\sin(\omega_0^*\tau_0) - 2S^2\tau_0\sin(2\omega_0^*\tau_0)$$
$$+ 2RS\omega_0^*\tau_0\cos(\omega_0^*\tau_0)$$
$$+ 2QS\tau_0\sin(\omega_0^*\tau_0) + 2S^2\tau_0\sin(2\omega_0^*\tau_0) - 2RS\omega_0^*\tau_0\cos(\omega_0^*\tau_0) - R^2\omega_0^*]$$
$$= \frac{\omega_0^*}{2N_0\left({\omega_0^*}^4 + P^2{\omega_0^*}^2 - S^2\right)}\left[8{\omega_0^*}^7 + 6P_1{\omega_0^*}^7 + 4P_2{\omega_0^*}^3 + 2P_3\omega_0^*\right]$$
$$= \frac{\omega_0^*}{2N_0\left({\omega_0^*}^4 + P^2{\omega_0^*}^2 - S^2\right)}f'(\omega_0^*)$$

Thus, we get the following results:

Theorem 3.2: Suppose that (22a), (22b) and (22c) hold. If equation (26) has no positive roots, then the two species equilibrium point $E_2 = (x^*, y^*, 0)$ is locally asymptotically stable for all $\tau \geq 0$.

Theorem 3.3: Suppose that (22a) and (22b) hold. If $\frac{m_1 m_2 y^*}{e_1^2 a_1 \beta x^*} < \frac{m_2 r x^*}{K c_1^2(1+m_2 y^*)} < \left(\frac{1}{x^*} - m_1 m_2 y^*\right)\frac{1}{e_1^2 a_1 \beta x^*}$ and sign $\frac{f'(\omega_0^*)}{\left({\omega_0^*}^4 + P^2{\omega_0^*}^2 - S^2\right)}$ positive, then the two species equilibrium point $E_2 = (x^*, y^*, 0)$ is locally asymptotically stable for $\tau \in (0, \tau_0^*)$.
And the system(1) undergoes a Hopf-bifurcation at equilibrium point $E_2 = (x^*, y^*, 0)$ when $\tau = \tau_0^*$.

Theorem 3.4: The two species equilibrium point $E_3 = (\bar{x}, 0, \bar{z})$ of the system (1) is locally asymptotically stable for all $t \geq 0$ if the following conditions are satisfied

$$\frac{e_1 a_1 \beta \bar{x}}{1 + m_1\bar{x}} < c_1 \tag{30a}$$
$$\frac{m_2}{e_2(1+m_3\bar{x})} < \frac{m_4}{1 + m_4\bar{z}}, \tag{30b}$$
$$\frac{c_2^2 m_3 \bar{z}}{e_2^2 a_2 \bar{x}} < \frac{r\bar{x}}{k(1 + m_4\bar{z})}, \tag{30c}$$

and the equation $h^4 + q_1 h^2 + q_2 h^2 + q_3 h + q_4 = 0$ (31)
has no positive root.
Where $q_1 = 2\bar{P}^2 - \bar{R}^2$, $q_2 = \bar{P}^4 + 2\bar{R}^2\bar{S} - \bar{Q}^2 - \bar{P}^2\bar{R}^2 - 2\bar{S}^2$
$q_3 = 4\bar{P}\bar{Q}\bar{R}\bar{S} - 2\bar{Q}^2\bar{S} - 2\bar{P}^2\mathrm{S}^2 - \bar{P}^2\bar{Q}^2 - \bar{R}^2\bar{S}^2$, $q_4 = \bar{S}^2(\bar{S}^2 - \bar{Q}^2)$
$\bar{P} = \frac{-c_2 m_3 \bar{z}}{e_2(1+m_3\bar{x})}$, $\bar{Q} = \left(\frac{1}{\bar{x}} - m_3 m_4\bar{z}\right)\frac{c_2^3\bar{z}}{e_2^2 a_2\bar{x}}$,and $\bar{R} = \frac{r\bar{x}}{K} + \frac{C_2 m_4 \bar{z}}{(1+m_4\bar{z})}$, $\bar{S} = \frac{c_2 m_4 r\bar{x}\bar{z}}{K(1+m_4\bar{z})}$.

Theorem 3.5: Suppose that (30a) and (30b) hold.
If $\frac{m_3 m_4 \bar{z}}{e_2^2 a_2 \bar{x}} < \frac{m_4 r\bar{x}}{K c_2^2(1+m_4\bar{z})} < \left(\frac{1}{\bar{x}} - m_1 m_2\bar{z}\right)\frac{1}{e_2^2 a_2\bar{x}}$ and sign $\frac{g'(\overline{\omega}_0)}{\left({\overline{\omega}_0}^4 + \bar{P}^2{\overline{\omega}_0}^2 - \bar{S}^2\right)}$ positive, then the two species equilibrium point $E_3 = (\bar{x}, 0, \bar{z})$ of the system (1) is locally asymptotically stable for $\tau \in (0, \overline{\tau}_0)$.
And the system (1) undergoes a Hopf-bifurcation at equilibrium point $E_3 = (\bar{x}, 0, \bar{z})$ when $\tau = \overline{\tau}_0$.

To study the local stability of positive equilibrium point $E_4 = (\hat{x}, \hat{y}, \hat{z})$ we have to find the Jacobian matrix $J(t)$ of $x(t), y(t), z(t)$, with variable t at the equilibrium point $E_4 = (\hat{x}, \hat{y}, \hat{z})$:

$$J_{(t)}(E_4) = \begin{bmatrix} \frac{c_1 m_1 \hat{y}}{e_1(1+m_1\hat{x})} + \frac{c_2 m_3 \hat{z}}{e_2(1+m_3\hat{x})} - \frac{r}{k}\hat{x}e^{-\lambda\tau} & \frac{-c_1}{e_1(1+m_2\hat{y})} & \frac{-c_2}{e_2(1+m_4\hat{z})} \\ \frac{c_1\hat{y}}{(1+m_1\hat{x})\hat{x}}e^{-\lambda\tau} & \frac{-c_1 m_2 \hat{y}}{1+m_2\hat{y}}e^{-\lambda\tau} & 0 \\ \frac{c_2\hat{z}}{(1+m_3\hat{x})\hat{x}}e^{-\lambda\tau} & 0 & \frac{-c_2 m_4\hat{z}}{1+m_4\hat{z}}e^{-\lambda\tau} \end{bmatrix}$$

Now let $u(t) = x(t) + \hat{x}$, $v(t) = y(t) + \hat{y}$, $w(t) = z(t) + \hat{z}$, then the linearization of system (1) become:

$$\frac{du(t)}{dt} = \left(\frac{c_1 m_1 \hat{y}}{e_1(1+m_1\hat{x})} + \frac{c_2 m_3 \hat{z}}{e_2(1+m_3\hat{x})}\right)u(t) - \frac{r}{k}\hat{x}u(t-\tau) - \frac{c_1}{e_1(1+m_2\hat{y})}v(t) - \frac{c_2}{e_2(1+m_4\hat{z})}w(t)$$
$$\frac{dv(t)}{dt} = \frac{c_1\hat{y}}{(1+m_1\hat{x})\hat{x}}u(t-\tau) - \frac{c_1 m_1\hat{y}}{1+m_2\hat{y}}v(t-\tau)$$
$$\frac{d\omega(t)}{dt} = \frac{c_2\hat{z}}{(1+m_3\hat{x})\hat{x}}u(t-\tau) - \frac{c_2 m_4\hat{z}}{1+m_4\hat{z}}w(t-\tau).$$

We choose the following Lyapunov-krosovskii functional candidate as following:

$$V(t) = u^2(t) + v^2(t) + w^2(t) + \int_{t-\tau}^{t} u^2(s)ds + \int_{t-\tau}^{t} v^2(s)ds + \int_{t-\tau}^{t} w^2(s)ds$$

Calculating the derivative of $V(t)$ along the trajectories then we get:

$$\frac{dV(t)}{dt} = 2u(t)\frac{du(t)}{dt} + 2v(t)\frac{dv(t)}{dt} + 2w(t)\frac{dw(t)}{dt} + u^2(t) - u^2(t-\tau) + v^2(t) - v^2(t-\tau) + w^2(t) - W^2(t-\tau)$$

$$\frac{dV(t)}{dt} = 2\left(\frac{c_1 m_1\hat{y}}{e_1(1+m_1\hat{x})\hat{x}} + \frac{c_2 m_3\hat{z}}{e_2(1+m_3\hat{x})}\right)u^2(t) - \frac{2r}{k}x^2u(t)u(t-\tau) - 2\frac{c_1}{e_1(1+m_2\hat{y})}u(t)v(t)$$
$$-2\frac{c_2}{e_2(1+m_4\hat{z})}u(t)w(t) + 2\frac{c_1\hat{y}}{(1+m_1\hat{x})\hat{x}}v(t)u(t-\tau) - 2\frac{c_1 m_1\hat{y}}{(1+m_2\hat{y})}v(t)v(t-\tau)$$
$$+2\frac{c_2\hat{z}}{(1+m_3\hat{x})\hat{x}}w(t)u(t-\tau) - 2\frac{c_2 m_4\hat{z}}{1+m_4\hat{z}}w(t)w(t-\tau) + u^2(t) - u^2(t-\tau) + v^2(t)$$
$$-v^2(t-\tau) + w^2(t) - w^2(t-\tau)$$

$$\frac{dV(t)}{dt} = 2Au^2(t) - 2Bu(t)u(t-\tau) - 2Cu(t)v(t) - 2Du(t)w(t) + 2Fv(t)u(t-\tau)$$
$$-2Gv(t)v(t-\tau) + 2Hw(t)u(t-\tau) - 2Mw(t)w(t-\tau) + u^2(t) - u^2(t-\tau) + v^2(t)$$
$$-v^2(t-\tau) + \omega^2(t) - \omega^2(t-\tau)$$

Where $A = \frac{c_1 m_1\hat{y}}{e_1(1+m_1\hat{x})\hat{x}} + \frac{c_2 m_3\hat{z}}{e_2(1+m_3\hat{x})}$, $B = \frac{2r}{k}\hat{x}$, $C = \frac{c_1}{e_1(1+m_2\hat{y})}$, $D = \frac{c_2}{e_2(1+m_4\hat{z})}$,
$F = \frac{c_1\hat{y}}{(1+m_1\hat{x})\hat{x}}$, $G = \frac{c_1 m_1\hat{y}}{1+m_2\hat{y}}$, $H = \frac{c_2\hat{z}}{(1+m_3\hat{x})\hat{x}}$, $M = \frac{c_2 m_4\hat{z}}{1+m_4\hat{z}}$

$$\frac{d\mathrm{V}(t)}{dt} = (2A+1)u^2(t) - 2Bu(t)u(t-\tau) - 2cu(t)v(t) - 2Du(t)w(t) + 2Fv(t)u(t-\tau)$$
$$- 2Gv(t)v(t-\tau) + 2Hw(t)u(t-\tau) - 2Mw(t)w(t-\tau) - u^2(t-\tau)$$
$$+ v^2(t) - v^2(t-\tau) + \omega^2(t) - \omega^2(t-\tau)$$

$\frac{dV(t)}{dt} = PRP^{\top}$; where $P = [u(t), v(t), w(t), u(t-\tau), v(t-\tau), w(t-\tau)]$ and

$$R = \begin{bmatrix} 2A+1 & -C & -D & -B & 0 & 0 \\ -C & 1 & 0 & F & -G & 0 \\ -D & 0 & 1 & H & 0 & -M \\ -B & F & H & -1 & 0 & 0 \\ 0 & -G & 0 & 0 & -1 & 0 \\ 0 & 0 & -M & 0 & 0 & -1 \end{bmatrix}$$

If $R < 0$, then the positive equilibrium point $E_4 = (\hat{x}, \hat{y}, \hat{z})$ is locally asymptotically stable [4].

4. Global stability of equilibria

We will discuss the global stability of equilibria in this section by constructing a suitable Lyapunov function and applying LaSalle's invariance principle.

Theorem 4.1 : Assume that the predators-free equilibrium point $E_1 = (K, 0,0)$ of the system (1) is locally asymptotically stable and let the following condition hold.

$$K < \min\left\{\frac{c_1\, G(x,y)}{e_1\, a_1\, \beta}\,,\ \frac{c_2\, H(x,z)}{e_2\, a_2}\right\} \quad (32)$$

where $G(x,y) = (1 + m_1 x)(1 + m_2 y)$ and $H(x,z) = (1 + m_3 x)(1 + m_4 z)$. Then E_1 is globally asymptotically stable.

Proof: Consider a suitable function: $V_1(x,y,z) = \left(x - K - K\ln\left(\frac{x}{K}\right)\right) + \frac{1}{e_1} y + \frac{1}{e_2} z$ (33)

Here, $V_1(x,y,z)$ is a positive definite function $\forall (x,y,z) \in R^3_+$. Then differentiating equation (33) with respect to t, we get:

$$\frac{dV_1(t)}{dt} = \left(1 - \frac{K}{x}\right)\left(rx\left(1 - \frac{x(t-\tau)}{K}\right) - \frac{a_1\,\beta\, x\, y}{G(x,y)} - \frac{a_2\, x\, z}{H(x,z)}\right) + \frac{1}{e_1}\left(\frac{e_1\, a_1\,\beta\, x(t-\tau)}{G_\tau(x,y)} - c_1\right) y$$
$$+ \frac{1}{e_2}\left(\frac{e_2\, a_2\; x(t-\tau)}{H_\tau(x,z)} - c_2\right) z$$

Where $G_\tau(x,y) = \left(1 + m_1\, x(t-\tau)\right)\left(1 + m_2\, y(t-\tau)\right)$ and $H_\tau(x,z) = \left(1 + m_3\, x(t-\tau)\right)\left(1 + m_4\, z(t-\tau)\right)$

$$\frac{dV_1(t)}{dt} = \frac{r}{K}(x - K)\left(K - x(t-\tau)\right) - \frac{a_1\,\beta\, x\, y}{G(x,y)} - \frac{a_2\, x\, z}{H(x,z)} + \frac{a_1\,\beta\, K\, y}{G(x,y)} + \frac{a_2\, K\, z}{H(x,z)} + \frac{a_1\,\beta\, y\, x(t-\tau)}{G_\tau(x,y)}$$
$$+ \frac{a_2\, z\, x(t-\tau)}{H_\tau(x,z)} - \frac{c_1}{e_1} y - \frac{c_2}{e_2} z$$

And for $V_2(x,y,z) = a_1\,\beta \int_{t-\tau}^{t} \frac{y\, x(s)}{G(x(s),y(s))} ds + a_2 \int_{t-\tau}^{t} \frac{z\, x(s)}{H(x(s),z(s))} ds$

$$\frac{dV_2(t)}{dt} = a_1\,\beta\left(\frac{x\, y}{G(x,y)} - \frac{y\, x(t-\tau)}{G_\tau(x,y)}\right) + a_2\left(\frac{x\, z}{H(x,z)} - \frac{z\, x(t-\tau)}{H_\tau(x,z)}\right)$$

Let $V_3(x,y,z) = V_1(x,y,z) + V_2(x,y,z)$ (34)

Here, $V_3(x,y,z)$ is a positive definite function $\forall (x,y,z) \in R^3_+$. Then differentiating equation (34) with respect to t, we get:

$$\frac{dV_3(t)}{dt} = \frac{r}{K}(x - \mathrm{K})\left(K - x(t-\tau)\right) - \frac{a_1\,\beta\, x\, y}{G(x,y)} - \frac{a_2\, x\, z}{H(x,z)} + \frac{a_1\,\beta\, K\, y}{G(x,y)} + \frac{a_2\, K\, z}{H(x,z)} + \frac{a_1\,\beta\, y\, x(t-\tau)}{G_\tau(x,y)}$$
$$+ \frac{a_2\, z\, x(t-\tau)}{H_\tau(x,z)} - \frac{c_1}{e_1} y - \frac{c_2}{e_2} z + \frac{a_1\,\beta\, x\, y}{G(x,y)} - \frac{a_1\,\beta\, y\, x(t-\tau)}{G_\tau(x,y)} + \frac{a_2\, x\, z}{H(x,z)} - \frac{a_2 z\, x(t-\tau)}{H_\tau(x,z)}$$
$$= \frac{-r}{K}(x - K)\left(x(t-\tau) - K\right) + \left(\frac{a_1\,\beta\, K}{G(x,y)} - \frac{c_1}{e_1}\right) y + \left(\frac{a_2\, K}{H(x,z)} - \frac{c_2}{e_2}\right) z < 0$$

Under condition (32), then it is globally asymptotically stable.

Theorem 4.2 : Assume that the prey-predators one equilibrium point $E_2 = (x^*, y^*, 0)$ of the system (1) is locally asymptotically stable in the Int R^3_+ and if the following conditions hold

$$\frac{G^*\left(G(x,y) - G^*\right) \mathrm{y}}{y^*\left(G(x,y)\right)^2} \geq 1\,, \quad (35)$$

$$\frac{e_2\, a_2\, x^*}{H(x,z)} < c_2\,,$$

(36)

then E_2 is globally asymptotically stable.

Proof: We construct the suitable Lyapunov function as follows:

$$V_4(x,y,z) = x - x^* - x^*\ln\left(\frac{x}{x^*}\right) + \frac{1}{e_1}\left(y - y^* - y^*\ln\left(\frac{y}{y^*}\right)\right) + \frac{1}{e_2} z. \quad (37)$$

Here, $V_4(x,y,z)$ is a positive definite function $\forall (x,y,z) \in R^3_+$. Then differentiating equation (37) with respect to t, we get:

$$\frac{dV_4(t)}{dt} = \left(1-\frac{x^*}{x}\right)\left(rx\left(1-\frac{x(t-\tau)}{K}\right)-\frac{a_1\,\beta\,x\,y}{G(x,y)}-\frac{a_2\,x\,z}{H(x,z)}\right)$$
$$+\frac{1}{e_1}\left(1-\frac{y^*}{y}\right)\left(\frac{e_1\,a_1\,\beta\;x(t-\tau)}{G_\tau\,(x,y)}-c_1\right)y+\frac{1}{e_2}\left(\frac{e_2\,a_2\;x(t-\tau)}{H_\tau\,(x,z)}-c_2\right)z$$
$$\frac{dV_4(t)}{dt} = \left(1-\frac{x^*}{x}\right)\left(rx\left(1-\frac{x(t-\tau)}{K}\right)-rx^*\left(1-\frac{x^*}{K}\right)+rx^*\left(1-\frac{x^*}{K}\right)-\frac{a_1\beta xy}{G(x,y)}-\frac{a_2xz}{H(x,z)}\right)$$
$$+\frac{1}{e_1}\left(1-\frac{y^*}{y}\right)\left(\frac{e_1\,a_1\,\beta\,x(t-\tau)}{G_\tau\,(x,y)}-c_1\right)y+\frac{1}{e_2}\left(\frac{e_2\,a_2\,x(t-\tau)}{G_\tau\,(x,z)}-c_2\right)z$$
$$=\frac{r}{K}(x-x^*)\left(K-x(t-\tau)\right)-rx^*\left(1-\frac{x^*}{x}\right)\left(1-\frac{x^*}{K}\right)+\frac{a_1\,\beta\,x^*\,y^*}{G^*}$$
$$-\frac{a_1\,\beta\,x\,y}{G(x,y)}-\frac{a_2\,x\,z}{H(x,z)}-\frac{x^*\,a_1\,\beta\,x^*y^*}{x\,G^*}+\frac{a_1\,\beta\,x^*y}{G(x,y)}+\frac{a_2\,x^*z}{H(x,z)}+\frac{a_1\,\beta\,y\,x(t-\tau)}{G_\tau(x,y)}$$
$$-\frac{a_1\,\beta\,x^*y}{G^*}-\frac{a_1\,\beta\,y^*\,x(t-\tau)}{G_\tau\,(x,y)}+\frac{a_1\,\beta\,x^*y^*}{G^*}+\frac{a_2\,z\,x(t-\tau)}{H_\tau\,(x,z)}-\frac{c_2}{e_2}z$$

Consider $V_5\,(x,y,z)=\;a_1\,\beta\int_{t-\tau}^{t}\left(\frac{x(s)y}{G(x(s),y(s))}-\frac{x^*y^*}{G^*}-\frac{x^*y^*}{P^*}\ln\left(\frac{G^*\,x(s)y}{x^*\,y^*\,G(x(s),y(s))}\right)\right)ds$

$$+a_2\int_{t-\tau}^{t}\frac{x(s)\,z}{H(x(s),z(s))}ds\,. \tag{38}$$

Here, $V_5(x,y,z)$ is a positive definite function $\forall(x,y,z)\in R_+^3$. Then differentiating equation (38) with respect to t, we get:

$$\frac{d\,V_5\,(t)}{dt}=\;a_1\,\beta\left[\frac{x\,y}{G(x,y)}-\frac{y\,x(t-\tau)}{G_\tau\,(x,y)}+\frac{x^*y^*}{G^*}\ln\left(\frac{G(x,y)\,x(t-\tau)}{x\,G_\tau\,(x,y)}\right)\right]+\frac{a_2\,x\,z}{H(x,z)}-\frac{a_2\,z\,x(t-\tau)}{H_\tau\,(x,z)}$$

Let $V_6\,(x,y,z)=V_4\,(x,y,z)+V_5\,(x,y,z)$. (39)

Then differentiating equation (39) with respect to t:

$$\frac{dV_6(t)}{dt}=\frac{r}{K}(x-x^*)(K-x(t-\tau))-rx^*\left(1-\frac{x^*}{x}\right)\left(1-\frac{x^*}{K}\right)+\frac{a_1\,\beta\,x^*\,y^*}{G^*}-\frac{a_1\,\beta\,x\,y}{G(x,y)}-\frac{a_2\,x\,z}{H(x,z)}$$
$$-\frac{x^*\,a_1\,\beta\,x^*\,y^*}{x\,G^*}+\frac{a_1\,\beta\,x^*\,y}{G(x,y)}+\frac{a_2\,x^*\,z}{H(x,z)}+\frac{a_1\,\beta\,y\,x(t-\tau)}{G_\tau\,(x,y)}-\frac{a_1\,\beta\,x^*\,y}{G^*}-\frac{a_1\,\beta\,y^*\,x(t-\tau)}{G_\tau\,(x,y)}+\frac{a_1\,\beta\,x^*\,y^*}{G^*}$$
$$+\frac{a_2\,z\,x(t-t)}{H_\tau\,(x,z)}-\frac{c_2}{e_2}z+\frac{a_1\,\beta\,x\,y}{G(x,y)}-\frac{a_1\,\beta\,y\,x(t-\tau)}{G_\tau\,(x,y)}+\frac{a_1\,\beta\,x^*\,y^*}{G^*}\ln\left(\frac{G(x,y)\,x(t-\tau)}{x\,G_\tau\,(x,y)}\right)+\frac{a_2\,x\,z}{H(x,z)}-\frac{a_2\,z\,x(t-\tau)}{H_\tau\,(x,z)}$$
$$=\frac{r}{K}(x-x^*)(K-x(t-\tau))-rx^*\left(1-\frac{x^*}{x}\right)\left(1-\frac{x^*}{K}\right)$$
$$-\frac{a_1\,\beta\,x^*\,y^*}{G^*}\left(\frac{G^*\,x(t-\tau)}{x^*\,G_\tau\,(x,y)}-1-\ln\left(\frac{G^*\,x(t-\tau)}{x^*\,G_\tau\,(x,y)}\right)\right)-\frac{a_1\,\beta\,x^*\,y^*}{G^*}\left(\frac{x^*}{x}-1-\ln\left(\frac{x^*}{x}\right)\right)$$
$$-\frac{a_1\,\beta\,x^*\,y^*}{G^*}\left(\frac{y}{y^*}-1-\ln\left(\frac{y}{y^*}\right)\right)+\frac{a_1\,\beta x^*\,y^*}{G^*}\left(\frac{G^*\,y}{G(x,y)\,y^*}-1-\ln\left(\frac{G^*\,y}{G(x,y)\,y^*}\right)\right)$$
$$+\frac{1}{e_2}\left(\frac{e_2\,a_2\,x^*}{H(x,z)}-c_2\right)z$$
$$=\frac{-r}{K}(x-x^*)(x(t-\tau)-K)-rx^*\left(1-\frac{x^*}{x}\right)\left(1-\frac{x^*}{K}\right)$$
$$-\frac{a_1\,\beta\,x^*\,y^*}{G^*}\left(\frac{G^*\,x(t-\tau)}{x^*\,G_\tau\,(x,y)}-1-\ln\left(\frac{G^*\,x(t-\tau)}{x^*\,G_\tau\,(x,y)}\right)\right)-\frac{a_1\,\beta x^*y^*}{G^*}\left(\frac{x^*}{x}-1-\ln\left(\frac{x^*}{x}\right)\right)$$
$$-\frac{a_1\,\beta\,x^*\,y^*}{G^*}\left(\frac{y\,G^*\,(G(x,y)-G^*)}{y^*\left(G(x,y)\right)^2}\frac{G(x,y)}{G^*}-1-\ln\left(\frac{G(x,y)}{G^*}\right)\right)+\frac{1}{e_2}\left(\frac{e_2\,a_2\,x^*}{H(x,z)}-c_2\right)z<0$$

under condition (35) and (36), then it is globally asymptotically stable.

Theorem 4.3 : Assume that the two spices equilibrium point $E_3=(\bar{x},0,\bar{z})$ of the system (1) is locally asymptotically stable in the Int R_+^3 and if the following conditions hold

$$\frac{z\,\bar{H}\,(\,\mathrm{H}(x,z)-\bar{H}\,)}{\bar{z}\,(\,H(x,z))^2}\;\geq\;1. \tag{40}$$

$$\frac{e_1\,a_1\,\beta\;\bar{x}}{(1+m_3\,\bar{x}\,)}<c_1\,, \tag{41}$$

where $\bar{H}=(1+m_3\,\bar{x})(1+m_4\,\bar{z})$, then E_3 is globally asymptotically stable.
The proof is similar to the proof of Theorem (4.2) so it is omitted.

Theorem 4.4 : Suppose that the positive equilibrium point $E_4 = (\hat{x}, \hat{y}, \hat{z})$ is locally asymptotically stable in the Int R_+^3 then it is globally asymptotically stable if the following conditions hold

$$\frac{y\,\hat{G}\,(\,G(x,y) - \hat{G}\,)}{\hat{y}\,(\,G(x,y)\,)^2} \geq 1\,. \quad (42)$$

$$\frac{z\,\hat{H}\,(\,H(x,z) - \hat{H}\,)}{\hat{z}\,(\,H(x,z))^2} \geq 1, \quad (43)$$

where; $\hat{G} = (1 + m_1\,\hat{x})\,(1 + m_2\,\hat{y})$ and $\hat{H} = (1 + m_3\,\hat{x})\,(1 + m_4\,\hat{z})$.

Proof: We construct the suitable Lyapunov function as follows:

$$V_{10}(x,y,z) = \left(x - \hat{x} - \hat{x}\ln\left(\frac{x}{\hat{x}}\right)\right) + \frac{1}{e_1}\left(y - \hat{y} - \hat{y}\ln\left(\frac{y}{\hat{y}}\right)\right) + \frac{1}{e_2}\left(z - \hat{z} - \hat{z}\ln\left(\frac{z}{\hat{z}}\right)\right)$$

(44)

Here, $V_{10}\,(x,y,z)$ is a positive definite function $\forall (x,y,z) \in R_+^3$. Then differentiating equation (44) with respect to t, we get:

$$\frac{dV_{10}\,(t)}{dt} = \left(1 - \frac{\hat{x}}{x}\right)\frac{dx}{dt} + \frac{1}{e_1}\left(1 - \frac{\hat{y}}{y}\right)\frac{dy}{dt} + \frac{1}{e_2}\left(1 - \frac{\hat{z}}{z}\right)\frac{dz}{dt}$$

$$\frac{dV_{10}\,(t)}{dt} = \left(1 - \frac{\hat{x}}{x}\right)\left[rx\left(1 - \frac{x(t-\tau)}{K}\right) - \frac{a_1\,\beta\,x\,y}{G(x,y)} - \frac{a_2\,x\,z}{H(x,z)}\right]$$
$$+ \frac{1}{e_1}\left(1 - \frac{\hat{y}}{y}\right)\left(\frac{e_1\,a_1\,\beta\,y\,x(t-\tau)}{G_\tau\,(x,y)} - c_1\,y\right) + \frac{1}{e_2}\left(1 - \frac{\hat{z}}{z}\right)\left(\frac{e_2\,a_2\,z\,x(t-\tau)}{H_\tau\,(x,z)} - c_2\,z\right)$$

$$\frac{dV_{10}\,(t)}{dt} = \left(1 - \frac{\hat{x}}{x}\right)\left[rx\left(1 - \frac{x(t-\tau)}{K}\right) + r\hat{x}\left(1 - \frac{\hat{x}}{K}\right) - r\hat{x}\left(1 - \frac{\hat{x}}{K}\right) - \frac{a_1\,\beta\,x\,y}{G(x,y)} - \frac{a_2\,x\,z}{H(x,z)}\right]$$
$$+ \left(1 - \frac{\hat{y}}{y}\right)\left(\frac{a_1\,\beta\,y\,x(t-\tau)}{G_\tau\,(x,y)} - \frac{c_1}{e_1}y\right) + \left(1 - \frac{\hat{z}}{z}\right)\left(\frac{a_2\,z\,x(t-\tau)}{H_\tau\,(x,z)} - \frac{c_2}{e_2}z\right)$$
$$= \frac{\mathrm{r}}{k}(x - \hat{x})\big(K - x(t-\tau)\big) - r\hat{x}\left(1 - \frac{\hat{x}}{x}\right)\left(1 - \frac{\hat{x}}{K}\right) + \frac{a_1\,\hat{\beta}\,\hat{x}\hat{y}}{\hat{G}} + \frac{a_2\hat{x}\hat{z}}{\hat{H}}$$
$$- \frac{\hat{x}\,a_1\,\beta\,\hat{x}\,\hat{y}}{x\,\hat{G}} - \frac{\hat{x}\,a_2\,\hat{x}\,\hat{z}}{x\,\hat{H}} - \frac{a_1\,\beta\,x\,y}{G(x,y)} - \frac{a_2\,x\,z}{H(x,z)} + \frac{a_1\,\beta\,\hat{x}\,y}{G(x,y)} + \frac{a_2\,\hat{x}\,z}{H(x,z)} + \frac{a_1\,\beta\,y\,x(t-\tau)}{G_\tau\,(x,y)}$$
$$- \frac{a_1\,\beta\,\hat{x}\,y}{\hat{G}} - \frac{a_1\,\beta\,\hat{y}\,x(t-\tau)}{G_\tau\,(x,y)} + \frac{a_1\,\beta\,\hat{x}\,\hat{y}}{\hat{G}} + \frac{a_2\,z\,x(t-\tau)}{H_\tau\,(x,z)} - \frac{a_2\,\hat{x}\,z}{\hat{H}} - \frac{a_2\,\hat{z}\,x(t-\tau)}{H_\tau\,(x,z)} + \frac{a_2\,\hat{x}\,\hat{z}}{\hat{H}}$$

Now consider

$$V_{11}\,(x,y,z) = a_1\,\beta\int_{t-\tau}^{t}\left(\frac{x(s)\,y}{G(x(s),y(s))} - \frac{\hat{x}\,\hat{y}}{\hat{G}} - \frac{\hat{x}\,\hat{y}}{\hat{G}}\ln\left(\frac{\hat{G}\,x(s)y}{\hat{x}\,\hat{y}\,G(x(s),y(s))}\right)\right)ds$$
$$+ a_2\int_{t-\tau}^{t}\left(\frac{x(s)\,z}{H(x(s),z(s))} - \frac{\hat{x}\,\hat{z}}{\hat{H}} - \frac{\hat{x}\,\hat{z}}{\hat{H}}\ln\left(\frac{\hat{H}\,x(s)\,z}{\hat{x}\,\hat{z}\,H(x(s),z(s))}\right)\right)ds \quad (45)$$

Then differentiating equation (45) with respect to t:

$$\frac{dV_{11}(t)}{dt} = \frac{a_1\,\beta\,x\,y}{G(x,y)} - \frac{a_1\,\beta\,y\,x(t-\tau)}{G_\tau\,(x,y)} + \frac{a_1\,\beta\,\hat{x}\,\hat{y}}{\hat{G}}\ln\left(\frac{G(x,y)\,x(t-\tau)}{x\,G_\tau\,(x,y)}\right)$$
$$+ \frac{a_2\,x\,z}{H(x,z)} - \frac{a_2\,z\,x(t-\tau)}{H_\tau\,(x,z)} + \frac{a_2\,\hat{x}\,\hat{z}}{\hat{H}}\ln\left(\frac{H(x,z)x(t-\tau)}{x\,H_\tau\,(x,z)}\right).$$

Consider $V(x,y,z) = V_{10}(x,y,z) + V_{11}(x,y,z)$. (46)

By differentiating equation (46) with respect to t, we get:

$$\frac{dV(t)}{dt} = \frac{\mathrm{r}}{K}(x - \hat{x})(K - x(t-\tau)) - r\hat{x}\left(1 - \frac{\hat{x}}{x}\right)\left(1 - \frac{\hat{x}}{K}\right) + \frac{a_1\,\beta\,\hat{x}\,\hat{y}}{\hat{G}} + \frac{a_2\,\hat{x}\,\hat{z}}{\hat{H}}$$
$$- \frac{\hat{x}\;a_1\,\beta\,\hat{x}\,\hat{y}}{x\,\hat{G}} - \frac{\hat{x}\,a_2\,\hat{x}\,\hat{z}}{x\,\hat{H}} - \frac{a_1\,\beta\,x\,y}{G(x,y)} - \frac{a_2\,x\,z}{H(x,z)} + \frac{a_1\,\beta\,\hat{x}\,y}{G(x,y)} + \frac{a_2\,\hat{x}\,z}{H(x,z)} + \frac{a_1\,\beta\,y\,x(t-\tau)}{G_\tau\,(x,y)}$$
$$- \frac{a_1\,\beta\,\hat{x}\,y}{\hat{G}} - \frac{a_1\,\beta\,\hat{y}\,x(t-\tau)}{G_\tau\,(x,y)} + \frac{a_1\,\beta\,\hat{x}\,\hat{y}}{\hat{G}} + \frac{a_2\,z\,x(t-\tau)}{H_\tau\,(x,z)} - \frac{a_1\,\hat{x}\,z}{\hat{H}} - \frac{a_2\,\hat{z}\,x(t-\tau)}{H_\tau\,(x,z)}$$
$$+ \frac{a_1\,\hat{x}\,\hat{z}}{\hat{G}} + \frac{a_1\,\beta\,x\,y}{G(x,y)} - \frac{a_1\,\beta\,y\,x(t-\tau)}{G_\tau\,(x,y)} + \frac{a_1\,\beta\,\hat{x}\,\hat{y}}{\hat{G}}\ln\left(\frac{G(x,y)\,x(t-\tau)}{x\,G_\tau\,(x,y)}\right) + \frac{a_2\,x\,z}{H(x,z)}$$
$$- \frac{a_2\,z\,x(t-\tau)}{H_\tau\,(x,z)} + \frac{a_2\,\hat{x}\,\hat{z}}{\hat{H}}\ln\left(\frac{H(x,z)\,x(t-\tau)}{x\,H_\tau\,(x,z)}\right) = \frac{\mathrm{r}}{K}(x - \hat{x})(K - x(t-\tau)) - r\hat{x}\left(1 - \frac{\hat{x}}{x}\right)\left(1 - \frac{\hat{x}}{K}\right)$$
$$- \frac{a_1\,\beta\,\hat{x}\,\hat{y}}{\hat{G}}\left(\frac{\hat{G}\,x(t-\tau)}{\hat{x}\,G_\tau\,(x,y)} - 1 - \ln\left(\frac{\hat{G}\,x(t-\tau)}{\hat{x}\,G_\tau\,(x,y)}\right)\right) - \frac{a_2\,\hat{x}\,\hat{z}}{\hat{H}}\left(\frac{\hat{H}\,x(t-\tau)}{\hat{x}\,H_\tau\,(x,z)} - 1 - \ln\left(\frac{\hat{H}\,x(t-\tau)}{\hat{x}\,H_\tau\,(x,z)}\right)\right)$$

$$-\frac{a_1\,\beta\,\hat{x}\,\hat{y}}{\hat{G}}\left(\frac{\hat{x}}{x}-1-\ln\left(\frac{\hat{x}}{x}\right)\right)-\frac{a_2\,\hat{x}\,\hat{z}}{\hat{H}}\left(\frac{x}{\hat{x}}-1-\ln\left(\frac{\hat{x}}{x}\right)\right)$$
$$+\frac{a_1\,\beta\,\hat{x}\,\hat{y}}{\hat{G}}\left(\frac{\hat{G}\,y}{\hat{y}\,G(x,y)}-1-\ln\left(\frac{\hat{G}\,y}{\hat{y}\,G(x,y)}\right)\right)-\frac{a_1\,\beta\,\hat{x}\,\hat{y}}{\hat{G}}\left(\frac{y}{\hat{y}}-1-\ln\left(\frac{y}{\hat{y}}\right)\right)$$
$$+\frac{a_2\,\hat{x}\,\hat{z}}{\hat{H}}\left(\frac{\hat{H}\,z}{\hat{z}\,H(x,z)}-1-\ln\left(\frac{\hat{H}\,z}{\hat{z}\,H(x,z)}\right)\right)-\frac{a_2\,\hat{x}\,\hat{z}}{\hat{H}}\left(\frac{z}{\hat{z}}-1-\ln\left(\frac{z}{\hat{z}}\right)\right)$$
$$=\frac{-r}{K}(x-\hat{x})(x(t-\tau)-K)-r\hat{x}\left(1-\frac{\hat{x}}{x}\right)\left(1-\frac{\hat{x}}{K}\right)$$
$$-\frac{a_1\,\beta\,\hat{x}\,\hat{y}}{\hat{G}}\left(\frac{\hat{G}\,x(t-\tau)}{\hat{x}\,G_\tau\,(x,y)}-1-\ln\left(\frac{\hat{G}\,x(t-\tau)}{\hat{x}\,G_\tau\,(x,y)}\right)\right)-\frac{a_2\,\hat{x}\,\hat{z}}{\hat{H}}\left(\frac{\hat{H}\,x(t-\tau)}{\hat{x}\,H_\tau\,(x,z)}-1-\ln\left(\frac{\hat{H}\,x(t-\tau)}{\hat{x}\,H_\tau\,(x,z)}\right)\right)$$
$$-\frac{a_1\,\beta\,\hat{x}\,\hat{y}}{\hat{G}}\left(\frac{\hat{x}}{x}-1-\ln\left(\frac{\hat{x}}{x}\right)\right)-\frac{a_2\,\hat{x}\,\hat{z}}{\hat{H}}\left(\frac{x}{\hat{x}}-1-\ln\left(\frac{\hat{x}}{x}\right)\right)$$
$$-\frac{a_1\,\beta\,\hat{x}\,\hat{y}}{\hat{G}}\left(\frac{\hat{G}\,y\,(\,G(x,y)-\hat{G}\,)}{\hat{y}\,(G(x,y))^2}\frac{G(x,y)}{\hat{G}}-1-\ln\left(\frac{G(x,y)}{\hat{G}}\right)\right)-\frac{a_1\,\beta\,\hat{x}\,\hat{y}}{\hat{G}}$$
$$-\frac{a_2\,\hat{x}\,\hat{z}}{\hat{H}}\left(\frac{\hat{H}\,z\,(H(x,z)-\hat{H})}{\hat{z}\,(H(x,z))^2}\,\frac{H(x,z)}{\hat{H}}-1-\ln\left(\frac{H(x,z)}{\hat{H}}\right)\right)-\frac{a_2\,\hat{x}\,\hat{z}}{\hat{H}}$$

Then $\frac{dV(t)}{dt}\;<\;0$ if condition (42) and (43) are holds. Then it is globally asymptotically stable.

5.1 Numerical simulation

In order to illustrate the theoretical results for system (1), we choose the parameters as follows:

$r=1.4$, $K=30$, $a_1=0.6$, $\beta=0.85$, $m_1=0.76$, $m_2=2.5$, $a_2=0.45$, $m_3=0.875$, $m_4=2.5$, $e_1=0.3$, $c_1=0.2$, $e_2=0.4$, $c_2=0.2$, $\tau=0.6$ (47)

Then $K<\min\{\frac{c_1}{e_1\,a_1\,\beta-m_1\,c_1}\,,\,\frac{c_2}{e_2\,a_2\,-m_3\,c_2}\}$ and $r\tau<\frac{\pi}{2}$. Then the predators-free equilibrium point E_1= (30,0,0) is locally asymptotically stable. In Fig1, we have shown the phase portrait and time series of the system (1).

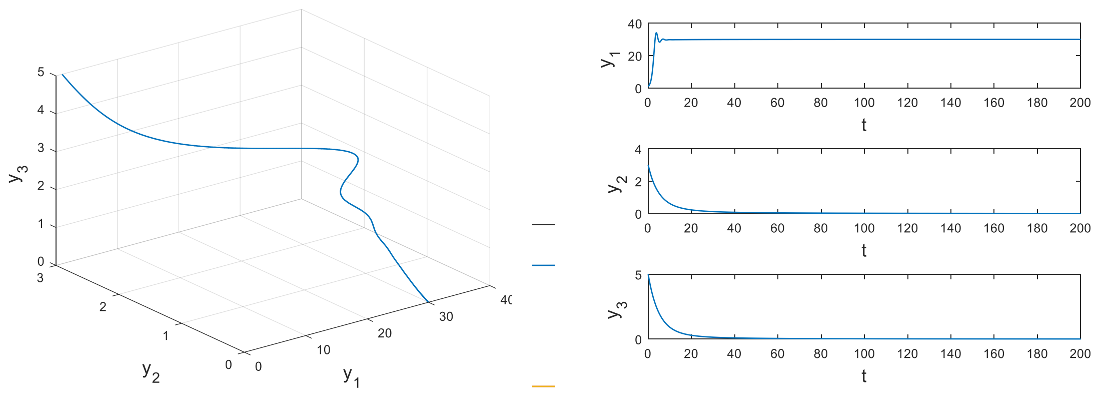

Figure1: Trajectories of the system (1) approaches asymptotically to the predators-free equilibriumpoint
E_1= (30,0,0) for the given data in equation (47) with initial point (1,3,5).

For the set value of parameters in equation (47) with change $m_1=0.3$, $c_1=0.04$ and fixed the value of other parameters, we can see that the system (1) has a prey-predator one equilibrium point $E_2=(29.6,4.184,0)$ which is locally asymptotically stable when $0<\tau<\tau_0^*=1.122$ and it is unstable if $\tau>\tau_0^*$, and it is undergoes a Hobf-bifurcation at E_2 with $\tau=$

τ_0^* . In Fig2, we have shown the phase portrait and time series of the system (1), respectively. Fig3 shows the periodic solution of E_2 with $\tau = 1.13$.

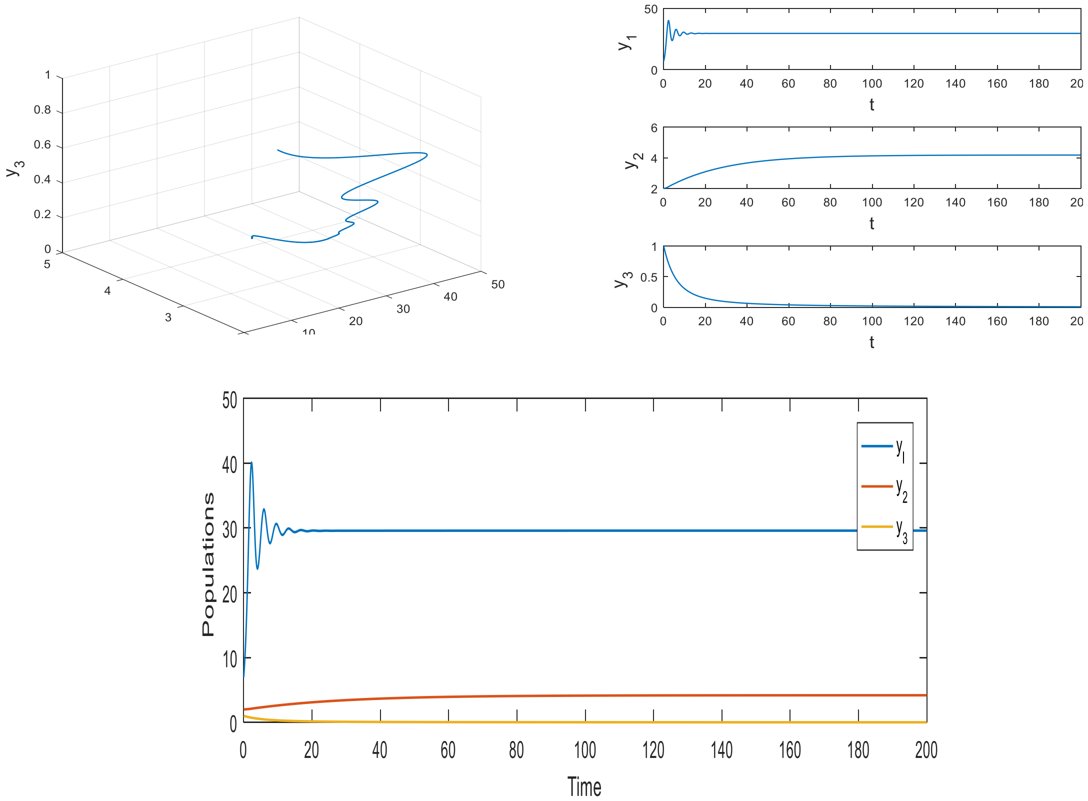

Figure 2: Trajectories of the system (1) approaches asymptotically to the prey-predator one equilibrium point E_2 with initial value (7,2,1) and $\tau = 1.05$.

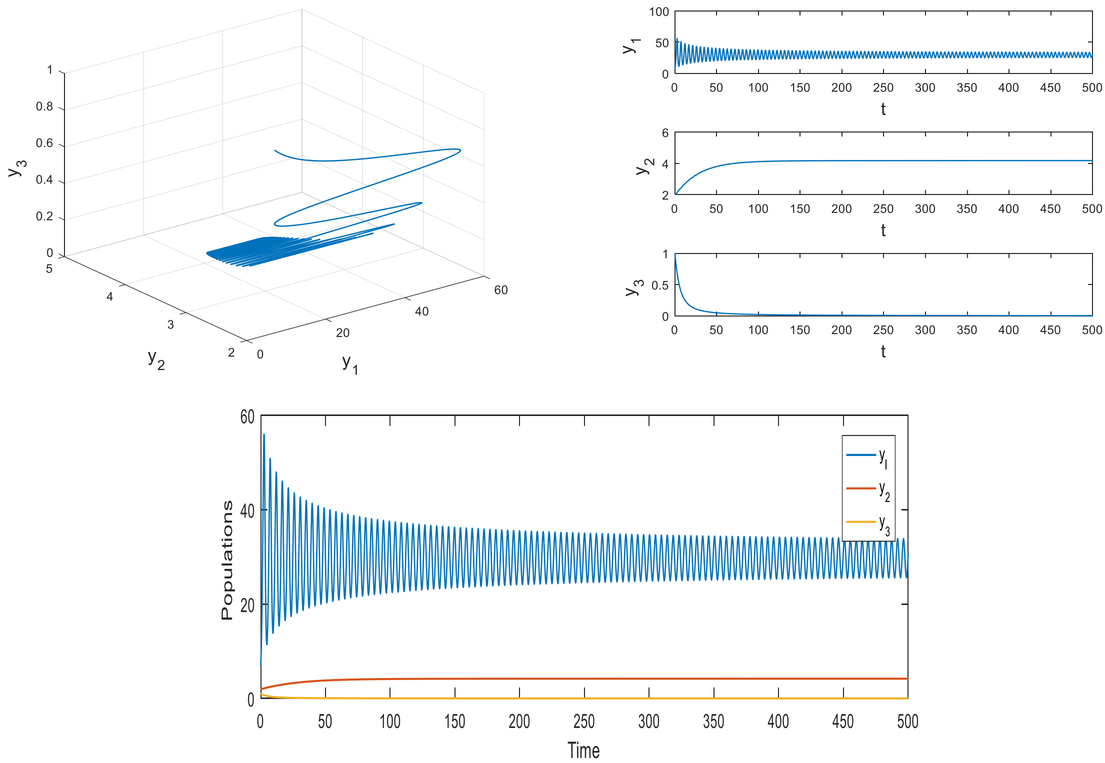

Figure 3: *Periodic solution of* E_2 *with* $\tau = 1.13$

Also, for the set of parameters in equation (47) with change $a_1 = 0.2$, $m_1 = 0.7$, $a_2 = 0.85$, $m_3 = 0.3$, $e_1 = 0.03$, $c_1 = 0.4$ and fixed the value of other parameters, the system (1) has two-species equilibrium point $E_3 = (29.4{,}0{,}1.636)$ which is locally asymptotically stable when $0 < \tau < \tau_a^* = 1.125$ and unstable if $\tau > \tau_a^*$. It is undergoes a Hopf-bifurcation at E_3 with $\tau = \tau_a^*$. In Fig4, we have shown the phase portrait and time series respectively of system (1). Fig5 shows the periodic solution of E_3 with $\tau = 1.14$.

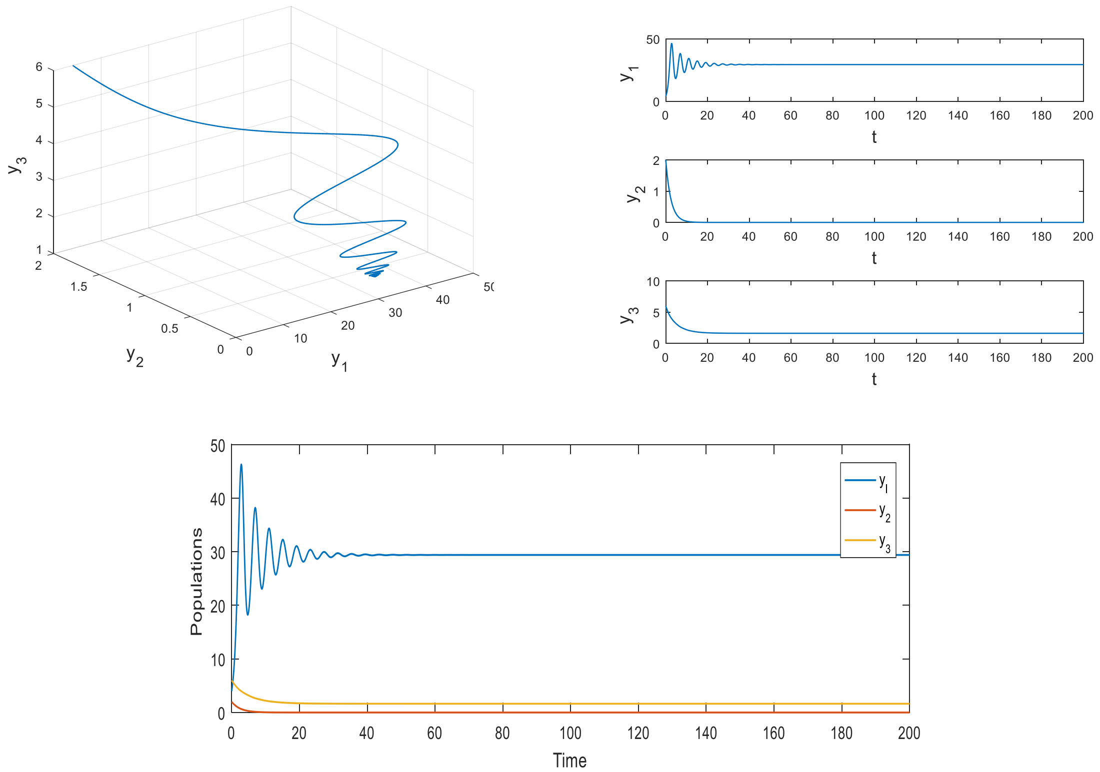

Figure 4: *Trajectories approaches asymptotically to the two-species equilibrium point* $E_3 =$ *(29.4,0,1.636 with initial value (4,2,6) and* $\tau = 0.95$.

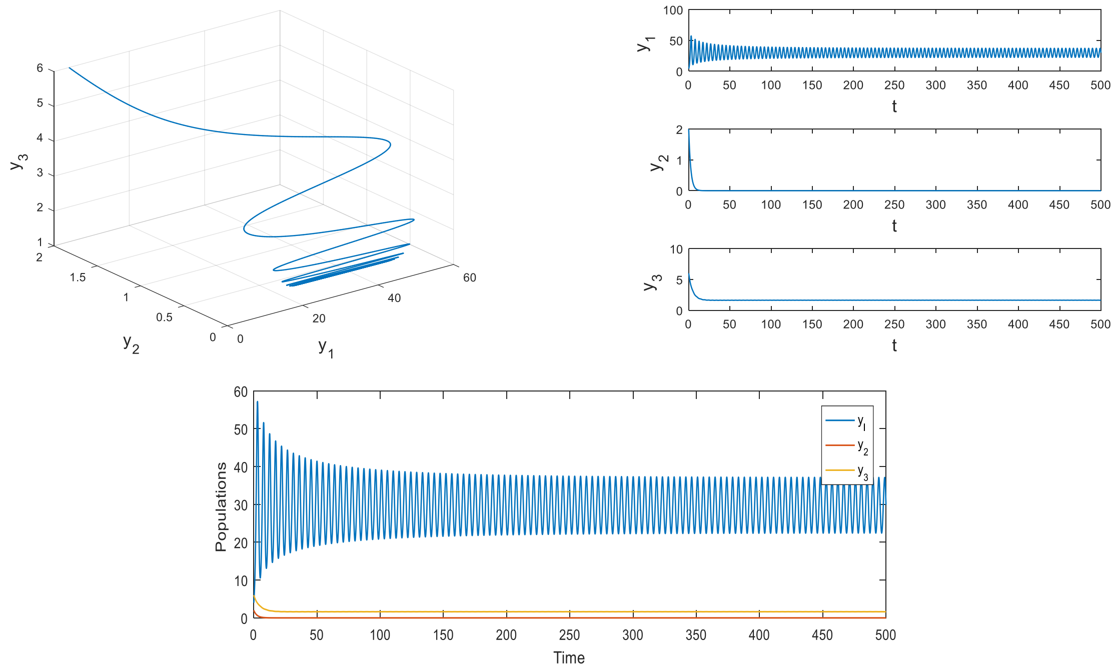

Figure 5: *Periodic solution for the equilibrium point* E_3 *with* $\tau = 1.14$.

Finally, for the set of parameters in equation (47) with change $a_1 = 0.4$, $m_1 = 0.02$, $m_3 = 0.02$, $e_1 = 0.8$, $c_1 = 0.1$ and fixed the value of other parameters, the system (1) has a positive equilibrium point $E_4 = (25.74, 18.09, 5.717)$ and satisfies the conditions that are indicated in equations (9), (11) and (12). It is asymptotically stable for $0 < \tau < \tau_p = 1.255$ and it is unstable for $\tau > 1.255$, it is undergoes a Hopf-bifurcation at E_4 with $\tau = \tau_p$. In Fig6, we have shown the phase portrait and time series respectively of the system (1). Fig7 shows the periodic solution of E_4 with $\tau = 1.26$.

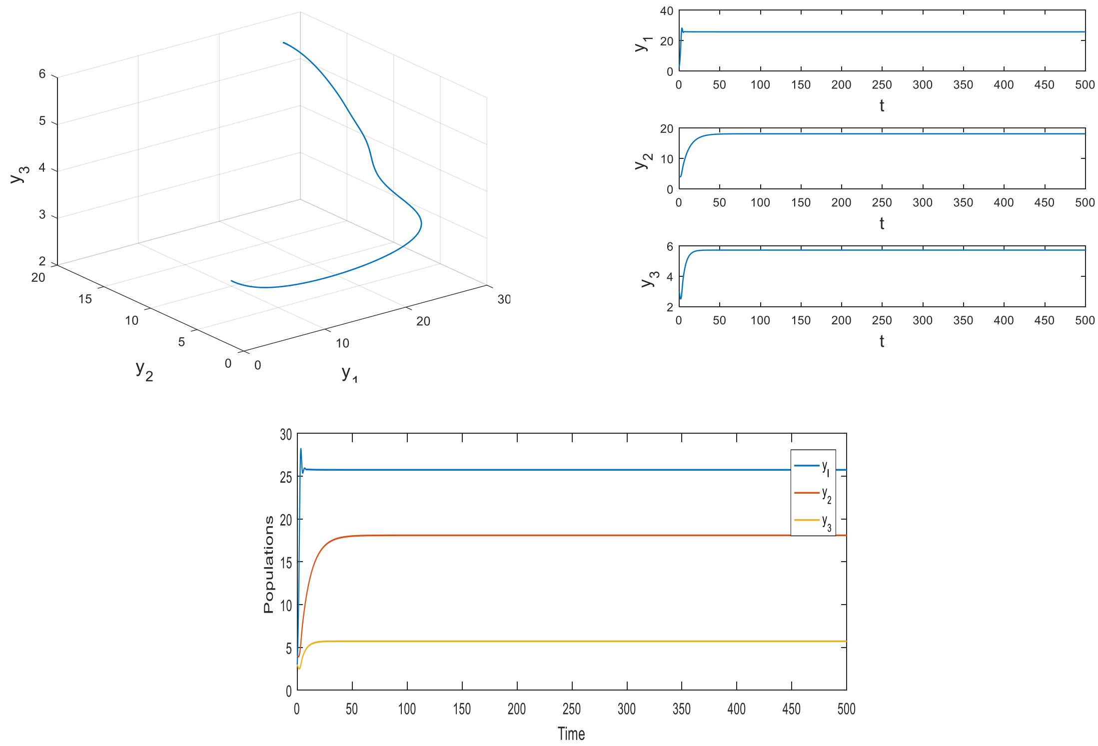

Figure 6: *Trajectories of the system (1) approaches asymptotically to positive equilibrium point E_4 with $\tau = 0.6$*

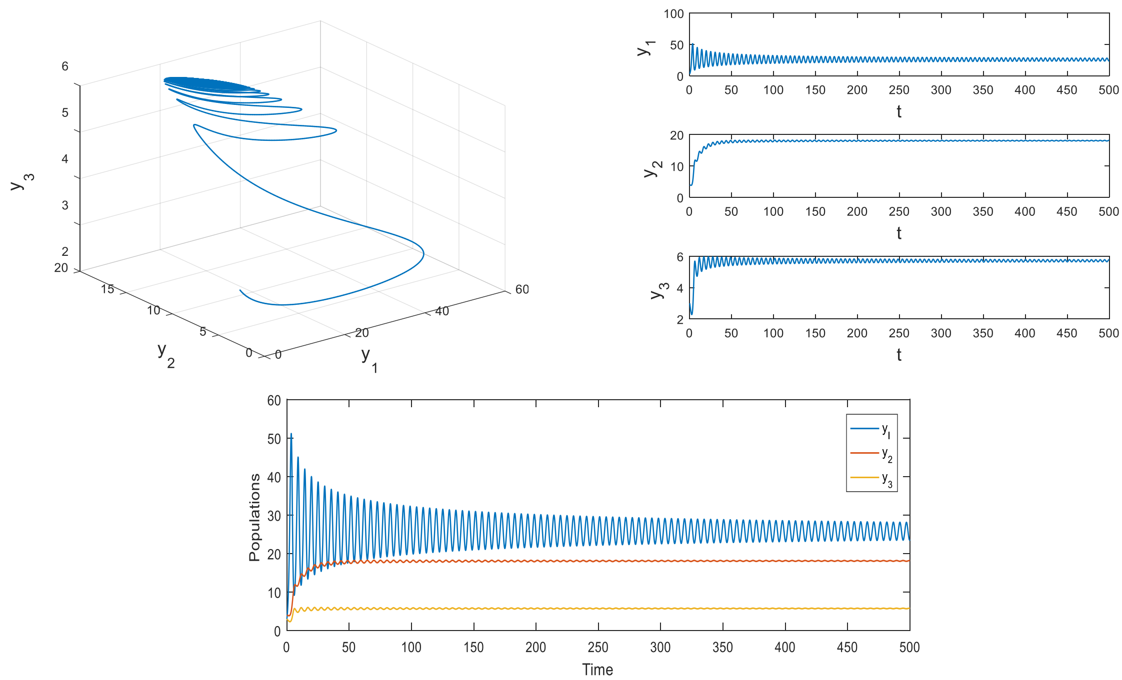

Figure 7: *Periodic solution for the positive equilibrium point, E_4 with $\tau = 1.26$.*

Conclusion:

A prey-predator model with time delay has been taken into consideration in this paper. Under some conditions, the existence and local stability have been derived. In addition, we see that when the time delay crosses some key levels, it can turn the stable equilibrium into an unstable one or even produce a Hopf bifurcation. It is shown that in numerical simulation. After that, we use an appropriate Lyapunov function and LaSalle's invariance principle to analyze the global stability of the equilibrium points under some sufficient conditions. Finally, numerical simulations of the system (1) are presented to illustrate our theoretical results.

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