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Classification of k-Sets in $PG(1,25)$, for $k = 4, \ldots, 13$

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Abstract
A k-set in the projective line is a set of k projectively distinct points. From the fundamental theorem over the projective line, all 3-sets are projectively equivalent. In this research, the inequivalent k-sets in $PG(1,25)$ have been computed and each k-set classified to its $(k-1)$-sets where $k = 5, \ldots, 13$. Also, the $PG(1,25)$ has been splitting into two distinct 13-sets, equivalent and inequivalent.

Keywords: Projective line, k-set.

تصنيف المجاميع-k في $PG(1,25)$ عندما $k = 4, \ldots, 13$

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الخلاصة
المجموعة-k في الخط الاسقاطي هي مجموعة من k من النقاط الاسقاطية المختلفة. من المبرهنة الاساسية على الخط الاسقاطي، كل المجموعات-3 هي متكافئة اسقاطيا. في هذا البحث، المجموعات-k الغير متكافئة في $PG(1,25)$ قد تم حسابها وكل واحدة من المجموعة - k صنف الى المجموعات - $(k-1)$ عندما $k = 5, \ldots, 13$. كذلك، الخط $PG(1,25)$ قد تم فصلها الى مجموعتين من المجموعات- 13، متكافئة وغير متكافئة.

1. Introduction

The structure of projective line over the finite field F_q, $PG(1,q)$, has been studied by many mathemations for small q. In 1998, the results about $PG(1,q)$for $2 \leq q \leq 13$ have been summarized by Hirschfeld in [1] where a full classification of $PG(1,11)$ has been done by Sadeh in [2] and of $PG(1,13)$ has been done by Ali in [3]. In [4], Al-Seraje gave a full classification of $PG(1,17)$ and gave the inequivalents k-sets only on $PG(1,16)$ and $PG(1,23)$ in [5, 6]. Al .Zangana in [7] studied the geometry of line of order nineteen and the conic, where a full classification and its application to error correcting codes have been given. Also, Al .Zangana using the relation between conic and projective line the spectrum sizes of k-sets on $PG(1,23)$ are given as a direct results from this relation in [8].
The aim of this research is to classify the projective line $PG(1,25)$ and then splitting the line into two 13-sets some of them are equivalent and others are not.

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2. Basic Definitions and Results

A projective line $PG(1,q)$ has $q+1$ points which are one-dimensional subspaces of a two-dimensional vector space $V(3,q)$ over the finite field F_q of q elements. These points also can be represented by $P(t_0,t_1), t_i \in F_q$.So,

$$PG(1,q) = \{P(t,1) \mid t \in F_q\} \cup \{P(1,0)\}.$$

Each point $P(t_0,t_1)$ with $t_0 \neq 0$ is determined by the non-homogeneous coordinate t_0/t_1. The coordinate for $P(1,0)$ is infinity, so the points of $PG(1,q)$ can be represented by the set

$$F_q \cup \{\infty\} = \{\infty, \lambda_1, \lambda_2, \dots, \lambda_q \mid \lambda_i \in F_q\}.$$

Definition 2.1[1]

A projectivity $PG(1,q)$ has given by 2×2 non-singular matrix A matrix F_q, denoted by $M(A)$, such that

$Y = AX$, where X=(x_0, x_1), Y=(y_0, y_1) and $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. If put $s = y_0/y_1$ and $t = x_0/x_1$, then the projectivity can be written as an equation

$$s = (at+b)/(ct+d).$$

Definition 2.2[1]

A k-set in the projective line $PG(1,q)$ is a set of k projectively distinct points.

Theorem 2.3[1].(The Fundamental Theorem of Projective Geometry)

If $\{P_0, \dots, P_{n+1}\}$ and $\{P'_0, \dots, P'_{n+1}\}$ are both subsets of $PG(n,q)$of cardinality $n+2$ such that no $n+1$ points chosen from the same set lie in a hyperplane, then there exists a unique projectivity τ such that $P'_i = P_i\tau$ for $i = 0, 1, \dots, n+1$.

According to above theorem in the projective line, all 3-sets are projectively equivalent.

The following groups occur in this work and for more details about them see [9].

Z_n = Cyclic group of order n.

V_4 = Klein 4- group which is the direct product of two copies of the cyclic group of order 2.

S_n = Symmetric group of degree n.

A_n = Alternating group of degree n.

D_n = Dihedral group of order $2n = < r, s | r^n = s^2 = (rs)^2 = 1 >$.

During this paper the notation SG-type is used for the stabilizer group type, *No.* for the number of reputation of that group and the symbol Ord(g) refers to order of group element g.

Definition 2.4[1]

The cross-ratio λ={P_1,P_2;P_3,P_4 } of four ordered points $P_1, P_2, P_3, P_4 \in PG(1,q)$ with coordinates t_1,t_2 ,t_3 ,t_4 is

$$\lambda = \{P_1,P_2;P_3,P_4\} = \{t_1,t_2;t_3,t_4\} = (t_1-t_3)(t_2-t_4)/(t_1-t_4)(t_2-t_3).$$

Lemma 2.5[1]

The Cross- ratio has the property that

(i) λ={t_1,t_2;t_3,t_4 }={t_2,t_1;t_4,t_3 }={t_3,t_4;t_1,t_2 }={t_4,t_3;t_2,t_1 }. So, {P_1, P_2; P_3,P_4 } is invariant under a projective group of order four, given by

$\{I, (P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3)\} \cong V_4$,

(ii) the cross-ratio takes just six value under all 24 permutations of {P_1,P_2 ,P_3 ,P_4 },

λ, $1/\lambda$, $1-\lambda$, $1/(1-\lambda)$, $(\lambda-1)/\lambda$, $\lambda/(\lambda-1)$,

(iii) λ={t_1,t_2;t_3,t_4 }takes the values ∞, 0 or 1 if and only if two of them are equal,

(iv) a projectivity is determined by the images of three points. Therefore, there exists a projectivity $T = M(A)$ such that $Q_i = P_iA, i = 1,2,3,4$ if and only if the cross-ratios of the two sets of four points in the corresponding order are equal.

During this research, a 3-set is called a triad, a4-sets is a tetrad, a 5-set a pentad, a 6-set a hexad, a 7-set a heptad, an 8-set an octad, a 9-set a nonad, a 10-set a decad.

Definition 2.6[1]

Let λ be the cross ratio of a given order of a tetrad. The tetrad is called

(i) harmonic, denoted by H, if λ=$1/\lambda$ or $\lambda = \lambda/(\lambda-1)$ or $\lambda = 1-\lambda$;

(ii) equianharmonic, denoted by E, if $\lambda = 1/(1-\lambda)$ or, equivalently,

$$\lambda = (\lambda - 1)/\lambda;$$

(iii) neither harmonic nor equianharmonic, denoted by N, if the cross-ratio is another value.

Lemma 2.7[1]

(i) The cross-ratio of any harmonic tetrad has the values $-1, 2, 1/2$.

(ii) The cross-ratio of a tetrad of type E satisfies the equation

$\lambda^2 + \lambda + 1 = 0.$ (1.1)

Therefore equianharmonic tetrads exist if and only if $\lambda^3 + 1 = 0$ has three solutions in F_q or $\lambda = -1$ is a unique solution of (1.1) in F_q.

In this research all tetrad containing the points ∞, 0, 1 because

1- the value ∞, 0, 1 cannot appear as the cross ratio of a tetrad whose four points are distinct,

2- three distinct points in $PG(1, q)$ are projectively equivalent.

the cross-ratio $\lambda = \{\infty, 0; 1, t\} = t$, it is necessary to consider the elements $t \in F_q/\{0,1\}$ and the corresponding tetrads $\{\infty, 0, 1, t\}$.

Hence there are three classes of tetrads:

$\chi_1 = \{\text{tetrads of type } H\}$,

$$\chi_2 = \{\text{tetrads of type } E\},$$
$$\chi_3 = \{\text{tetrads of type } N\}.$$

Lemma 2.8[1]

(i) in $PG(1, q)$, $q = p^h, p > 3$, the number of harmonic tetrads n_H is

$$q(q^2 - 1)/8$$

and the stabilizer group $G$of each one is D_4.

(ii) in $PG(1, q)$, $q \equiv 1 (\text{mod } 3)$, the number of equianharmonic tetrads n_E is

$$q(q^2 - 1)/12$$

and the stabilizer group G of each one is A_4.

(iii) The stabilizer group of any tetrads in χ_3 is of type V_4.

3. Algorithms

In this section, the algorithms that needed are described. Algorithm **A** describe the matrix transformation between two tetrads, Algorithm **B** describes the way to compute the inequivalent k-sets and Algorithm **C** describes the way to compute the stabilizer group of k-set.

Algorithm A

A projectivity $\mathcal{T} = M(A)$ in $PG(1, q)$ is given by the equation

$tY = XA$,

where $Y = (y_0, y_1)$, $X = (x_0, x_1)$, $A = (t_{ij})$, $t \in F_q \backslash \{0\}$; that is,

$x_0 t_{00} + x_1 t_{10} = t y_0$,

$$x_1 t_{10} + x_2 t_{11} = t y_1$$

Since any two triads are projective inequivalent to find a projectivity maps

$P(1,0)$to $P(a_0, a_1)$,

$P(0,1)$to $P(b_0, b_1)$,

$P(1,1)$to $P(c_0, c_1)$,

the following procedure can be used.

Let $\alpha, \rho \in F_q \backslash \{0\}$ and

$(1,0)A = \alpha(a_0, a_1)$,

$(0,1)A = \rho(b_0, b_1)$.

Then

$A = \begin{pmatrix} \alpha a_0 & \alpha a_1 \\ \rho b_0 & \rho b_1 \end{pmatrix}$.

Also , there is $\gamma \in F_q \backslash \{0\}$, such that$(1,1)A = \gamma(c_0, c_1)$. This gives a non- homogeneous system

$\begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \end{pmatrix} \begin{pmatrix} \alpha \\ \rho \end{pmatrix} = \begin{pmatrix} \gamma c_0 \\ \gamma c_1 \end{pmatrix}$, and this system has a unique solution given by

$$\frac{\alpha}{D_1}=\frac{\rho}{D_2}=\frac{\gamma}{D_3}$$

where

$D_1=\begin{vmatrix} c_0 & b_0 \\ c_1 & b_1 \end{vmatrix}, D_2=\begin{vmatrix} a_0 & c_0 \\ a_1 & c_1 \end{vmatrix}, D_3=\begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix}\neq 0.$

Thus,

$$\frac{D_3}{\gamma}A=\begin{pmatrix} D_1a_0 & D_1a_1 \\ D_2b_0 & D_2b_1 \end{pmatrix}$$

and $\tau=M(A)$.Therefore, the tetrad

$$K=\{P(1,0),P(0,1),P(1,1),P(k_0,k_1)\}$$

equivalent to

$$K^*=\{P(a_0,a_1),P(b_0,b_1),P(c_0,c_1),P(d_0,d_1)\}$$

if and only if

$(k_0,k_1)A=t(d_0,d_1)$, $t\in F_q\backslash\{0\}$.

Algorithm B

Input: A_k

Output: Λ_k

1: $A_{k+1}=\emptyset$

2: **for all A∈A_kdo**

3: **for all** $\boldsymbol{B}(\neq \boldsymbol{A})\in \boldsymbol{A\_kdo}$

4if CR(A)=CR(B) and |S_A |=|S_B| and $Clas(A)=Clas(B)$ **then** $Clas$(H) is $(k-1)$-set types of H

5: Construct matrix transformation T_i from the tetrad t^* of A to tetrads t_i of B

6: **if** $AT_i \nrightarrow B$ for all i **then**

7: Add B to Λ_k

8: **end if**

9: **end if**

10: **end for**

11: **end for**

Algorithm C.

Let $Par(A)$ be the set all distinct tetrads in a k-set A.

Input:A

Output:S_A

1: $S_A=\emptyset$

4: **for all**$t_i\in Par(A)$**do**

5: Construct matrix transformation T_i from $t^*\in A$ to tetrads t_i

6: **if**$AT_i\longrightarrow A$**then**

7: Add T_i to S_A

8: **end if**

9: **end for**

4. Classification of The Projective Line $\boldsymbol{PG(1,25)}$

Lemma 2.5 turns out that among the $\binom{26}{4}=14950$ defrents tetrads in $PG(1,25)$, there are exactly five classes of tetrads as shown below:

$M_1=$ {the class of H tetrads} $\{\infty,0,1,a\}$ for $a=\beta^6,\beta^{12},\beta^{18}$;

$M_2=$ {the class of Etetrads}$\{\infty,0,1,b\}$ for $b=\beta^4,\beta^{20}$;

$M_3=$ {the class of N_1tetrads} $\{\infty,0,1,c\}$ for $c=\beta,\beta^5,\beta^8,\beta^{16},\beta^{19},\beta^{23}$;

$M_4=$ {the class of N_2tetrads} $\{\infty,0,1,d\}$ for $d=\beta^2,\beta^{11},\beta^{13},\beta^{21},\beta^{22}$;

$M_5 =$ {the class of N_3tetrads}$\{\infty, 0,1, e\}$ for $e = \beta^7, \beta^9, \beta^{15}\beta^{14}, \beta^{10}, \beta^{17}$.
From Lemma2.6 deduced that $|M_1|= 1950$, $|M_2|= 1300$, $|M_3|= |M_4| = |M_5| = 3900$.
A represented one has been chosen from each class as shown below.
The tetrad $H = \{\infty, 0,1, \beta^{12}\}$ chosen from M_1.
The tetrad $E = \{\infty, 0,1, \beta^4\}$ chosen from M_2.
The tetrad $N_1 = \{\infty, 0,1, \beta\}$ chosen from M_3.
The tetrad $N_2 = \{\infty, 0,1, \beta^2\}$ chosen from M_4.
The tetrad $N_3 = \{\infty, 0,1, \beta^7\}$ chosen from M_5.
Theorem 4.1. On $PG(1,25)$, there are
(i) five projective distinct tetrads, see Table-1,
(ii) 8projectively distinct pentads, see Table-2,
(iii) 28projectively distinct hexads, see Table-3,
(iv) 54projectively distinct heptads, see Table -4,
(v) 131 projectively distinct octads, see Table-5,
(vi) 225projectively distinct nonads , see Table-6,
(vii) 398 projectively distinct decads , see Table-7,
(viii) 531 projectively distinct t 11-sets, see Table-8,
(ix) 692 projectively distinct 12- sets, see Table-9,
(x) 714 projectively distinct 13- sets, see Table-10.

Table 1- Distinct tetrads on $PG(1,25)$

Type	The tetrads	SG-type
H	$\{\infty,0,1,\beta^{12}\}$	$D_4 = \langle(\beta^{12}t + \beta^{12})/(t + \beta^{12}), 1/\beta^{12}t\rangle$
E	$\{\infty, 0,1, \beta^4\}$	$A_4 = \langle(\beta^8 t + 1), \beta^4/t\rangle$
N_1	$\{\infty, 0,1, \beta\}$	$V_4 = \langle\beta/t, (\beta^{12}t + 1)/(\beta^{11}t + 1)\rangle$
N_2	$\{\infty, 0,1, \beta^2\}$	$V_4 = \langle\beta^2/t, (\beta^{12}t + 1)/(\beta^{10}t + 1)\rangle$
N_3	$\{\infty, 0,1, \beta^7\}$	$V_4 = \langle\beta^7/t, (\beta^{12}t + 1)/(\beta^5 t + 1)\rangle$

Table 2-Inequivalent pentads

Type	The pentads	SG-type
$\mathcal{P}_1$	$\{\infty ,0,1 ,\beta^{12},\beta^6\}$	$Z_5 \rtimes Z_4 = \langle 1/(t + \beta^{12}), (t\beta^{18} + \beta^{12})\rangle$
$\mathcal{P}_2$	$\{\infty,0 ,1 ,\beta^{12},\beta\}$	I
$\mathcal{P}_3$	$\{\infty, 0,1,\beta^{12},\beta^2\}$	$Z_2 = \langle(t + 1)/(t + \beta^{12})\rangle$
$\mathcal{P}_4$	$\{\infty, 0,1,\beta^{12},\beta^3\}$	I
$\mathcal{P}_5$	$\{\infty, 0,1, \beta^4,\beta^2\}$	$Z_2 = \langle\beta^4/t\rangle$
$\mathcal{P}_6$	$\{\infty, 0,1, \beta^4,\beta^5\}$	$S_3 = \langle(\beta^8 t + 1) , \beta^5 t/(t + \beta^{17})\rangle$
$\mathcal{P}_7$	$\{\infty, 0,1, \beta ,\beta^2\}$	$Z_2 = \langle\beta^2/t\rangle$
$\mathcal{P}_8$	$\{\infty, 0,1, \beta,\beta^8\}$	$Z_2 = \langle t/(t + \beta^{12})\rangle$

Table 3-Inequivalent of hexads

Type	The hexad	Type of pentads	SG-type
H_1	$\{\infty,0,1,\beta^{12},\beta^{6},\beta^{18}\}$	$\mathcal{P}_1\mathcal{P}_1\mathcal{P}_1\mathcal{P}_1\mathcal{P}_1\mathcal{P}_1$	$S_5=\langle(t+1),\beta^{12}t\rangle$
H_2	$\{\infty,0,1,\beta^{12},\beta^{6},\beta\}$	$\mathcal{P}_1\mathcal{P}_2\mathcal{P}_2\mathcal{P}_4\mathcal{P}_4\mathcal{P}_3$	I
H_3	$\{\infty,0,1,\beta^{12},\beta,\beta^{2}\}$	$\mathcal{P}_2\mathcal{P}_3\mathcal{P}_7\mathcal{P}_2\mathcal{P}_7\mathcal{P}_3$	$Z_2=\langle\beta^{12}(t+1)/(\beta^{11}t+1)\rangle$
H_4	$\{\infty,0,1,\beta^{12},\beta,\beta^{3}\}$	$\mathcal{P}_2\mathcal{P}_4\mathcal{P}_7\mathcal{P}_3\mathcal{P}_4\mathcal{P}_8$	I
H_5	$\{\infty,0,1,\beta^{12},\beta,\beta^{4}\}$	$\mathcal{P}_2\mathcal{P}_4\mathcal{P}_4\mathcal{P}\mathcal{P}_8\mathcal{P}_3$	I
H_6	$\{\infty,0,1,\beta^{12},\beta,\beta^{5}\}$	$\mathcal{P}_2\mathcal{P}_2\mathcal{P}_6\mathcal{P}_4\mathcal{P}_6\mathcal{P}_4$	$Z_2=\langle(t+1)/(t+\beta^{12})\rangle$
H_7	$\{\infty,0,1,\beta^{12},\beta,\beta^{7}\}$	$\mathcal{P}_2\mathcal{P}_2\mathcal{P}_4\mathcal{P}_4\mathcal{P}_4\mathcal{P}_4$	$Z_2=\langle(t+\beta^{12})/\beta^{12}(t+1)\rangle$
H_8	$\{\infty,0,1,\beta^{12},\beta,\beta^{8}\}$	$\mathcal{P}_2\mathcal{P}_4\mathcal{P}_8\mathcal{P}_4\mathcal{P}_2\mathcal{P}_8$	$Z_2=\langle(\beta^{12}t+\beta^{8})/(t+1)\rangle$
H_9	$\{\infty,0,1,\beta^{12},\beta,\beta^{9}\}$	$\mathcal{P}_2\mathcal{P}_4\mathcal{P}_2\mathcal{P}_7\mathcal{P}_7\mathcal{P}_4$	$Z_2=\langle(\beta^{12}t+1)/(\beta^{11}t+1)\rangle$
H_{10}	$\{\infty,0,1,\beta^{12},\beta,\beta^{10}\}$	$\mathcal{P}_2\mathcal{P}_3\mathcal{P}_8\mathcal{P}_3\mathcal{P}_8\mathcal{P}_2$	$Z_2=\langle(t+1)/(\beta^{14}t+\beta^{12})\rangle$
H_{11}	$\{\infty,0,1,\beta^{12},\beta,\beta^{11}\}$	$\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2$	$S_3=\langle(\beta^{11}(t+1)/(\beta^{11}t+1),1/\beta^{12}t\rangle$
H_{12}	$\{\infty,0,1,\beta^{12},\beta,\beta^{13}\}$	$\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2\mathcal{P}_8\mathcal{P}_8$	$V_4=\langle\beta^{12}t,\beta/t\rangle$
H_{13}	$\{\infty,0,1,\beta^{12},\beta,\beta^{14}\}$	$\mathcal{P}_2\mathcal{P}_3\mathcal{P}_2\mathcal{P}_7\mathcal{P}_2\mathcal{P}_8$	I
H_{14}	$\{\infty,0,1,\beta^{12},\beta,\beta^{15}\}$	$\mathcal{P}_2\mathcal{P}_4\mathcal{P}_8\mathcal{P}_5\mathcal{P}_6\mathcal{P}_8$	I
H_{15}	$\{\infty,0,1,\beta^{12},\beta,\beta^{16}\}$	$\mathcal{P}_2\mathcal{P}_4\mathcal{P}_2\mathcal{P}_5\mathcal{P}_4\mathcal{P}_5$	$Z_2=\langle(t+\beta^{13})/(t+\beta^{12})\rangle$
H_{16}	$\{\infty,0,1,\beta^{12},\beta,\beta^{20}\}$	$\mathcal{P}_2\mathcal{P}_4\mathcal{P}_6\mathcal{P}_7\mathcal{P}_7\mathcal{P}_5$	I
H_{17}	$\{\infty,0,1,\beta^{12},\beta,\beta^{21}\}$	$\mathcal{P}_2\mathcal{P}_4\mathcal{P}_4\mathcal{P}_5\mathcal{P}_2\mathcal{P}_5$	$Z_2=\langle(t+\beta^{9})/(t+\beta^{12})\rangle$
H_{18}	$\{\infty,0,1,\beta^{12},\beta,\beta^{22}\}$	$\mathcal{P}_2\mathcal{P}_3\mathcal{P}_7\mathcal{P}_5\mathcal{P}_8\mathcal{P}_5$	I
H_{19}	$\{\infty,0,1,\beta^{12},\beta,\beta^{23}\}$	$\mathcal{P}_2\mathcal{P}_2\mathcal{P}_7\mathcal{P}_7\mathcal{P}_2\mathcal{P}_2$	$V_4=\langle1/t,(\beta t+\beta^{12})/(t+\beta^{13})\rangle$
H_{20}	$\{\infty,0,1,\beta^{12},\beta^{2},\beta^{4}\}$	$\mathcal{P}_3\mathcal{P}_4\mathcal{P}_5\mathcal{P}_4\mathcal{P}_3\mathcal{P}_5$	$Z_2=\langle(\beta^{12}t+\beta^{4})/(t+1)\rangle$
H_{21}	$\{\infty,0,1,\beta^{12},\beta^{2},\beta^{9}\}$	$\mathcal{P}_3\mathcal{P}_4\mathcal{P}_5\mathcal{P}_3\mathcal{P}_4\mathcal{P}_5$	$Z_2=\langle(\beta^{10}t+1)/(\beta^{22}t+\beta^{22})\rangle$
H_{22}	$\{\infty,0,1,\beta^{12},\beta^{2},\beta^{10}\}$	$\mathcal{P}_3\mathcal{P}_3\mathcal{P}_4\mathcal{P}_4\mathcal{P}_4\mathcal{P}_4$	$V_4=\langle1/\beta^{12}t,(t+1)/(t+\beta^{12})\rangle$
H_{23}	$\{\infty,0,1,\beta^{12},\beta^{2},\beta^{14}\}$	$\mathcal{P}_3\mathcal{P}_3\mathcal{P}_3\mathcal{P}_3\mathcal{P}_5\mathcal{P}_5$	$V_4=\langle\beta^{12}t,\beta^{2}/t\rangle$
H_{24}	$\{\infty,0,1,\beta^{12},\beta^{2},\beta^{15}\}$	$\mathcal{P}_4\mathcal{P}_4\mathcal{P}_4\mathcal{P}_4\mathcal{P}_6\mathcal{P}_6$	$V_4=\langle\beta^{2}t,\beta^{3}/t\rangle$
H_{25}	$\{\infty,0,1,\beta^{12},\beta^{2},\beta^{16}\}$	$\mathcal{P}_4\mathcal{P}_4\mathcal{P}_7\mathcal{P}_5\mathcal{P}_5\mathcal{P}_7$	$Z_2=\langle(t+\beta^{12})/\beta^{12}(t+1)\rangle$
H_{26}	$\{\infty,0,1,\beta^{12},\beta^{2},\beta^{20}\}$	$\mathcal{P}_4\mathcal{P}_4\mathcal{P}_5\mathcal{P}_8\mathcal{P}_5\mathcal{P}_8$	$Z_2=\langle(t+1)/(t+\beta^{12})\rangle$
H_{27}	$\{\infty,0,1,\beta,\beta^{2},\beta^{13}\}$	$\mathcal{P}_7\mathcal{P}_7\mathcal{P}_7\mathcal{P}_7\mathcal{P}_7\mathcal{P}_7$	$S_3=\langle(\beta^{4}t+1)/(\beta^{3}t+1),\beta^{3}/t\rangle$
H_{28}	$\{\infty,0,1,\beta,\beta^{8},\beta^{15}\}$	$\mathcal{P}_8\mathcal{P}_8\mathcal{P}_8\mathcal{P}_8\mathcal{P}_8\mathcal{P}_8$	$S_3=\langle1/\beta^{6}(t+\beta^{3}),(t+\beta^{13})/(t+\beta^{12})\rangle$

Table 4-Stabilizer group type of heptads

SG-type	*No.*
I	32
Z_2	18
Z_3	3
Z_6	1

Table 5-Stabilizer group type of octads

SG-type	*No.*
I	78
Z_2	39
V_4	8
S_3	2
D_4	1
D_6	1
D_8	1
S_4	1

Table 6-Stabilizer group type of nonads

SG-type	*No.*
I	180
Z_2	37
S_3	3
Z_3	1
Z_4	1
Z_8	1

Table 7-Stabilizer group type of decads

SG-type	*No.*
I	294
Z_2	2
Z_3	6
V_4	10
D_4	2
D_5	1
A_4	1
D_8	1

Table 8-tabilizer group type of 11- sets

SG-type	*No.*
I	463
Z_2	62
Z_3	2
S_3	3
D_5	1

Table 9-Stabilizer group type of 12- sets

SG-type	*No.*
I	559
Z_2	110
Z_3	2
V_4	15
S_3	3
D_6	1
D_{12}	2

Table 10-Stabilizer group type of 13- sets

SG-type	*No.*
I	626
Z_2	74
Z_3	8
Z_6	1
Z_{12}	1
Z_4	3
D_{13}	1

In the following examples, some k-sets have been chosen where $k = 9,\cdots,13$ with unique largest size of stabilizer group.

Example 4.2

(i) There is unique nonads $\mathcal{K} = \{\infty, 0,1, \beta, \beta^4, \beta^5, \beta^6, \beta^{12}, \beta^{20}\}$ with stabilizer group of type Z_8 as given below.

$$Z_8 =< \beta(\beta^8 t + 1) >$$

(ii) There is a unique decad

$\mathcal{R} = \{\infty, 0,1, \beta, \beta^2, \beta^5, \beta^6, \beta^{10}, \beta^{12}, \beta^{18}\}$ with stabilizer group of type

$D_8 =< \beta^3 t/(\beta^2 t + 1), (\beta^{12} t + \beta^{12}) >$.

(iii) There is a uniqe11-set

$\mathcal{H} = \{\infty, 0,1, \beta, \beta^2, \beta^4, \beta^6, \beta^7, \beta^{12}, \beta^{16}, \beta^{18}\}$ with stabilizer group of type

$D_5 =< 1/(t + 1), (\beta^{12} t + \beta^4)/(\beta^{11} t + 1) >$.

(vi) There is a uniqe12-set

$\mathcal{J} = \{\infty, 0,1, \beta, \beta^2, \beta^3, \beta^6, \beta^9, \beta^{12}, \beta^{14}, \beta^{18}, \beta^{19}\}$ with stabilizer group of type

$D_6 =< (\beta^{18} t + \beta^6)/(\beta^{18} t + 1), (\beta^6 t + \beta^{12})/(t + \beta^{18}) >$.

(iv) There is a unique 13-set

$\mathcal{F} = \{\infty, 0,1, \beta, \beta^2, \beta^3, \beta^4, \beta^6, \beta^{11}, \beta^{12}, \beta^{16}, \beta^{17}, \beta^{22}\}$ with stabilizer group of type

$D_{13} =< 1/\beta^8(t + \beta^{13}), (\beta^{12} t + \beta)$.

5. Splitting

Each 13-set K_i and its complement ${K_i}^c$splitting $PG(1,25)$. The stabilizer group G_{K_i} of K_i also fixes the complement ${K_i}^c$. If $PG(1,25)$ split into two 13-sets $K = \{K_i, {K_i}^c\}$, then the stabilizer group of the partition K is as follows.

(i) If K_i projectively inequivalent to its complement ${K_i}^c$, then $G_{{K_i}^c}$ is G_{K_i} and the stabilizer group of the splitting is also G_{K_i}.

(ii) If K_i projectively equivalent to its complement ${K_i}^c$ then the stabilizer group of the splitting isG_{K_i} union of all linear transformation between K_i and${K_i}^c$. In this case, the stabilizer of the splitting generated always by two element one of them belong to $G_{K_i}$and other is projectivity between K_i and${K_i}^c$.

Theorem 5.1

The projective line $PG(1,25)$ has

(i) 158 projectively distinct partitions into two equivalent 13-sets(EQ).

(ii) 556 projectively distinct partition into inequivalent 13-sets (NEQ).

The partitions details are given in the following table.

Table 11-Partition of $PG(1,25)$ into two 13-sets

EQ:$\{K_i \cong {K_i}^c\}$	NEQ:$\{K_i \ncong {K_i}^c\}$
Z_2 :120	I :506
V_4:26	Z_2:48
S_3:6	Z_3:2
D_4:3	
D_6:1	
D_{12}:1	
G_{52}:1	

The group G_{52} has one element of order 1, 27 element of order 2, 12 element of order 13, 12 element of order 26.

Example 5.2

(i) The unique 13-set $K_{j_1} = \{\infty, 0, 1, \beta, \beta^2, \beta^3, \beta^6, \beta^8, \beta^9, \beta^{12}, \beta^{14}, \beta^{18}, \beta^{19}\}$ which has stabilizer group of type $Z_6 = \langle(\beta^{18}t + \beta^6)/(\beta^{18}t + 1)\rangle$ formed with its complement ${K_{j_1}}^c = \{\beta^4, \beta^5, \beta^7, \beta^{10}, \beta^{11}, \beta^{13}, \beta^{15}, \beta^{16}, \beta^{17}, \beta^{20}, \beta^{21}, \beta^{22}, \beta^{23}\}$ splitting as the projective line such that $K_{j_1} \cong {K_{j_1}}^c$. The projective equation which maps K_{j_1} to ${K_{j_1}}^c$ is given as follows.

$$\frac{\beta^5(t + \beta^9)}{t + \beta^{17}}.$$

This splitting has stabilizer group of type D_6 is generated by the following two elements:

$$a = \frac{\beta^{18}t + \beta^6}{\beta^{18}t + 1}, b = \frac{\beta^5(t + \beta^9)}{t + \beta^{17}}.$$

(ii) The 13-set $K_{j_2} = \{\infty, 0, 1, \beta, \beta^2, \beta^3, \beta^4, \beta^6, \beta^7 \beta^{12}, \beta^{14}, \beta^{16}, \beta^{18}\}$ has stabilizer group of type Z_3formed with its complement $K_{j_2}^c = \{\beta^5, \beta^8, \beta^9, \beta^{10}, \beta^{11}, \beta^{13}, \beta^{15}, \beta^{17}, \beta^{19}, \beta^{20}, \beta^{21}, \beta^{22}, \beta^{23}\}$ splitting the projective line such that $K_{j_2} \ncong K_{j_2}^c$.

HereZ_3 is generated by the element

$$c = \frac{t + \beta^{15}}{\beta^{14}(\beta^8 t + 1)}.$$

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